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Constructing Box Splines by using subdivisions

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Abstract: *In this paper, we construct and draw the graph of some linear box splines using subdivision. The functions were introduced by Prautzsch in [3]. We focus in a particular example of box splines are the B-splines with equidistant knots. Box splines consist of regularly arranged polynomial pieces. A particular interest in linear box spline surfaces that consist of triangular polynomial pieces. The technique involves control points which can be computed iteratively using Matlab from the initial control points of well dened recursion.*

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I. Introduction

Box Splines were widely and explicitly studied by Boor, *Höllig* and Riemenschneider [1]. An S-variate box spline $B(x|v_1 \cdots v_k)$ is determined by some k directions v_i in R^s. For simplicity, we will assume that $k \geq s$ and that $v_1, \ldots, v_s$ are linearly independent.
A box spline $B_k(x) = B(x|v_1 \ldots v_k)$ can be constructed geometrically as shown in Figure 1 for k = 4 and s = 2. Any box spline surface

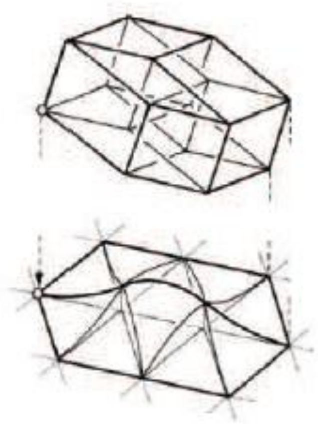

Figure 1: Box splines are shadows of boxes.

$$s(x) = \sum_{i \in Z^s} c_i B(x - i). \quad (1)$$

is an affine combination of its control points c_i and this surface representation is affinely invariant meaning that under any affine map the control point images control the surface image.
Since the box splines are non-negative, s(x) is even a convex combination of its control points and lies in their convex hull.

II. Convexity

Let $e_1, \ldots, e_s$ denote the unit directions and let $e = e_1 + \ldots + e_s$. Further, let

$$s(x) = \sum_{i \in Z^s} c_i B(x - i | e_1 \ldots e_1 \ldots e_s \ldots e_s e \ldots e). \tag{2}$$

be a box spline surface with these directions. The piecewise linear box spline surface

$$c(x) = \sum_{i \in Z^s} c_i B(x - i | e_1 \ldots e_1 \ldots e_s e), \tag{3}$$

is said to be the control net of the surface s(x).
If the control net c(x) is a scalar valued and convex, then the surface s(x) is also a convex function [2, 4]. Furthermore, any control net of s(x) obtained under subdivision as described next is convex [5].

III. Box splines

Any box spline $B_k(x) = B(x|v_1 \cdots v_k)$ can also be constructed geometrically as in Figure 3 ..... for k = 4 and s = 2.
Let π be the orthogonal projection

$$\pi : [t_1 \cdots t_k]^t \mapsto [t_1 \cdots t_s]^t,$$

and

$$\beta_k = [u_1 \cdots u_k][0, 1)^k,$$

be a parallelepiped such that $v_i = \pi u_i$.
Then $B_k(x)$ represents the density of the "shadow" of β_k, i.e.,

$$B_k(x) = \frac{1}{vol_k \beta_k} vol_{k-s} \beta_k(x),$$

where

$$\beta_k(x) = \pi^{-1} x \bigcap \beta_k.$$

For K=3 and s=2, the corresponding geometric construction illustrated in Figure 4. It is due to [6], while the idea of polyhedral shadows can be traced back to [7, 8].

IV. Some properties of Box splines

The Box spline $B(x)$= have the following properties:

- does not depend on the ordering of the directions v_i,
- is positive over the convex set $[v_1 \cdots v_k][0,1)^k$,
- has the support suppB(x) $= [v_1 \cdots v_k][0,1]^k$,
- is symmetric with respect to the center to its support.
- The box spline B(x) is polynomial of degree $\leq k - s$ over each tile of this partition.

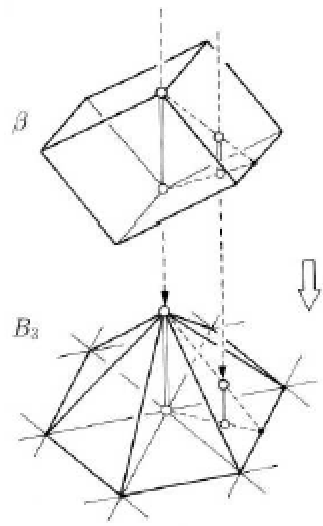

Figure 2: Geometric construction of a piecewise linear box spline over a triangular grid.

V. Box spline surface

Any Box spline surface

$$s(x) = \Sigma_{i \in Z^s} c_i B(x - i),$$

is an affine combination of its control points c_i and the surface representation is affinely invariant meaning that under any affine map the control point images control the surface image.
Since the box splines are non-negative, s(x) is even a convex combination of its control points and lies in their convex hull.

VI. Subdivision

Any box $\beta = [u_1 \cdots u_k][0,1)^k$ in $\mathbb{R}^k$ can be partitioned into 2^k translates of the scaled box $\hat{\beta} = \beta/2$ spanned by the half directions $\hat{u}_i = u_i/2$, see Figure 7.

Based on this observation Prautzsch [9] concluded in 1993 that the non-normalised "shadow" $M_\beta(x) = vol_{k-s}(\pi^- x \bigcap \beta)$ of β under the projection

$$\pi : [t_1 \ldots\ldots t_k]^t \longmapsto [t_1 \ldots t_s]^t$$

can be written as a linear combination of translates of the scaled box spline $M_{\hat{\beta}}(x) = 2^{s-k} M_\beta(2x)$.

Consequently, if the projections $v_i = \pi u_i$ lie in $\mathbb{Z}^s$, then any box spline surface

$$s(x) = \Sigma_{i \in Z^s} c_i^1 B(x - i),$$

with $B(x) = B(x|v_1 \ldots v_k)$ has also a "finer" representation

$$s(x) = \Sigma_{i \in Z^s} c_i^2 B(2x - i).$$

The new control points c_i^2 can be computed iteratively from the initial control points c_i^1 by the recursion

$$d^0(i) = \begin{cases} 0 & \text{if } \; i/2 \notin Z^s \\ c^1_{i/2} & \text{if } \; i/2 \in Z^s \end{cases},$$

$$d^r(i) = (d^{r-1}(i) + d^{r-1}(i - v_r))/2, \quad r = 1 \ldots k,$$

$$c_i^2 = 2^s d^k(i).$$

VII. Algorithm

Step 1:

Initial coefficient will be given

Directional vectors $v_1 = (1,0)$, $v_2 = (0,1)$ and $v_3 = (1,1)$.

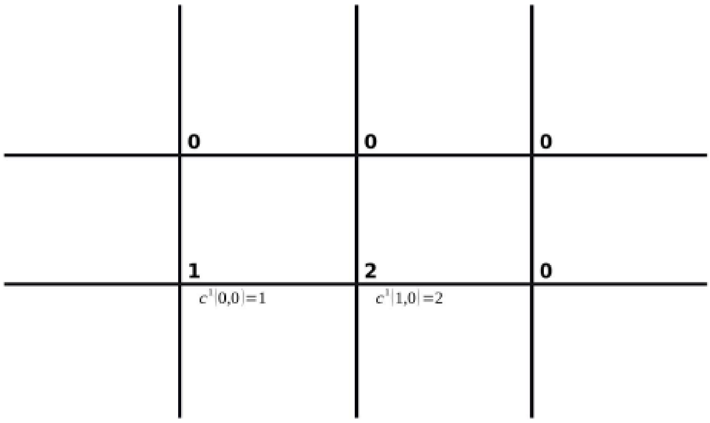

Figure 3: c^1 Control Points.

$$c^1_{(0,0)} = 1 \qquad c^1_{(1,0)} = 2,$$

$$d^0(j,k) = 0 \qquad if\ j\ \ is\ odd\ or\ k\ is\ odd.$$

$$d^0(2j,2k) = c^1_{(j,k}.$$

$$d^0(0,0) = c^1_{(0,0)} = 1,$$

$$d^0(1,0) = 0,$$

$$d^0(2,0) = c^1_{(1,0)} = 2,$$

We have three directional vectors: $v_1(1,0), v_2 = (0,1)$ and $v_3 = (1,1)$.

Direction $v_1(1,0)$:

$$d^1(i) = \frac{d^0(i) + d^0(i-(1,0))}{2},$$

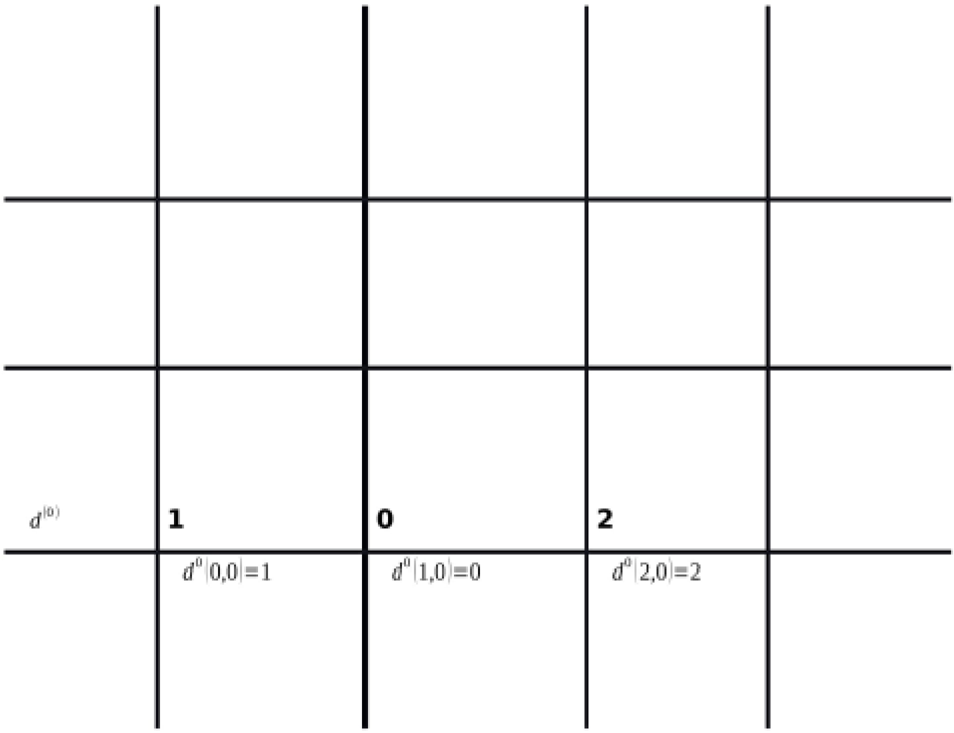

Figure 4: d^0 Control Points.

where i is a vector.

$$d^1(0,0) = \frac{d^0(0,0) + d^0(-1,0)}{2} = \frac{1}{2}.$$

$$d^1(1,0) = \frac{d^0(1,0) + d^0(0,0)}{2} = \frac{1}{2}.$$

$$d^1(2,0) = \frac{d^0(2,0) + d^0(1,0)}{2} = 1.$$

$$d^1(3,0) = \frac{d^0(3,0) + d^0(2,0)}{2} = 1.$$

Direction $v_2(0,1)$:

$$d^2(i) = \frac{d^1(i) + d^1(i-(0,1))}{2},$$

where i is a vector.

$$d^2(0,0) = \frac{d^1(0,0) + d^1(0,-1)}{2} = \frac{1}{4}.$$

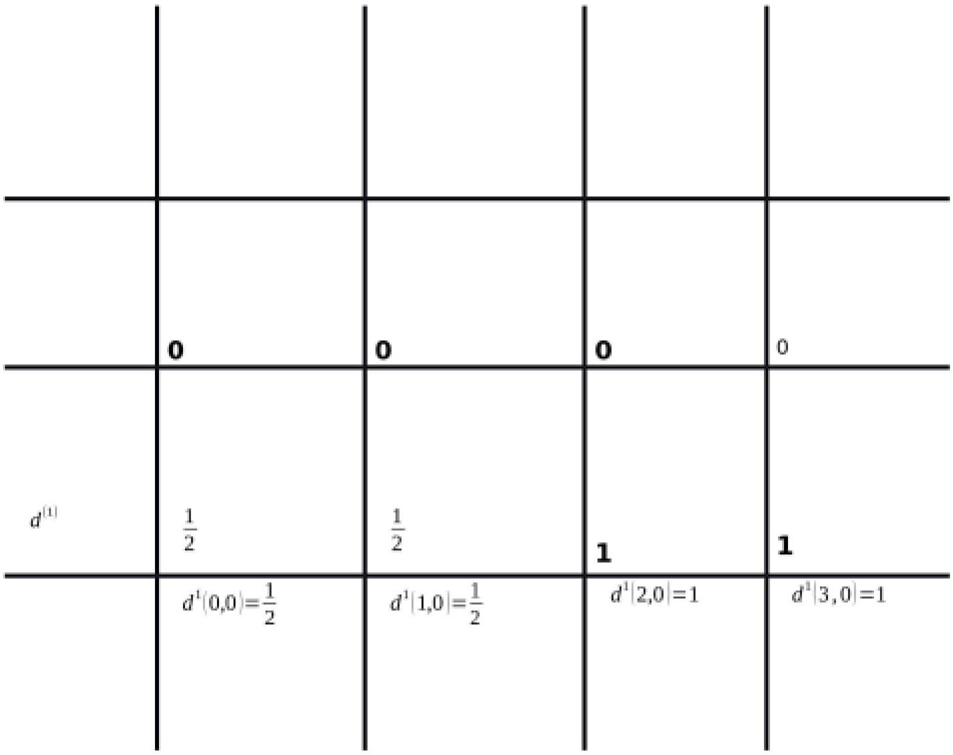

Figure 5: d^1 Control Points.

$$d^2(1,0) = \frac{d^1(1,0) + d^1(1,-1)}{2} = \frac{1}{4}.$$

$$d^2(2,0) = \frac{d^1(2,0) + d^1(2,-1)}{2} = \frac{1}{2}.$$

$$d^2(3,0) = \frac{d^1(3,0) + d^1(3,-1)}{2} = \frac{1}{2}.$$

$$d^2(0,1) = \frac{d^1(0,1) + d^1(0,0)}{2} = \frac{1}{4}.$$

$$d^2(1,1) = \frac{d^1(1,1) + d^1(1,0)}{2} = \frac{1}{4}.$$

$$d^2(2,1) = \frac{d^1(2,1) + d^1(2,0)}{2} = \frac{1}{2}.$$

$$d^2(3,1) = \frac{d^1(3,1) + d^1(3,0)}{2} = \frac{1}{2}.$$

Such that
(0,0) - (0,1) = (0,-1)

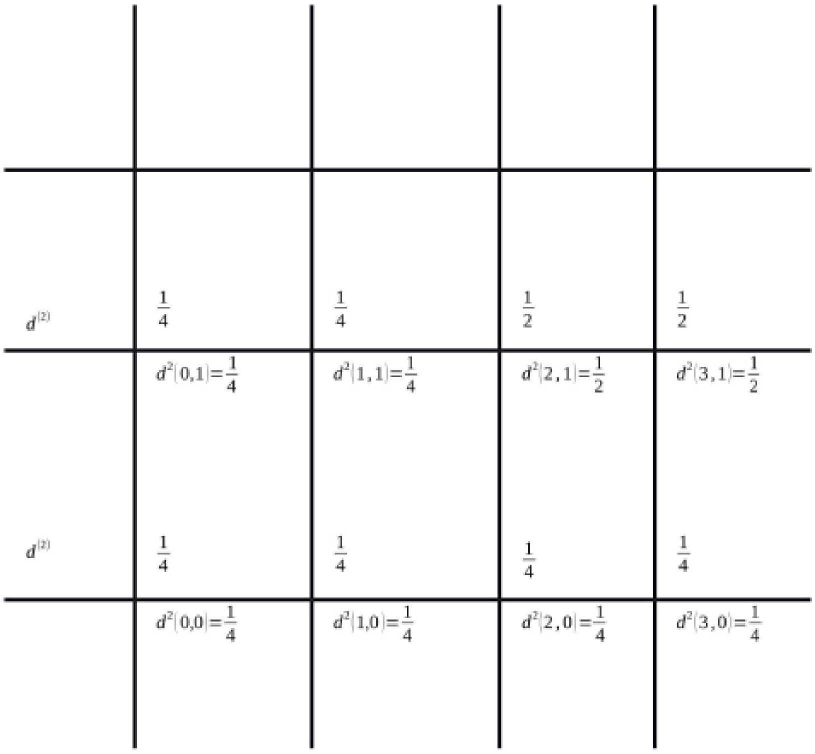

Figure 6: d^2 Control Points.

(1,0) - (0,1) = (1,-1)
(2,0) - (0,1) = (2,-1)
(3,0) - (0,1) = (3,-1)
(0,1) - (0,1) = (0,0)
(1,1) - (0,1) = (1,0)
(2,1) - (0,1) = (2,0)
(3,1) - (0,1) = (3,0).

Direction $v_3(1,1)$:

$$d^3(i) = \frac{d^2(i) + d^2(i-(1,1))}{2},$$

where i is a vector.

$$d^3(0,0) = \frac{d^2(0,0) + d^2(-1,-1)}{2} = \frac{1}{8},$$

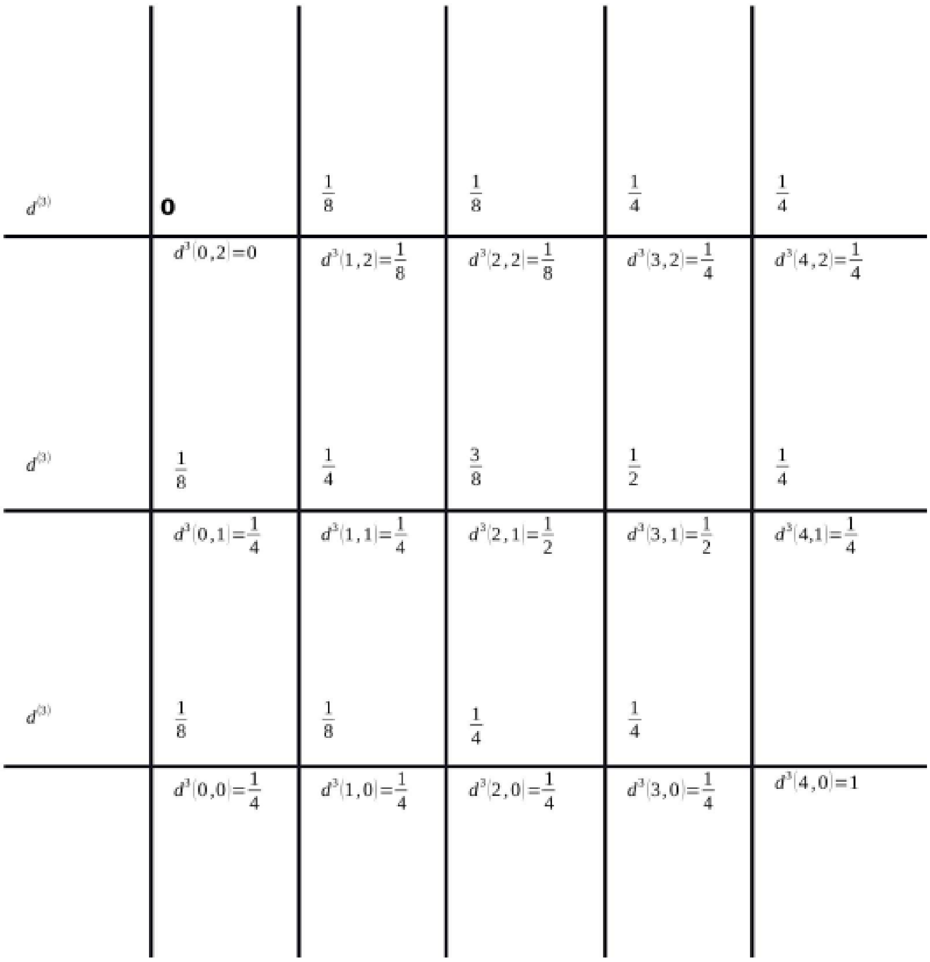

Figure 7: d^3 Control Points.

$$d^3(1,0) = \frac{d^2(1,0) + d^2(0,-1)}{2} = \frac{1}{8},$$
$$d^3(2,0) = \frac{d^2(2,0) + d^2(1,-1)}{2} = \frac{1}{4},$$
$$d^3(3,0) = \frac{d^2(3,0) + d^2(2,-1)}{2} = \frac{1}{4},$$

such that
(0,0) - (1,1) = (-1,-1)
(1,0) - (1,1) = (0,-1)
(2,0) - (1,1) = (1,-1)
(3,0) - (1,1) = (2,-1).

Computing control points c_i^2:

$$c_i^2 = 4d^3(i).$$

For example

$$c_{(0,0)}^2 = 4d^3(0,0) = 4\frac{1}{8} = \frac{1}{2}.$$

c_i^1	i(x,y)			
c_i^2	$\frac{i}{2}$		c_i^2	$\frac{i}{2}$
c_i^3	$\frac{i}{4}$		$c_{(0,0)}^2$	(0,0)
c_i^4	$\frac{i}{8}$		$c_{(1,0)}^2$	$(\frac{1}{0}, 0)$
c_i^k	$\frac{i}{2^{k-1}}$		$c_{(2,0)}^2$	(1,0)

We evaluate all of the control points and the resulting vectors v_i where $i = 1 \dots 15$.

$\frac{i}{2^{k-1}}$	c_i^k	i(x,y)			
$\frac{1}{2}$	$c_{(0,0)}^2$	(0,0)		$(0,0,\frac{1}{2})$	v_1
$\frac{1}{2}$	$c_{(1,0)}^2$	$(\frac{1}{2}, 0)$		$(\frac{1}{2},0, \frac{1}{2})$	v_2
1	$c_{(2,0)}^2$	(1,0)		(1,0,1)	v_3
1	$c_{(3,0)}^2$	$(\frac{3}{2},0)$		$(\frac{3}{2},0,1)$	v_4
0	$c_{(4,0)}^2$	(2,0)		(2,0,0)	v_5
$\frac{1}{2}$	$c_{(0,1)}^2$	$(0,\frac{1}{2})$		$(0,\frac{1}{2},\frac{1}{2})$	v_6
1	$c_{(1,1)}^2$	$(\frac{1}{2},\frac{1}{2})$		$(\frac{1}{2},\frac{1}{2}, 1)$	v_7
$\frac{3}{2}$	$c_{(2,1)}^2$	$(1,\frac{1}{2})$		$(1,\frac{1}{2},\frac{3}{2})$	v_8
2	$c_{(3,1)}^2$	$(\frac{3}{2},\frac{1}{2})$		$(\frac{3}{2},\frac{1}{2},2)$	v_9
1	$c_{(4,1)}^2$	$(2,\frac{1}{2})$		$(2,\frac{1}{2},1)$	v_10
0	$c_{(0,2)}^2$	(0,1)		(0,1,0)	v_11
$\frac{1}{2}$	$c_{(1,2)}^2$	$(\frac{1}{2}, 2)$		$(\frac{1}{2},1, \frac{1}{2})$	v_12
$\frac{1}{2}$	$c_{(2,2)}^2$	(1,1)		$(1,1,\frac{1}{2})$	v_13
1	$c_{(3,2)}^2$	$(\frac{3}{2},1)$		$(\frac{3}{2},1,1)$	v_14
1	$c_{(4,2)}^2$	(2,1)		(2,1,1)	v_15

VIII. Using Matlab

We use Matlab to find all the control points and plot the graph of the resulting linear Box splines with initial four control points.

$$c^1 = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$d^0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}$$

$$d^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.5 & 1.5 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 2.5 & 2.5 & 3 \end{pmatrix}$$

$$d^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.75 & 0.75 & 1 \\ 0 & 0.75 & 0.75 & 1 \\ 0 & 1.25 & 1.25 & 1.5 \end{pmatrix}$$

$$d^3 = \begin{pmatrix} 0 & 0 & 0.375 & 0.375 \\ 0 & 0.375 & 0.75 & 0.875 \\ 0 & 0.375 & 1 & 1.125 \\ 0 & 1.25 & 1.25 & 1.5 \end{pmatrix}$$

$$c^2 = 4 * d^3 = \begin{pmatrix} 0 & 0 & 1.5 & 1.5 \\ 0 & 1.5 & 3 & 3.5 \\ 0 & 1.5 & 4 & 4.5 \\ 0 & 5 & 5 & 6 \end{pmatrix}$$

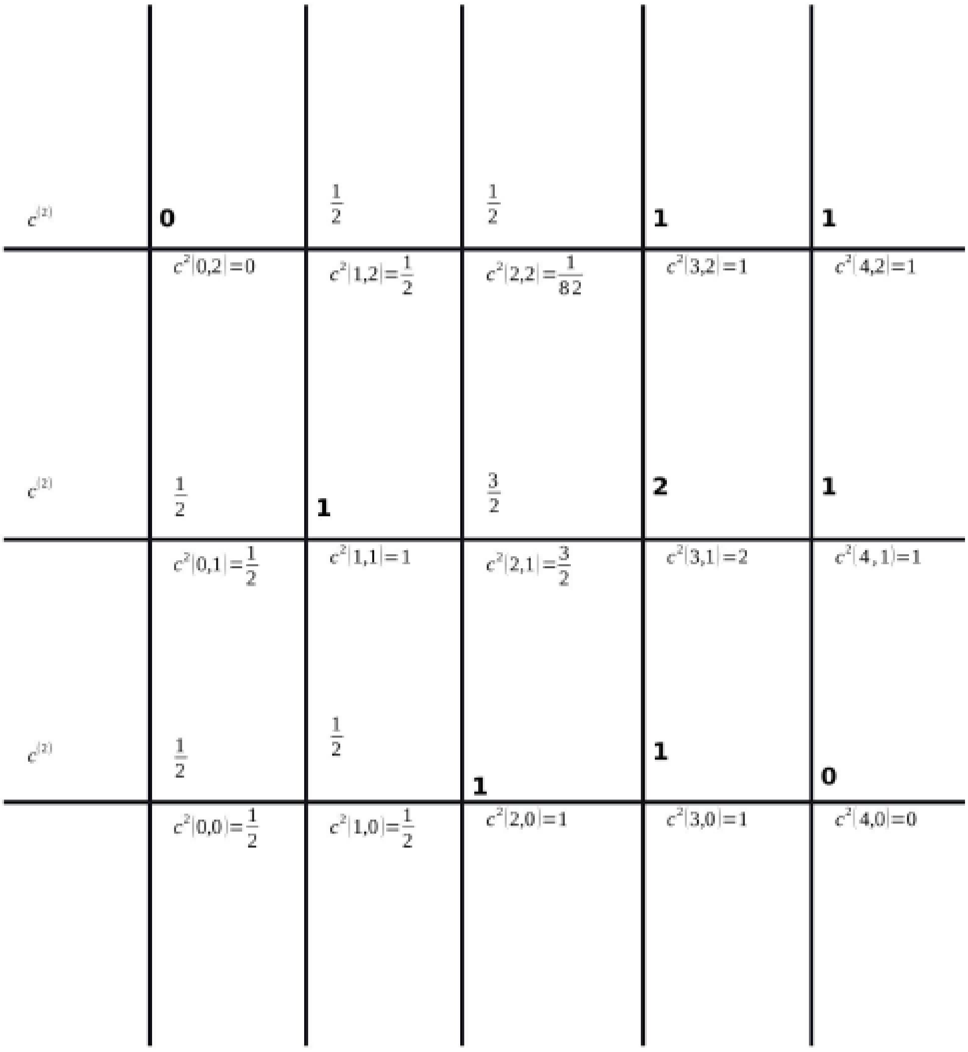

Figure 8: c^2 Control Points.

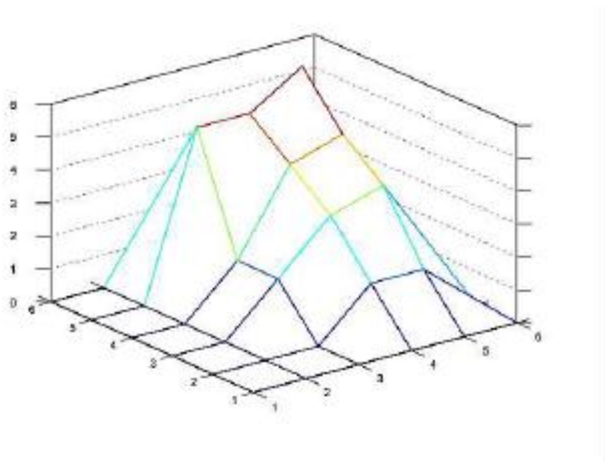

Figure 9: The Linear Box spline.

Bibliography

[1]. C. de Boor, K. H□ollig, and S. Riemenschneider. Box Splines, Springer-Verlag.
[2]. W. Dahmen and C. A. Micchelli. Convexity of multivariate Bernstein polynomials and box spline surfaces. Studia Math., 23:265-287, 1988.
[3]. H. Prautzsch, and W. Boehm. Box Splines.Handbook of Computer Aided Geometric Design,Farin, Hoschek, Kim, eds.
[4]. H. Prautzsch. The location of the control points in the case of box splines. IMA J. Numer. Anal., 6:43-49, 1986.
[5]. H. Prautzsch. On convex Bezier triangles. Mathematical Modelling and Numerical Analysis, 26:23-36, 1992. [6] C. de Boor and R. DeVore. Approximation by smooth multivariate splines. Trans. Amer. Math. Soc., 276: 775-788, 1983.
[6]. I. J. Schoenberg. Contributions to the problem of approximation of equidistant data by analytic functions, Part A: On the problem of smoothing of graduation. a rst class of analytic approximation. Quart. Appl. Math., 3:45-99, 1946.
[7]. A. Sommerfeld. Eine besondere anschauliche Ableitung des Gaussischen Fehlergesetzes. In Geburtstage, Verlag von J.A. Barth, Festschrift Ludwing Boltzmann gewidmet zum 60, pages 848-859. Leipzig, 1904.
[8]. H. Prautzsch. Unterteilungsalgorithmen f □ur multivariate Splines ein geometrischer Zugang. Dissertation, Univ. Braunschwig, 1984.

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