Analyticity of Rank of Operators on A Banach Space

Nahid H. k. Abdelgader

Corresponding Author: Nahid H. k. Abdelgader

ABSTRACT. If $G(z)$ is an analytic family of operators on a Banach space which is of finite rank for each $z$, then rank $G(z)$ is constant except for isolated points.

In this note we consider the analytic group $G(z)$ of operators on a complex Banach space $x$, such that the rank of $G(z)$ is finite for each $z$. We show that the rank of $G(z)$ is constant on the domain of analyticity, unless for separated points.

Definition 1 Let $X$ be a real vector space. The complexification of $X$ is the complex vector space $X_C := X \otimes \mathbb{C}$, with scalar multiplication defined by $\alpha(x \otimes \beta) := x \otimes \alpha \beta (\alpha \beta \in \mathbb{C})$

Lemma 1 If $G \in \mathbb{B}(x)$, then rank $G \geq N$ iff there exist bounded projections $P$ and $Q$ of dimension $N$ such that $PFQ$ has rank $N$.

Proof. If rank $G < N$, then rank $PGQ \leq \text{rank}G < N$. Conversely, if rank $G \geq N$, there are $X_1, \ldots, x_N$ such that $Gx_1, \ldots, Gx_N$ are linearly independent. If $P$ projects on the span of $Gx_1, \ldots, Gx_N$ and $Q$ on the span of $X_1, \ldots, x_N$, then $PGQ$ has rank $N$.

Now we show that if $G(z)$ is analytic on a domain $\Omega$ and rank $G(z)$ is finite for each $z$, then there is an integer $n$ such that rank $G(z) = n$ except at some points where $n \geq \text{rank}G(z)$.

Proof. For each $k \leq 0$, let $E_{j-1} = \{z \in \Omega | \text{rank}G(z) \leq j - 1\}$. Since $\Omega = \bigcup_{j}^{\infty} E_{j-1}$, $E_{j-1}$ is uncountable for some integer $k$, and so there is a smallest integer $n$ such that $E_n$ has a point of accumulation within $\Omega$.

If $P$ and $Q$ are arbitrary projections with dim $P = \text{dim}Q > n$, then the determinant $d(z)$ of $PG(z)Q$, computed with respect to fixed bases of $Px$ and $Qx$, vanishes on $E_n$, and hence on all of $\Omega$. Since $P$ and $Q$ are arbitrary, the following lemma satisfying $E_n = \Omega$. Since $n$ is minimal, $E_{(n-1)}$ consists of isolated points.

This proof also shows that the rank of $G(z)$ is determined by its values on any set with an accumulation point in $\Omega$, and hence that no analytic continuation of $G(z)$ can have rank exceeding $n$.

When we refer to the lemma we find that the norm and rank $G_n \leq n$, then rank $G \leq n$. For if $P$ and $Q$ have the same dimension exceeding $m$, then $\det PGQ = \lim \det PG_nQ = 0$. The hypothesis of previous theorem can be weakened by assuming only that the set of points at which $G(z)$ has finite rank is uncountable; however, it does not suffice to assume only that $G(z)$ has finite
rank on a set with an accumulation point in \( \Omega \), for if \( G(z) \) is the infinite diagonal matrix \( G(z) \) with diagonal elements \( a_1(z), a_2(z), a_3(z), \ldots a_m(z) \), where \( a_m(z) = (z-1)(z-1/2) \ldots (z-1/m) \), then \( G(z) \) is analytic for \( |z| < 1 \), while \( \text{rank} G(1/n) = m - 1 \). If \( G \in B(X) \) has finite rank, then we let \( \beta(G) \) denote the operator norm of \( G \), and

\[
\tau(G) = \inf \sum_{i=1}^{n} |x_i^*||x_i|
\]

where the infimum is taken over all representations \( G = \sum_{i=1}^{n} (x_i^*, \cdot)x_i \) of \( G \). \( \tau \) is a norm, and

\[
|trG| \leq \tau(F),
\]

\( B(G) \leq \tau(G) \leq \beta(G) \text{ rank } G \) \( \tag{1} \)

and

\[
\tau(AG) \leq B(A)\tau(G) \text{ for any } A \text{ in } B(x). \tag{2}
\]

**Theorem** if \( G(z) \) is analytic and the rank of \( G(z) \) is finite for all \( z \) in \( \Omega \), then \( trG(z) \) is analytic, and \( tr\frac{G(z)}{dz} = trG'(z) \)

**Proof.** the rank \( G(z) \) \( \leq n \leq \infty \) for some integer \( n \). The rank of \( D(z, h) = h^{-1}[F(z + h) - F(z)] \) cannot exceed \( 2n \), so that

\[
\mid trG(z + h) - trG(z) - trG'(z) \mid = \mid trh^{-1}[F(z + h) - F(z)] \mid \leq \tau(D(z, h) - G'(z)) \leq 4n\beta(D(z, h) - G'(z)).
\]

But the final term tends to zero as \( h \to 0 \), since \( G(z) \) is analytic in norm.

**References**


[2]. JAMES S. HOWLAND ,Analyticity of Determinants of operators on a Banach Space.