One Wave Solution to the Boussinesq and the Kudryashov–Sinelshchikov equations

Md. Khorshed Alam\textsuperscript{1}, Shahanaj Pervin\textsuperscript{2}

\textsuperscript{1,2}Department of Arts and Sciences, Bangladesh Army University of Science and Technology, Bangladesh

Corresponding Author: Md. Khorshed Alam

Abstract: The main goal of this paper is to demonstrate the use of multiple exp-function method for Boussinesq equation and Kudryashov–Sinelshchikov equation (KSE). With the help of Maple, applying the approach to these equations yields some new exact explicit travelling one wave solutions. The significance of obtained solutions gives credence to the explanation and understanding of related physical phenomena.

Keywords: The Boussinesq equation and the Kudryashov–Sinelshchikov equations, Multiple exp-function method, One Wave Solution.

I. Introduction

Many physical, mechanical, chemical, biological, engineering and some economic laws and relations appear mathematically in the form of differential equations which are linear or nonlinear, homogeneous or inhomogeneous. Almost all differential equations relating physical phenomena are nonlinear. Methods of solutions of linear differential equations are reasonably easy and well avowed. In contrast, the techniques of solutions of nonlinear differential equations are less obtainable and in general, approximations are generally used. Nonlinearity is a fascinating element of nature, today; many scientists observe nonlinear science as the most important frontier or the fundamental understanding of nature. The analytical solutions of such equations are of fundamental importance to reveal the inner structure of the phenomena. The world around us is inherently nonlinear. Nonlinear evolution equations (NLEEs) are able to designate the real applications in a plethora of areas in science, technology and engineering [1–25]. Some analytic methods have been utilized to investigate NLEEs, such as the Hirota bilinear method, the Hietarinta approach, the Bäcklund transformation method, multiple exp-function algorithm, Darboux transformation, Pfaffian technique, the inverse scattering method and the generalized symmetry method [1–10]. The Hirota’s bilinear method [6], the multiple exp-function algorithm [4,5] and the Hereman–Nuseir simplified form [7] are rather heuristic and the most frequently used in the literature. These approaches have powerful properties that make it conceivable for the purpose of constructing multiple soliton solutions for variety nonlinear evolution equations [11–16]. The multiple exp-function algorithm and the simplified Hirota’s method are independent in constructing bilinear forms. It undertakes that the multisoliton solutions can be expressed as polynomials of exponential functions. The multiple exp-function algorithm, is basically a generalization of Hirota’s perturbation scheme. Furthermore, the subsequent solutions contain generic phase shifts and wave frequencies.

Our aim in this paper is to present an application the multiple exp-function method to the Boussinesq equation and Kudryashov–Sinelshchikov equation (KSE) to be solved by this method for the first time.

We consider the Boussinesq equation in the following form

\[ u_t - u_{xx} - (u^2)_{xx} - \mu u_{xxxx} = 0 \]  

(1.1)

and the Kudryashov–Sinelshchikov equation (KSE) given by [26,27],

\[ u_t + \alpha uu_x + \beta u_{xxx} + \gamma (uu_x)_x + du_{xx} = 0 \]  

(1.2)

The rest of the paper is organized as follows: In Section 2, we give the description the multiple exp-function method. In Section 3, we apply this method to the Boussinesq equation and the Kudryashov–Sinelshchikov equation (KSE) with discussion and Conclusions are given at last section.
II. Multiple Exp-Function Algorithm

We first describe the procedure for constructing multiple wave solutions to nonlinear equations [4], by considering the \((1 + 1)\)-dimensional evolution equation:

\[ p(x,t,u_x,u_t, \ldots) = 0 \]  

(2.1)

Step 1. Defining solvable differential equations:

We introduce new variables, \( \eta_i = \eta_i(x,t) \), \( 1 \leq i \leq n \), by solvable partial differential equations, for example, the linear ones:

\[ \eta_{ix} = k_i \eta_i, \quad \eta_{it} = -\omega_i \eta_i, \quad 1 \leq i \leq n, \]  

(2.2)

where \( k_i, 1 \leq i \leq n \), are the angular wave numbers and \( \omega_i, 1 \leq i \leq n \), are the wave frequencies. Solving such linear equations yields the exponential function solutions:

\[ \eta_i = c_i e^{\epsilon_i}, \quad \xi_i = k_i - \omega_i t, \quad 1 \leq i \leq n, \]  

(2.3)

where \( c_i, 1 \leq i \leq n \), are any constants, positive or negative. This explains why we called the approach the multiple exp-function method.

The arbitrariness of the constants brings \( c_i, 1 \leq i \leq n \), more choices for solutions than we used to [28, 29]. Each of the functions \( \eta_i, 1 \leq i \leq n \), describes a single wave, and a multiple wave solution is a combination using all those single waves.

We remark that the linear differential relations in (2.2) are extremely helpful while transforming differential equations to algebraic equations and carrying out necessary computations by computer algebra systems.

Step 2. Transforming nonlinear PDEs

Consider rational solutions in the new variables \( \eta_i, \eta_i(x,t), 1 \leq i \leq n \):

\[ u(x,t) = \frac{p(\eta_1, \eta_2, \ldots, \eta_n)}{q(\eta_1, \eta_2, \ldots, \eta_n)}, \quad p = \sum_{r,s=1}^{n} \sum_{j=0}^{M} p_{r,s,j} \eta_i \eta_j, \quad q = \sum_{r,s=1}^{n} \sum_{j=0}^{N} q_{r,s,j} \eta_i \eta_j, \]  

(2.4)

where \( p_{r,s,j} \) and \( q_{r,s,j} \) are all constants to be determined from the original equation (2.1). By manipulating differential relations in (2.2), we can express all partial derivatives of \( u \) with \( x \) and \( t \) in terms of \( \eta_i, 1 \leq i \leq n \). For example, we can have

\[ u_x = \frac{q \sum_{i=1}^{n} p_{i,i} \eta_i - p \sum_{i=1}^{n} q_{i,i} \eta_i}{q^2} = \frac{-q \sum_{i=1}^{n} \omega_i p_{i,i} \eta_i - p \sum_{i=1}^{n} \omega_i q_{i,i} \eta_i}{q^2} \]  

(2.5)

and

\[ u_t = \frac{q \sum_{i=1}^{n} p_{i,i} \eta_i - p \sum_{i=1}^{n} q_{i,i} \eta_i}{q^2} = \frac{-q \sum_{i=1}^{n} k_i p_{i,i} \eta_i - p \sum_{i=1}^{n} k_i q_{i,i} \eta_i}{q^2} \]  

(2.6)
where \( p_{\eta_i} \) and \( q_{\eta_i} \) are partial derivatives of \( p \) and \( q \) with respect to \( \eta_i \). Substituting (2.4) and its derivatives leads to rational function equation in the new variables \( \eta_i, 1 \leq i \leq n \)

\[
Q(x,t,\eta_1,\eta_2,\ldots,\eta_n) = 0.
\]

(2.7)

This is called the transformed equation of the original equation (2.1).

Step 3. Solving algebraic systems

Now we set the numerator of the resulting rational function \( Q(x,t,\eta_1,\eta_2,\ldots,\eta_n) = 0 \) to zero. This yields a system of algebraic equations of all variables \( k_i, \omega_i, p_{\xi_{ij}}, q_{\xi_{ij}} \). We solve this system to determine two polynomials \( p \) and \( q \) and the wave exponents \( \xi_i, 1 \leq i \leq n \).

As a result, the multiple wave solution \( u \) is computed and given by

\[
u(x,t) = \frac{p(c_1 e^{k_1 x - \alpha_1 t}, \ldots, c_n e^{k_n x - \alpha_n t})}{q(c_1 e^{k_1 x - \alpha_1 t}, \ldots, c_n e^{k_n x - \alpha_n t})}.
\]

(2.8)

### 2.1 Formation Of Solution With Discussion

In 1872, the Boussinesq equation [30] was established by Joseph Boussinesq to describe the propagation of small amplitude, long waves on the surface of shallow water. Kudryashov and Sinelshchikov [31] reported a more common nonlinear partial differential equation for describing the pressure waves within a mixture liquid and gas bubbles, by considering the viscosity of liquid and the heat transfer in 2010.

#### 3.1 One wave solution of Boussinesq equation

Let us apply the multiple exp-function method to the Boussinesq equation (1.1)

We require the linear conditions,

\[
\begin{align*}
\eta_{1,x} &= k_1 \eta_1, \\
\eta_{1,t} &= -\omega_1 \eta_1,
\end{align*}
\]

(3.1)

where \( k_1 \) and \( \omega_1 \) are constants. We then try a pair of polynomials of degree one,

\[
\begin{align*}
p(\eta_1) &= a_0 + a_1 \eta_1, \\
q(\eta_1) &= b_0 + b_1 \eta_1,
\end{align*}
\]

(3.2)

where \( a_0, a_1, b_0, \) and \( b_1 \) are constants to be determined. By the multiple exp-function method and using the differential relations in (3.1), we obtain the following solution to the resulting algebraic system with Maple,

\[
\begin{align*}
b_1 &= \frac{a_1 b_0}{a_0}, \\
\omega_1 &= \sqrt{b_0 (\mu b_0 k_1^2 + 6 a_0 + b_1)} k_1,
\end{align*}
\]

(3.3)

and all other constants are arbitrary. Since we can have an exponential function solution to (3.1),

\[
\eta_1 = e^{k_1 x - \alpha_1 t}.
\]

(3.4)

the corresponding one-wave solutions read

\[
u = u(x,t) = \frac{p}{q} = \frac{a_0 + a_1 e^{k_1 x - \alpha_1 t}}{b_0 + b_1 e^{k_1 x - \alpha_1 t}}.
\]

(3.5)

where \( b_1 \) and \( \omega_1 \) are defined by (3.3) and all other involved constants are arbitrary.
3.2 One wave solution of Kudryashov–Sinelschikov equation

Applying the multiple exp-function algorithm [4] to the Kudryashov–Sinelschikov equation (1.2) and following the same procedure we can have two solutions to the resulting algebraic system with Maple

\begin{equation}
  b_1 = \frac{a_1 b_0}{a_0}, \quad k_1 = \frac{\sqrt{-\gamma \alpha}}{\gamma}, \quad \omega_1 = \frac{\alpha \beta}{\gamma}
\end{equation}

and

\begin{equation}
  b_1 = 0, \quad k_1 = \frac{\sqrt{(d+2\gamma)\alpha}}{(d+2\gamma)}, \quad \omega_1 = \frac{\alpha(-\beta b_0 + d a_0 + \gamma a_0)}{(d+2\gamma)} k_1.
\end{equation}

Then, the two corresponding one-wave solutions are determined by (3.5) where $b_1, k_1$ and $\omega_1$ are defined by (3.6) and (3.7) and all other involved constants are arbitrary.

Many authors have investigated the Boussinesq equation and the to the Kudryashov–Sinelschikov equation by means of different methods. For example, Kaptsov [32] implemented Hirota’s bilinear method, Baratos [33] approached by using the method of lines, A.M. Wazwaz [34] used adomian decomposition method, Demiray et al.,[35] applied $(G'/G,1/G)$ and $(1/G)$-expansions method, Inc et al., [36] used Riccati–Bernoulli sub-ODE method, Li and Chen [37] applied dynamical system method, He et al., [38] employed the modified exp-function method, Tua et al., [39] exert the Lie symmetry method to attain traveling wave solutions of these above equations. But in this article, we construct more general and new exact traveling wave solutions by using the multiple exp-function method. The obtained solutions are more understandable and simple which describe the mechanism of the complicated nonlinear physical phenomena.

III. Conclusion

This study shows that the multiple exp-function method is quite efficient and practically well-suited for use in finding exact traveling wave solutions to the Boussinesq equation and Kudryashov–Sinelschikov equation (KSE). The result shows that this method is a mathematical powerful tool for obtaining exact travelling wave solution of these equations. With the aid of Maple, we have assured the correctness of the obtained solutions by putting them back into the original equations. It is also a promising method to solve other nonlinear partial differential equations arising in various scientific real time application arenas.

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