Alternative Methods to Prove Theorem of Basis And Dimensions

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Abstract: In this paper, we study about alternative methods by which we can prove the theorem. In a vector space if \( \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \) generates \( V \) and if \( \{W_1, W_2, \ldots, W_m\} \) is linearly independent (LI), then \( m \leq n \), where \( \dim W = m \) and \( \dim V = n \).

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I. Introduction

1.1. Basis of A Vector Space:
If \( V(F) \) is a vector space and S is any subset of \( V(F) \), then \( S \) is called a basis for \( V(F) \) if:
1. \( S \) is LI.
2. Every vector of \( V(F) \) is expressible as the linear combination of vectors of \( S \) uniquely.

i.e. \( S \) generates \( V(F) \) \( \Rightarrow \) \( L(S) = V(F) \).

1.2. Dimension of A Vector Space:
The number of vectors in the basis for a vector space \( V(F) \) is called dimension of \( V(F) \). It is denoted by \( \dim V \).

1.3. Linear Dependence of Vectors:
Let \( V(F) \) be a vector space and the set \( S = \{W_1, W_2, \ldots, W_m\} \) is finite set of vector in \( V(F) \), then \( S \) is called linearly dependent if there exists scalars \( x_1, x_2, \ldots, x_m \) not all zero such that \( x_1W_1 + x_2W_2 + \cdots + x_mW_m = 0 \), briefly written as LD.

1.4. Linear Independence of Vectors:
Let \( V(F) \) be a vector space and the set \( S = \{W_1, W_2, \ldots, W_m\} \) is finite set of vector in \( V(F) \), then \( S \) is called linearly independent if there exists scalars \( x_1, x_2, \ldots, x_m \) all are zero such that \( x_1W_1 + x_2W_2 + \cdots + x_mW_m = 0 \), briefly written as LI.

1.5. Linear Combination of Vectors:
Let \( V(F) \) be a vector space and \( W_1, W_2, \ldots, W_m \) be m-vectors and \( x_1, x_2, \ldots, x_m \) are m-scalars, then a vector \( W = x_1W_1 + x_2W_2 + \cdots + x_mW_m = \sum_{i=1}^{m} x_iW_i \) is called Linear Combination of Vectors.

1.6. Linear Span:
If \( V(F) \) is a vector space and \( S \) is any subset of \( V(F) \), then the set of all Linear Combination of elements of \( S \) is called Linear Span of \( S \) and is denoted by \( L(S) \).

\( L(S) = \{ W \mid W = \sum_{i=1}^{m} x_iW_i, x_i \in F \text{ and } W_i \in S \} \)
Here, \( L(S) \) also means that \( S \) generates.

II. Alternative Methods

2.1. Method 1
To prove this theorem, it is sufficient to show that every subset \( S \) of \( V \) which contains more than \( n \) vectors is linearly dependent (LD).

Suppose \( S = \{W_1, W_2, \ldots, W_m\} \) where \( m > n \) and all the vectors of \( S \) are distinct. Since \( \{a_1, a_2, \ldots, a_n\} \) generates \( V \) or span \( V \), so that there exists scalars \( a_i \) in \( F \) such that \( W_j = \sum_{i=1}^{n} a_{ij}W_i \).

For any scalars \( x_1, x_2, \ldots, x_m \), we have
\( x_1W_1 + x_2W_2 + \cdots + x_mW_m = \sum_{j=1}^{m} x_jW_j \)

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= \sum_{d=1}^{m} \sum_{j=1}^{n} x_j (\sum_{i=1}^{n} a_{ij} a_i).
(Since, W_j = \sum_{i=1}^{n} a_{ij} a_i)

Since, we know that if A is a n x n matrix and n x m then the homogeneous system of linear equation AX = 0 has non-trivial solution. Hence, for m > n, implies that there exists scalars x_1, x_2, ..., x_m not all zero such that \sum_{j=1}^{m} a_{ij} x_j = 0, 1 \leq i \leq n.

Therefore, every element of V(F) be a linear combination of elements of S.
Also, W is given subspace of V(F) so clearly W \subseteq V.
We can now repeat the above process of replacement with the vector W_j.
This relation shows that any vector which is expressible as a linear combination of \alpha_1, \alpha_2, ..., \alpha_n can be expressed as a linear combination of the W_j.

Hence, \alpha_1, \alpha_2, ..., \alpha_n are LLI.

We repeat this process with W_{j+1} and so on. At each step we are able to add one W_j and delete one of the \alpha’s in generating set.
If m \leq n, then we finally obtain a generating set or spanning set of the form \{W_1, W_2, ..., W_m, a_1, a_2, ..., a_n\}.
Lastly, we show that m > n is not possible. Otherwise, after n of the above steps, we obtain the generating sets \{W_1, W_2, ..., W_n\} which contradicts the hypothesis that S is linearly independent (LI).

Hence, m \geq n i.e. m \leq n.

Thus, we can’t have more LLI vectors than the number of elements in a set of generators.

2.2. Method 2
Given dim W = m and dim V = n.
So let set S = \{x_1, x_2, ..., x_n\} be a basis for V(F) and also we have L(S) = V(F).
Therefore, S be linearly independent (LI).
Here, S is linearly independent (LI).
Therefore, either S is a basis of W or any subset of S be a basis for W.
Thus, basis of W cannot contain more than n elements.
Hence, dim W \leq dim V or m \leq n.

2.3. Method 3
If A = \{\alpha_1, \alpha_2, ..., \alpha_n\} generates a span V and if S = \{W_1, W_2, ..., W_m\} is LI, then we have shown that m \leq n.

Since, we know that if A is a n x n matrix and n x m then the homogeneous system of linear equation AX = 0 has non-trivial solution. Hence, for m > n, implies that there exists scalars x_1, x_2, ..., x_m not all zero such that \sum_{j=1}^{m} a_{ij} x_j = 0, 1 \leq i \leq n.

Therefore, every element of V(F) be a linear combination of elements of S.
Also, W is given subspace of V(F) so clearly W \subseteq V.
We can now repeat the above process of replacement with the vector W_j.
This relation shows that any vector which is expressible as a linear combination of \alpha_1, \alpha_2, ..., \alpha_n can be expressed as a linear combination of the W_j.

Hence, \alpha_1, \alpha_2, ..., \alpha_n are LLI.

We repeat this process with W_{j+1} and so on. At each step we are able to add one W_j and delete one of the \alpha’s in generating set.
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Lastly, we show that m > n is not possible. Otherwise, after n of the above steps, we obtain the generating sets \{W_1, W_2, ..., W_n\} which contradicts the hypothesis that S is linearly independent (LI).

Hence, m \leq n.

References
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