A note on norm attaining operators

Fatima O.Alnoor
Corresponding Author: Fatima O.Alnoor

Abstract: We show the norm attaining quadraticallyhyponormal weighted shift is subnormal . Also , We show that there is a Banach space X such that the set of norm attaining operators from Xto any infinite dimensional space L₁(μ) is not dense .

I. Introduction

Let ℋ be a complex Hilbert space and let L(ℋ) denote the algebraof bounded linear operators on ℋ. An operator A ∈ L(ℋ) is said tobe normal if A* A = AA* and subnormal if A = N|ₐ, where N is normal on some Hilbert space K ⊆ ℋ. Anoperator A ∈ L(ℋ) is said to be paranormal if ∥A²x∥ ≥ ∥Ax∥² for allunit vector x ∈ H. An operator A ∈ L(ℋ) is called normattaining if there is an x ∈ ℋ with ∥x∥ = 1 and ∥Ax∥ = ∥A∥.

The Bishop -Plehmstheorem, the origin of the so-called “perturbation minimization principles,” asserts that the set of norm-attaining functionals on a Banach space is norm dense in the set of all bounded functionals . Given Banach spaces X and Y, let us consider the Banach space L(X, Y), of bounded linearoperators from X into Y and let us denote by NA(X, Y), the set ofnorm-attaining operators; that is, A ∈ NA(X, Y), if for some element x in the unit sphere X, such that ∥Ax∥ = ∥A∥. In mentioned paper by E.Bishop and R.Plehmstauther raised the problem if NA(X, Y), is norm dense in L(X, Y).

II. Results

We start from a basic criterion for norm attaining operators:

Lemma 1. If A ∈ L(ℋ) is a norm attaining operator if and only if ∥A∥² ∈ σ₁(A*A) where σ₁(S) denote the point spectrum of S ∈ L(ℋ).

Proof. Observe that ∥Ax∥ = ∥A∥∥x∥ if and only if (A*A − ∥A∥²)x, x) = 0 since A* A − ∥A∥² is hermitian, we can see that (A*A − ∥A∥²)x, x) = 0 if and only if A*Ax − ∥A∥² x or equivalently, x ∈ Ker(A* A − ∥A∥²). Thus A is a norm attaining operator if and only if ∥A∥² ∈ σ₁(A*A).

Let (βₙ)ₙ₌₀ be a bounded sequence of positive real numbers, and letAₜ: ℓ²(ℤ₊) → ℓ²(ℤ₊) be the associated unilateral weighted shift, defined byAₜgₙ = βₙgₙ₊₁ (all n ≥ 0), where {gₙ}ₙ₌₀ is the canonical orthonormal basis in ℓ²(ℤ₊). It is well-known that Aₜ is hyponormal if and only if βₙ < βₙ₊₁ for all n ≥ 0.

Theorem 2. Aₜ is norm attaining if and only if ∥Aₜ∥ = β₁ for some i.

Proof. SinceAₜAₜ = diag(β₀, β₁, ..., ) we have σₙ(AₜAₜ) = {β₀, β₁, ..., }. The desired result now follows from Lemma 1.

In addition to its usefulness to produce examples of hyponormal weighted shifts T for which Aₜ + λAₜ² is not hyponormal (for some complex number λ),

For, if β₀ < β₁ < β₂ = β₃... , one knows that the associated Aₜ can't be subnormal, so one could use the freedom in β₀ and β₁ to build such an example. However, such an attempt is doomed to fail, as the following theorem shows. First, we need a definition.

Definition 3: Let Aₜ be a Hilbert space operator. We call Aₜ quadratically hyponormal if Aₜ + λAₜ² is hyponormal for every complex number λ.

Theorem 4: Let Aₜ be a subnormal weightedshift with weight sequence {βₙ}ₙ₌₀ if βₙ = βₙ₊₁ = ... for some t ≥ 0 then

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Corollary 5: Let $A_\beta$ be a norm attaining hyponormal weighted shift. Then $\beta_n = \beta_{n+1} = \ldots$ for some $n \geq 0$.

**Proof.** By Theorem 4, we have that $\|A_\beta\| = \beta_n = \max_i \beta_i$ for some $n \geq 0$. But since $A_\beta$ is hyponormal, the corresponding weight sequence is monotonically increasing. Thus, $\beta_n = \beta_{n+1} = \ldots$ for some $n \geq 0$.

Theorem 6: If $A_\beta$ is 2-hyponormal and $\beta_n = \beta_{n+1}$ for some $n$, then $\beta_1 = \beta_2 = \beta_3 = \ldots$, $A_\beta$ is subnormal.

Although the norm attaining operators are dense in $\mathcal{L}(\mathcal{H})$, we can not expect that every hyponormal operator is a norm attaining operator.

Corollary 7: Let $\beta \equiv \{\beta_n\}_{n=0}^{\infty}$ be a strictly increasing bounded sequence. Then $A_\beta$ is hyponormal (and hence paranormal), but not norm attaining.

### III. Lorentz Spaces

Let us start by recalling the definition of Lorentz sequence spaces and preduals, a family of classical Banach spaces.

By an *admissible sequence* $w$, we shall mean a decreasing sequence $w = (w(n))$ of positive numbers such that $w(1) = 1$ and $w \in c_0 \setminus \ell_1$, the Banach space of all sequences of scalars $b = (b(n))$ for which

$$\|b\| = \sup_{\pi} \left( \sum_{j=1}^{\infty} |b(\pi(n))|^p w(n) \right)^{1/p},$$

where $\pi$ ranges over all permutations of the integers. Denote by $d_*(w, p)$ is called Lorentz sequence if $p = 1$ it is known [8,15] that $d(w, 1)$ has predual $d_*(w, 1)$ which is defined by

$$d_*(w) = \left\{ b \in c_0 : \lim_{n \to \infty} \sum_{j=1}^{n} b^*(j) = 0 \right\},$$

where $b^*$ is the decreasing rearrangement of $\{|b(n)|\}$. The norm of $d_*(w)$ is given by

$$\|b\| = \sup_n \frac{\sum_{j=1}^{n} b^*(j)}{\sum_{j=1}^{n} w(j)}.$$

Thus $\|b\| \leq 1$ if and only if

$$\sum_{k \in J} b(k) \leq \sum_{j=1}^{n} w(j).$$

For any $n \in \mathbb{N}$ and any set $J \subset \mathbb{N}$ with $n$ elements.

$d_*(w)$ predual of the Lorentz sequence space $d(w, 1)$, and $d_*(w)$ is space with symmetric basis $\{e_n\}$ that shares some properties of $c_0$. Also the space $d_* \left( \frac{1}{n} \right)$ was used by [GO] to get an example of a space $X$ such that the closure of the $NA(X, \ell_p)$ is the set of compact operators for any $1 < p < \infty$ and so the set of norm attaining operators is not dense, since the space $d_* \left( \frac{1}{n} \right)$ is a subset of $\ell_p$.

We do have:

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Theorem 8: Let \( w \notin \ell_1 \) be a decreasing sequence of positive real numbers and \( \mu \) any positive measure. The following assertions hold:

i) \( NA(d_*(w,1)L_1(\mu)) = K(d_*(w,1)L_1(\mu)) \)

ii) If \( \mu \) is purely atomic, the set of norm attaining operators from \( d_*(w,1) \) to \( L_1(\mu) \) is dense.

iii) If \( \mu \) is not purely atomic and \( \sigma \)-finite, then

\[
NA(d_*(w,1)L_1(\mu)) = L(d_*(w,1)L_1(\mu)) \iff w \notin \ell_1
\]

Theorem 9: Assume that \( w \in \ell_2 \setminus \ell_1, \) For the complex Lorentz sequence space \( d(w,1) \) and its canonical predual \( d_*(w,1) \), then \( NA(d_*(w,1),d(w,1)) \) is not dense in \( L(d_*(w,1),d(w,1)) \).

Reference
