Lacunary Arithmetic Statistical Convergence For Double Sequences.

A. M. Brono¹ M. M. Karagama² And F. B. Ladan³

¹Department of Mathematical Sciences, University of Maiduguri, Borno State, Nigeria
Email: bronaoahmadu@unimaid.edu.ng

²Department of Mathematical Sciences, University of Maiduguri, Borno State, Nigeria
Email: mustaphakaragama@gmail.com

³Department of Mathematical Sciences, University of Maiduguri, Borno State, Nigeria
Email: falmatabladan@gmail.com

Abstract: This paper extends the recently introduced summability concept of convergence namely: arithmetic statistical convergence and lacunary arithmetic statistical convergence, to double sequences. We shall also investigate the relationship between these concepts and prove some inclusion theorems.

Keywords and Phrases: Summability, Arithmetic statistical convergence, lacunary arithmetic statistical convergence and double sequences.

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I. Introduction:

The concept of statistical convergence was introduced by Fast [4] and it was further investigated from the sequence space point of view and linked with summability theory by Fridy [2], Connor [3], Fridy and Orhan [1], Salát [5] and many others.

The idea of arithmetic convergence was introduced by Ruckle [9]. Yaying and Hazarika [8] used this concept of arithmetic convergence introduced arithmetic statistical convergence and lacunary arithmetic statistical convergence of single sequence. We shall use the concept of statistical convergence of double sequences. [see Mursaleen (6)] to extend the results of Yaying and Hazarika [8] to double sequences.

II. Lacunary Arithmetic Statistical Convergence.

Definition 2.1: (Yaying and Hazarika [2017]) A sequence \( x = (x_k) \) is called arithmetically convergent if for each \( \varepsilon > 0 \) there is an integer \( l \) such that for every integer \( k \) we have \( |x_k - x_{(k,l)}| < \varepsilon \), where the symbol \( (k,l) \) denotes the greatest common divisor of two integers \( k \) and \( l \). We denote the sequence space of all arithmetic convergent sequence by AC.

Definition 2.2: (Fridy and Orhan [1993]) Let \( \theta = (k_r) \) be a lacunary sequence. A number sequence \( x = (x_k) \) is said to be lacunary statistically convergent to \( l \) if, for each \( \varepsilon > 0 \),

\[
\lim_{r \to \infty} \frac{1}{k_r} \left\{ \{ k \in I_r : |x_k - l| \geq \varepsilon \} \right\} = 0
\]

In this case, one writes \( S_{k_r} - \lim x_k = l \) or \( x_k \Rightarrow l(S_{k_r}) \). The set of all lacunary statistically convergence sequences is denoted by \( S_\theta \).

Definition 2.3: (Yaying and Hazarika [2017]) A sequence \( x = (x_k) \) is said to be arithmetic statistically convergent if for each \( \varepsilon > 0 \), there is an integer \( l \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \left\{ \{ k \in n : |x_k - x_{(k,l)}| \geq \varepsilon \} \right\} = 0
\]

We shall use \( ASC \) to denote the set of all arithmetic statistically convergence sequences. Thus for \( \varepsilon > 0 \) and integer \( l \)

\[
ASC = \left\{ (x_k) : \lim_{n \to \infty} \frac{1}{n} \left\{ \{ k \in n : |x_k - x_{(k,l)}| \geq \varepsilon \} \right\} = 0 \right\}.
\]

We shall write \( ASC - \lim x_k = x_{(k,l)} \) to denote the sequence \( (x_k) \) is arithmetic statistically convergent to \( x_{(k,l)} \).

Definition 2.4: (Yaying and Hazarika [2017]) Let \( \theta = (k_r) \) be a lacunary sequence. The number sequence \( x = (x_k) \) is said to be lacunary arithmetically statistically convergent if for each \( \varepsilon > 0 \) there is an integer \( l \) such that

\[
\lim_{r \to \infty} \frac{1}{l_r} \left\{ \{ k \in l_r : |x_k - x_{(k,l)}| \geq \varepsilon \} \right\} = 0
\]
We shall write

\[ \text{ASC}_\theta = \{ x = (x_k): \lim_{r \to +\infty} \frac{1}{h_r} \sum_{k \leq l_r} |x_k - x_{(k,l_r)}| \geq \varepsilon \} \].

We shall write \( \text{ASC}_\theta = \lim x_k = x_{(k,l)} \) to denote the sequence \( (x_k) \) is lacunary statistically convergent to \( x_{(k,l)} \).

**Definition 2.5:** (Yaying and Hazarika [2017]) Let \( \theta = (k_r) \) be a lacunary sequence. A lacunary refinement of \( \theta \) is a lacunary sequence \( \theta' = (k'_r) \) satisfying \( k'_r \subseteq (k_r) \). (See Freedman et al. [ ].

**Definition 2.6:** (Yaying and Hazarika [2017]) A function \( f \) defined on a subset \( E \) of \( \mathbb{R} \) is said to be lacunary arithmetic statistical continuous if it preserves lacunary arithmetic statistical convergence i.e. if

\[ \text{ASC}_\theta - \lim x_k = x_{(k,l)} \Rightarrow \text{ASC}_\theta - \lim f(x_k) = f(x_{(k,l)}). \]

**Theorem 2.1:** (Yaying and Hazarika [2017]) Let \( x = (x_k) \) and \( y = (y_k) \) be two sequences.

(i) If \( \text{ASC}_\theta - \lim x_k = x_{(k,l)} \) and \( a \in \mathbb{R} \), then \( \text{ASC}_\theta - \lim ax_k = ax_{(k,l)} \).

(ii) If \( \text{ASC}_\theta - \lim x_k = x_{(k,l)} \) and \( \text{ASC}_\theta - \lim y_k = y_{(k,l)} \), then \( \text{ASC}_\theta - \lim (x_k + y_k) = (x_{(k,l)} + y_{(k,l)}) \).

**Theorem 2.2:** (Yaying and Hazarika [2017]) Let \( x = (x_k) \) and \( y = (y_k) \) be two sequences.

(i) If \( \text{ASC}_\theta - \lim x_k = x_{(k,l)} \) and \( a \in \mathbb{R} \), then \( \text{ASC}_\theta - \lim cx_k = cx_{(k,l)} \).

(ii) If \( \text{ASC}_\theta - \lim x_k = x_{(k,l)} \) and \( \text{ASC}_\theta - \lim y_k = y_{(k,l)} \), then \( \text{ASC}_\theta - \lim (x_k + y_k) = (x_{(k,l)} + y_{(k,l)}) \).

**Theorem 2.3:** (Yaving and Hazarika [2017]) If \( \theta' = (k'_r) \) is a lacunary refinement of a lacunary sequence \( \theta = (k_r) \) and \( (x_k) \in \text{ASC}_\theta \), then \( (x_k) \in \text{ASC}_{\theta'} \).

**Theorem 2.4:** (Yaying and Hazarika [2017]) Suppose \( \beta = (l_r) \) is a lacunary refinement of a lacunary sequence \( \theta = (k_r) \). Let \( l_r = (k_{r-1}, k_r) \) \( r \in \mathbb{R} \). If \( \text{ASC}_\theta - \lim x_k = x_{(k,l)} \), then \( \text{ASC}_{\beta} - \lim x_k = x_{(k,l)} \).

**Theorem 2.5:** (Yaving and Hazarika [2017]) Suppose \( \beta = (l_r) \) and \( \theta = (k_r) \) are two lacunary sequences.

Let \( l_r = (k_{r-1}, k_r) \) \( r \in \mathbb{R} \). If \( \text{ASC}_{\beta} - \lim x_k = x_{(k,l)} \), then \( \text{ASC}_{\theta} - \lim x_k = x_{(k,l)} \).

**Theorem 2.6:** (Yaving and Hazarika [2017]) Let \( \theta = (k_r) \), \( r = 1, 2, 3, \ldots \) be a lacunary sequence. If \( \lim \inf \theta > 1 \), then \( \text{ASC} \subseteq \text{ASC}_{\theta} \).

**Theorem 2.7:** (Yaving and Hazarika [2017]) For \( \lim \sup \theta < \infty \), we have \( \text{ASC}_\theta \subseteq \text{ASC} \).

We shall now use analogy to extend the above concepts and results to double sequences;

### III. Lacunary Arithmetic Statistical Convergence For Double Sequences.

**Definition 3.1:** A double sequence \( x = (x_{k,m}) \) is called arithmetically convergent if for each \( \varepsilon > 0 \) there is an integer \( l, m \) such that for every integer \( k, m \) we have \( |x_{k,m} - x_{(k,l,m)}| < \varepsilon \), where the symbol \((k, l, m, n)\) denotes the greatest common divisor of four integers \( k, l, m, n \). We denote the double sequence space of all arithmetic convergent sequences by \((AC)_2\).

**Note:** \( g = (\{(k, l), (m, n)\}) \) where \( g \) denotes the greatest common divisor (gcd) for double sequences. Therefore we shall use \( g \) as the above equality throughout this paper.

**Definition 3.2:** Let \( \theta = (k_{r,s}) \) be a lacunary double sequence. A double sequence \( x = (x_{k,m}) \) is said to be lacunary statistically convergent to \( l \text{or} \ S_{\theta_{r,s}} - \lim x_{k,m} \) convergent to \( l \), if, for each \( \varepsilon > 0 \),

\[ \lim_{r,s \to +\infty} \frac{1}{h_{r,s}} \sum_{k \leq l_{r,s}} |x_{k,m} - l| \geq \varepsilon = 0 \]

In this case, one writes \( S_{\theta_{r,s}} - \lim x_{k,m} = l \text{ or} \ S_{\theta_{r,s}} - \lim x_{k,m} = l \text{or} \ S_{\theta_{r,s}} - \lim x_{k,m} = l \text{or} \ S_{\theta_{r,s}} - \lim x_{k,m} = l \).

**Definition 3.3:** A double sequence \( x = (x_{k,m}) \) is said to be arithmetically statistically convergent if for each \( \varepsilon > 0 \), there is an integer \( l, m \) such that

\[ \lim_{n \to +\infty} \frac{1}{n} \sum_{k \leq l_{r,s}} |x_{k,m} - x_{g}| \geq \varepsilon = 0 \]

We shall use \((ASC)_2\) to denote the set of all arithmetic statistical convergent double sequences. Thus for \( \varepsilon > 0 \) and integer \( l, m \).
**Lacunary Arithmetic Statistical Convergence For Double Sequences.**

\((ASC)_{2} = \{ (x_{k,m}) : \lim_{n \to \infty} \frac{1}{u} \left| \{ k, m \in n : |x_{k,m} - x_{\varepsilon}| \geq \varepsilon \} \right| = 0 \} .\)

We shall write \((ASC)_{2} - \lim x_{k,m} = x_{\varepsilon}\) to denote the double sequence \((x_{k,m})\) is arithmetic statistically convergent to \(x_{\varepsilon}\)

**Definition 3.4:** Let \(\theta = (k_{r,s})\) be a lacunary double sequence. The double sequence \(x = (x_{k,m})\) is said to be lacunary arithmetic statistically convergent for double sequences if for each \(\varepsilon > 0\) there is an integer \(l, n\) such that for every integer \(k, m \geq l, n\)

\[
\lim_{r,s \to \infty} \frac{1}{h_{r,s}} \left| \{ k, m \in I_{r,s} : |x_{k,m} - x_{\varepsilon}| \geq \varepsilon \} \right| = 0
\]

We shall write

\(ASC_{\theta_{r,s}} = \left\{ x = (x_{k,m}) : \lim_{r,s \to \infty} \frac{1}{h_{r,s}} \left| \{ k, m \in I_{r,s} : |x_{k,m} - x_{\varepsilon}| \geq \varepsilon \} \right| = 0 \right\} .\)

We shall write \(ASC_{\theta_{r,s}} - \lim x_{k,m} = x_{\varepsilon}\) to denote the double sequence \((x_{k,m})\) is lacunary arithmetic statistically convergent to \(x_{\varepsilon}\)

**Definition 3.5:** Let \(\theta = (k_{r,s})\) be a lacunary double sequence. A lacunary refinement of \(\theta\) is a lacunary double sequence \(\theta' = (k'_{r,s})\) satisfying \((k'_{r,s}) \subseteq (k_{r,s})\). (See Freedman et al. [7].)

**Theorem 3.1:** Let \(x = (x_{k,m})\) and \(y = (y_{k,m})\) be two double sequences.

(i) If \((ASC)_{2} - \lim x_{k,m} = x_{(k,l)(m,n)}\) and \(a \in \mathbb{R}\), then \((ASC)_{2} - \lim ax_{k,m} = ax_{(k,l)(m,n)}\).

(ii) If \((ASC)_{2} - \lim x_{k,m} = x_{(k,l)(m,n)}\) and \((ASC)_{2} - \lim y_{k,m} = y_{(k,l)(m,n)}\), then \((ASC)_{2} - \lim (x_{k,m} + y_{k,m}) = (x_{(k,l)(m,n)} + y_{(k,l)(m,n)})\).

**Proof 3.1:**

(i) The result is obvious when \(a = 0\). Suppose \(a \neq 0\), then for integer \(l, n\)

\[
\frac{1}{uv} \left| \{ k \leq u, m \leq v : |ax_{k,m} - ax_{\varepsilon}| \geq \varepsilon \} \right| = \frac{1}{uv} \left| \{ k \leq u, m \leq v : |x_{k,m} - x_{\varepsilon}| \geq \frac{\varepsilon}{|a|} \} \right|
\]

Which gives the result

The result of (ii) follows from

\[
\frac{1}{uv} \left| \{ k \leq u, m \leq v : |(x_{k,m} + y_{k,m}) - (x_{(k,l)(m,n)} + y_{(k,l)(m,n)})| \geq \varepsilon \} \right| \\
\leq \frac{1}{uv} \left| \{ k \leq u, m \leq v : |x_{k,m} - x_{(k,l)(m,n)}| \geq \frac{\varepsilon}{2} \} \right| + \frac{1}{uv} \left| \{ k \leq u, m \leq v : |y_{k,m} - y_{(k,l)(m,n)}| \geq \frac{\varepsilon}{2} \} \right|
\]

Thus we defined a related concept of convergence in which the set \(\{ k, m : k, m \leq uv \}\) is replaced by the set \(\{ k, m : k_{r-1,s-1} \leq k, m \leq k_{r,s} \}\), for some lacunary double sequence \((k_{r,s})\). (see definition 3.4)

**Theorem 3.2:** Let \(x = (x_{k})\) and \(y = (y_{k})\) be two sequences.

(iii) If \(\forall \theta_{r,s} - \lim x_{k} = x_{(k,l)}\) and \(a \in \mathbb{R}\), then \(\forall \theta_{r,s} - \lim ax_{k} = ax_{(k,l)}\)

(iv) If \(\forall \theta_{r,s} - \lim x_{k} = x_{(k,l)}\) and \(\forall \theta_{r,s} - \lim y_{k} = y_{(k,l)}\), then \(\forall \theta_{r,s} - \lim (x_{k} + y_{k}) = (x_{(k,l)} + y_{(k,l)})\)

**Proof 3.2:**

(i) The result is obvious when \(a = 0\). Suppose \(a \neq 0\), then for integer \(l, n\)

\[
\frac{1}{hr,s} \left| \{ k \in I_{r,s} : |ax_{k,m} - ax_{\varepsilon}| \geq \varepsilon \} \right| = \frac{1}{hr,s} \left| \{ k \in I_{r,s} : |x_{k,m} - x_{\varepsilon}| \geq \frac{\varepsilon}{|a|} \} \right|
\]

Which gives the result

The result of (ii) follows from
\[
\frac{1}{h_{r,s}} \left| \left( k, m \in I_{r,s} : \left| (x_{k,m} + y_{k,m} - (x_g + y_g) \right| \geq \varepsilon \right) \right| \\
\leq \frac{1}{w_u} \left| \left( k \leq u, m \leq v : |x_{k,m} - x_g| \geq \frac{\varepsilon}{2} \right) \right| + \frac{1}{w_v} \left| \left( k \leq u, m \leq v : |y_{k,m} - y_g| \geq \frac{\varepsilon}{2} \right) \right| \\
\leq \frac{1}{h_{r,s}} \left| \left( k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \frac{\varepsilon}{2} \right) \right| + \frac{1}{h_{r,s}} \left| \left( k, m \in I_{r,s} : |y_{k,m} - y_g| \geq \frac{\varepsilon}{2} \right) \right|
\]

\[\square\]

**Theorem 3.3:** If \( \hat{\Theta}' = (k_{r,s}) \) is a lacunary refinement of a lacunary double sequence \( \Theta = (k_{r,s}) \) and \( (x_{k,m}) \in ASC_{\hat{\Theta}'_{r,s}} \), then \( (x_{k,m}) \in ASC_{\Theta_{r,s}} \).

**Proof 3.3:**
Suppose for each \( I_{r,s} \) of \( \Theta \) contains the point \((k_{r,s}) \) \( \mu_{r,s} \) of \( \Theta' \) such that \( k_{r-1,s-1} < k_{r,s} < k_{r,s+1} < \cdots < k_{r,s} \).

Where \( I_{r,s} = (k_{r,x-1}, k_{r,s}) \).

Since \( (k_{r,s}) \subseteq (k_{r,s}) \), so \( r, s \mu_{r,s} \geq 1 \).

Let \( (I_{r,s})_{r,s}^{\infty} \) be the double sequence of interval \( (I_{r,s}) \) ordered by increasing right end points. Since \( (x_{k,m}) \in ASC_{\Theta_{r,s}} \), then for each \( \varepsilon > 0 \) and integer \( l, n \)

\[
\lim_{I_{r,s} \subseteq I_{r,s}^*} \sum_{I_{r,s} \subseteq I_{r,s}^*} \frac{1}{h_{r,s}} \sum_{I_{r,s} \subseteq I_{r,s}^*} \left| \left( k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \right) \right| = 0
\]

Also since \( h_{r,s} = k_{r,s} - k_{r,s-1}, s \), \( h_{r,s} = k_{r,s} - k_{r-1,s} \).

For each \( \varepsilon > 0 \) and integer \( l, n \)

\[
\frac{1}{h_{r,s}} \left| \left( k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \right) \right| = \frac{1}{h_{r,s}} \sum_{I_{r,s} \subseteq I_{r,s}^*} \frac{1}{h_{r,s}} \sum_{I_{r,s} \subseteq I_{r,s}^*} \left| \left( k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \right) \right| \\
\to 0 \text{ as } r, s \to \infty
\]

This implies \( (x_{k,m}) \in ASC_{\Theta_{r,s}} \).

**Theorem 3.4:** Suppose \( \gamma = (l_{r,s}) \) is a lacunary refinement of a lacunary double sequences \( \Theta = (k_{r,s}) \). Let \( l_{r,s} = (k_{r,x-1}, k_{r,s}) \) and \( j_{r,s} = (l_{r,x-1}, l_{r,s}) \), \( r, s \). \( \Theta \) is \( \mu_{r,s} \) such that \( \gamma_{r,s} \geq \delta \) for every \( j_{r,s} \subseteq I_{l,s} \).

Then \( (x_{k,m}) \in ASC_{\Theta_{r,s}} \Rightarrow (x_{k,m}) \in ASC_{\gamma_{r,s}} \).

**Proof 3.4:**
For any \( \varepsilon > 0 \) and integer \( l, n \), every \( j_{r,s} \) we can find \( I_{l,s} \) such that \( j_{r,s} \subseteq I_{l,s} \), then we have

\[
\frac{1}{|j_{r,s}|} \left| \left( k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \right) \right| = \left( \frac{|j_{l,s}|}{|j_{r,s}|} \right) \left( \frac{1}{|j_{l,s}|} \right) \left| \left( k, m \in I_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \right) \right| \\
\leq \left( \frac{|I_{l,s}|}{|j_{r,s}|} \right) \left( \frac{1}{|I_{l,s}|} \right) \left| \left( k, m \in I_{l,s} : |x_{k,m} - x_g| \geq \varepsilon \right) \right| \\
\leq \left( \frac{1}{\beta} \right) \left( \frac{1}{|I_{l,s}|} \right) \left| \left( k, m \in I_{l,s} : |x_{k,m} - x_g| \geq \varepsilon \right) \right| \square
\]

Which gives the result.

**Theorem 3.5:** Suppose \( \gamma = (l_{r,s}) \) and \( \Theta = (k_{r,s}) \) are two lacunary double sequences. Let \( l_{r,s} = (k_{r,x-1}, k_{r,s}) \) and \( j_{r,s} = (l_{r,x-1}, l_{r,s}) \), \( r, s \in \{1, 2, \ldots\} \) and \( l_{a,b} = I_{a,x} \cap I_{b,z} \) \( a, b = 1, 2, 3, \ldots \) and where \( a \) = \( wz \) and \( b \) \( yz \). If there exists \( \delta > 0 \) such that \( \gamma_{a,b} \geq \delta \) for every \( y, z, a, b \), then \( (x_{k,m}) \in ASC_{\Theta_{r,s}} \Rightarrow (x_{k,m}) \in ASC_{\gamma_{r,s}} \).

**Proof 3.5:**
Let \( \gamma \cup \Theta \). Then \( \mu \) is a lacunary refinement of \( \Theta \). The interval sequence of \( \mu \) is \( \{l_{a,b} = I_{a,x} \cap I_{b,z} : l_{a,b} \neq \emptyset, \text{ where } a = wz \text{ and } b \text{ } yz\} \). Using theorem 3.4 and the condition \( \gamma_{a,b} \geq \delta \) gives \( (x_{k,m}) \in ASC_{\Theta_{r,s}} \Rightarrow (x_{k,m}) \in ASC_{\gamma_{r,s}} \).

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Let \( x_{k,m} \in \text{ASC}_{r,s} \). Since \( \mu \) is a lacunary refinement of the lacunary double sequences, from theorem 3.3, we have \( (x_{k,m}) \in \text{ASC}_{r,s} \Rightarrow (x_{k,m}) \in \text{ASC}_{r,s} \).

**Theorem 3.6:** Let \( \theta = (k_{r,s}) \), \( r,s = 1,2,3, \ldots \), be a lacunary double sequences. If \( \liminf r_{r,s} > 1 \), then \( (\text{ASC})_2 \subseteq \text{ASC}_{\theta_{r,s}} \).

**Proof 3.6:**

Let \( (x_{k,m}) \in (\text{ASC})_2 \) and \( \liminf r_{r,s} > 1 \). Then there exist \( \alpha > 1 \) such that \( q_{r,s} = \frac{k_{r,s}}{k_{r-1,s-1}} \geq 1 + \alpha \) for sufficiently large \( r,s \) which implies that \( \frac{h_{r,s}}{k_{r,s}} \geq \frac{\alpha}{1+\alpha} \).

Then, for sufficiently large \( r,s \) and integer \( k,m \):

\[
\frac{1}{k_{r,s}} \left[ \{ k, m \in k_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \} \right] \geq \frac{1}{k_{r,s}} \left[ \{ k, m \in l_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \} \right]
\]

\[
\geq \frac{\alpha}{1+\alpha} \frac{1}{h_{r,s}} \left[ \{ k, m \in l_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \} \right]
\]

Thus \( x = (x_{k,m}) \in (\text{ASC})_2 \Rightarrow (x_{k,m}) \in \text{ASC}_{\theta_{r,s}} \).

**Theorem 3.7:** For \( \limsup q_{r,s} < \infty \), we have \( \text{ASC}_{\theta_{r,s}} \subseteq (\text{ASC})_2 \).

**Proof 3.7:**

Let \( \limsup q_{r,s} < \infty \) then there exist \( \omega > 0 \) such that \( q_{r,s} < \omega \) for every \( r,s \). Let \( r_{r,s} = \left\lfloor \frac{k_{r,s}}{k_{r-1,s-1}} \right\rfloor \). Then for an integer \( l,n \):

\[
\frac{1}{p} \left[ \{ k, m \in p : |x_{k,m} - x_g| \geq \varepsilon \} \right] \leq \frac{1}{\frac{h_{r,s}}{r_{r,s}}} \left[ \{ k, m \in r_{r,s} : |x_{k,m} - x_g| \geq \varepsilon \} \right]
\]

\[
= \frac{1}{\frac{h_{r,s}}{r_{r,s}}} \left\lfloor \left( \frac{\tau_{r,s}}{h_{r,s}} \right) \left( h_{r,s} + \cdots + h_{r,s} \right) \right\rfloor
\]

\[
\leq \frac{\frac{k_{r,s}}{h_{r,s}} + 1}{\frac{k_{r,s}}{h_{r,s}}} \left( \frac{\tau_{r,s}}{r_{r,s}} \left( h_{r,s} + \cdots + h_{r,s} \right) \right)
\]

\[
\leq \frac{\frac{k_{r,s}}{h_{r,s}} + 1}{\frac{k_{r,s}}{h_{r,s}}} T + \epsilon q_{r,s}
\]

Which gives \( (x_{k,m}) \in (\text{ASC})_2 \).

**Corollary 3.1.**

From there 2.6 and 2.7, if \( \theta = (k_r) \) be a lacunary double sequences and if

\[
1 < \liminf q_r \leq \limsup q_r < \infty
\]

Then \( (\text{ASC})_2 = \text{ASC}_{\theta} \).

In (2016) Yaying and Hazarika introduced lacunary arithmetic convergent sequence \( AC_{\theta} \) as follow:

\[
AC_{\theta} = \left\{ (x_k) : \lim_{r \to \infty} h_r \left( \sum_{k \in \ell_r} |x_k - x_{[k,l]}| = 0 \text{ for integer } l \right) \right\}
\]

Analogously, we define double lacunary arithmetic convergence.

From theorem 3.6 and 3.7, if \( \theta = (k_{r,s}) \) be a lacunary double sequences and if

\[
1 < \liminf q_{r,s} \leq \limsup q_{r,s} < \infty
\]

Then \( (\text{ASC})_2 = \text{ASC}_{\theta_{r,s}} \).

Now we introduce lacunary arithmetic convergent sequence \( AC_{\theta_{r,s}} \) as follow:

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\[
AC_{\theta_{r,s}} = \left\{ (x_{k,m}) : \lim_{r,s \to \infty} \frac{1}{h_{r,s}} \sum_{k \in I_{r}, m \in I_{s}} |x_{k,m} - x_{g}| = 0 \text{ some integers } l, n \right\}
\]

In relation to this we shall introduce for double sequences space and give some relation between the double spaces \(AC_{\theta_{r,s}}\) and \(ASC_{\theta_{r,s}}\).

**Theorem 3.8:** Let \(\theta = (k_{r,s})\) be a lacunary double sequence; then if \((x_{k,m}) \in (AC_{\theta})_{2}\) then \((x_{k,m}) \in (ASC_{\theta})_{2}\)

**Proof 3.8:** Let \((x_{k,m}) \in (AC_{\theta})_{2}\) and \(\varepsilon > 0\). We can write, for an integer \(l, n\)

\[
\sum_{k,m \in I_{l,r}} |x_{k,m} - x_{g}| \geq \sum_{k,m \in I_{l,r}, |x_{k,m} - x_{g}| \geq \varepsilon} |x_{k,m} - x_{g}| = \sum_{k,m \in I_{l,r}, |x_{k,m} - x_{g}| < \varepsilon} |x_{k,m} - x_{g}| + \sum_{k,m \in I_{l,r}, |x_{k,m} - x_{g}| \geq \varepsilon} |x_{k,m} - x_{g}|
\]

\[
\geq \sum_{k,m \in I_{l,r}, |x_{k,m} - x_{g}| \geq \varepsilon} |x_{k,m} - x_{g}|
\]

\[
\geq \varepsilon \left\{ (k, m \in I_{r,s} : |x_{k,m} - x_{g}| \geq \varepsilon) \right\}
\]

Which gives the result.

**References**


