The M-step iterative procedure for finite family of uniformly continuous asymptotically Nonexpansive in the intermediate sense Maps.

Chika Moore, Agatha Chizoba Nnubia And Nfor Marybruno
Department of Mathematics, Nnamdi Azikiwe University, P. M. B. 5025, Awka, Anambra State, Nigeria.

Abstract: Let K be a closed convex nonempty subset of a uniformly convex Banach space E and let \( \{T_i\}_{i=1}^m \) be a finite family of self maps on K such that \( T_1 \) is uniformly continuous asymptotically Nonexpansive in the intermediate sense Maps. with \( F=\bigcup_{i=1}^m F(T_i) \neq \emptyset \), an m-step iteration process was used and sufficient conditions for the strong convergence of the process to a common fixed point of the family are proved.


Key words/phrases:- uniformly continuous; asymptotically Nonexpansive in the intermediate sense Maps; finite family, common fixed point; M-step iteration process; strong convergence.

I. Introduction

The Mann iteration scheme [Error! Reference source not found.], introduced in 1953, was used to prove the convergence of the sequence to the fixed points of mappings of which the Banach principle is not applicable. In 1974, Ishikawa [Error! Reference source not found.] devised a new iteration scheme and established it’s convergence to a fixed point of Lipschitzian pseudocontractive map when Mann iteration process failed to converge. Noor [Error! Reference source not found.] introduced the three-step iteration process for solving nonlinear operator equations in real Banach spaces as follows.

Let \( E \) be a real Banach space, \( K \) a nonempty convex subset of \( E \) and \( T:K \to K \), a mapping. For an arbitrary \( x_o \in K \), the sequence \( \{x_n\} \subseteq K \) defined by

\[
\begin{align*}
x_{n+1} &= (1-\alpha_n) x_n + \alpha_n T y_n, \\
y_n &= (1-\beta_n) x_n + \beta_n T z_n, \\
z_n &= (1-\gamma_n) x_n + \gamma_n T x_n
\end{align*}
\]

(1)

Where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \([0,1]\) is called the three-step iteration (or the Noor iteration). When \( \gamma_n = 0 \), then the three-step iteration reduces to the Ishikawa iterative sequence. If \( \beta_n = \gamma_n = 0 \), then the three-step iteration reduces to the Mann iteration.

Glowinski and Le Tallec (1989) used the three-step iterative schemes to solve elastoviscoplasticity, liquid crystal and eigen-value problems. They have shown that the three-step approximation scheme performs better than the two-step and one-step iterative methods. Haubrueg et al (1998) have studied the convergence analysis of three-step iterative schemes and applied these three-step iteration to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations also lead to highly parallelized algorithms under certain conditions. Thus, it is clear that three-step schemes play an important part in solving various problems, which arise in pure and applied sciences.

In 2013, Chika Moore et al [Error! Reference source not found.] introduced the M-step process and proved strong convergence of the process to the common fixed point of finite family of hemicontractive maps. Let \( K \) be
The M-step iterative procedure for finite family of uniformly continuous asymptotically nonexpansive in the intermediate sense Maps when such point are known to exist.

The following definitions is vital in this work. Let $E$ be a normed linear space and let $J:E\rightarrow 2E^*$ be the normalized duality mapping defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = |x|^2 = |f|^2 \}; \forall x \in E$$

where $E^*$ denotes the dual space of $E$ and $\langle , \rangle$ denotes the generalised duality pairing between $E$ and $E^*$. The single-valued normalised duality mapping is denoted by $j$. Let $K$ be a nonempty closed convex subset of $E$ and $T:K \rightarrow K$ be a map. The mapping $T$ is said to be uniformly L-Lipschitzian if there exists a constant $L>0$ such that for any $x,y \in K$ and $\forall n \geq 1$

$$|T^n x - T^n y| \leq L|x-y|$$

Let $K$ be a nonempty subset of a real Hilbert space $H$. A mapping $T:K \rightarrow K$ is called nonexpansive if and only if for all $x,y \in K$, we have that

$$|Tx - Ty| \leq |x-y|.$$ (5)

The mapping $T$ is called asymptotically nonexpansive mapping if and only if there exists a sequence $\{\mu_n\}_{n \geq 1} \subseteq [0, +\infty)$, with $\lim_{n \rightarrow +\infty} \mu_n = 0$ such that for all $x,y \in K$,

$$|T^n x - T^n y| \leq (1 + \mu_n)|x-y| \quad \forall n \in \mathbb{N}$$ (6)

$T$ is called asymptotically nonexpansive in the intermediate sense (ANIS) if and only if there exist two sequences $\{\mu_n\}_{n \geq 1}, \{\eta_n\}_{n \geq 1} \subseteq [0, +\infty)$, with $\lim_{n \rightarrow +\infty} \mu_n = 0 = \lim_{n \rightarrow +\infty} \eta_n$ such that for all $x,y \in K$,
The class of asymmetrically nonexpansive mappings was introduced by Goebel and Kirk [1] as a generalisation of nonexpansive mappings. As further generalisation of class of nonexpansive mappings, Alber, Chidume and Zegeye [Error! Reference source not found.] introduced the class of total asymmetrically nonexpansive mappings, where a mapping $T:K \to K$ is called total asymmetrically nonexpansive (TAN) if and only if there exist two sequences $(\mu_n)_{n \geq 1}, (\eta_n)_{n \geq 1}$ such that

$$|T^n_kx-T^n_ky|\leq (1+\mu_n)|x-y|+\eta_n \quad n\geq 1$$

The above asymptotic results have been extended to the intermediate sense mappings. The class of asymmetrically nonexpansive mappings is properly contained in the class of total asymmetrically nonexpansive mappings. As further generalisation of class of nonexpansive mappings, Alber, Chidume and Zegeye introduced the class of total asymmetrically nonexpansive mappings. The class of asymmetrically nonexpansive mappings includes the class of mappings which are asymptotically nonexpansive in the intermediate sense. These classes of mappings had been studied extensively by several authors (see e.g. [1], [Error! Reference source not found.]).

We shall make use of the following lemmas. We need the following lemmas in this work.

**Lemma 2.1** [Error! Reference source not found.] Let $(\mu_n), (\beta_n), (\gamma_n)$ be sequences of nonnegative numbers satisfying the conditions:

$$\sum_{n=0}^{\infty} \beta_n = \infty, \quad \beta_n \to 0 \quad as \quad n \to \infty$$

$$\frac{2}{\beta_{n+1}} \leq \beta_n < \frac{2}{\beta_n} \quad n=1, 2, \ldots$$

where $\gamma: [0, 1) \to [0, 1)$ is a strictly increasing function with $\gamma(0)=0$. Then $\mu_n \to 0$ as $n \to \infty$.

**3 Main Result**

**Proposition 3.1** Let $H$ be a real Hilbert space, let $K$ be a nonempty closed convex subset of $H$ and let $T_i:K \to K$ be $m$ uniformly continuous asymmetrically nonexpansive in the intermediate sense mapping from $K$ into itself, then there exist sequences $(\mu_n)_{n \geq 1}, (\eta_n)_{n \geq 1} \subset [0, +\infty)$ such that

$$|T^n_kx-T^n_ky|\leq (1+\mu_n)|x-y|+\eta_n \quad n\geq 1, \forall i \in I$$

**Proof** Since $T_i:K \to K$ where $i \in I$ are asymmetrically nonexpansive in the intermediate sense maps, there exist sequences $(\mu_n)_{n \geq 1}, (\eta_n)_{n \geq 1} \subset [0, +\infty)$ such that $\lim_{n \to \infty} \mu_n = 0 = \lim_{n \to \infty} \eta_n$ and $\forall x,y \in K$,

$$|T^n_kx-T^n_ky|\leq (1+\mu_n)|x-y|+\eta_n \quad n\geq 1, \forall i \in I.$$  

Now, setting $\mu_n = \max_{i \in I} \mu_i, \eta_n = \max_{i \in I} \eta_i$. We have that $(\mu_n)_{n \geq 1}, (\eta_n)_{n \geq 1} \subset [0, +\infty)$ ,

$$\lim_{n \to \infty} \mu_n = 0 = \lim_{n \to \infty} \eta_n \quad and \quad \forall x,y \in K,$$

$$|T^n_kx-T^n_ky|\leq (1+\mu_n)|x-y|+\eta_n \quad n\geq 1, \forall i \in I.$$ 

Thus, completing the proof:

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\[ \lim_{n \to \infty} T_n^{n-1} y_n \in K \]  where \( \alpha_n \in (0,1) \) satisfying the following conditions:

\[ \sum_{n=1}^{\infty} \alpha_n < \infty, 0 < \alpha_n < \frac{1}{2}, n \geq 1 \]

Let \( x^* \in F \) then \( \lim_{n \to \infty} |x_n - x^*| \) exists for \( x^* \in F \)

\[ y_{n,j} - x^* \leq (1 - \alpha_n) |y_n - x^*| + \alpha_n |T_n^{n-1} y_{n,j} - x^*| \]

\[ y_{n,1} - x^* \leq (1 + \alpha_n (\mu_{n,1} - 1)) |y_n - x^*| + \alpha_n \eta_{n,1} \]

\[ y_{n,2} - x^* \leq (1 - \alpha_n) |y_n - x^*| + \alpha_n \mu_{n,2} \left( 1 + \alpha_n (\mu_{n,1} - 1) \right) |y_n - x^*| + \alpha_n \eta_{n,1} \]

\[ y_{n,3} - x^* \leq \left[ 1 + \alpha_n (\mu_{n,1} - 1) + \alpha_n \mu_{n,2} (\mu_{n,2} - 1) + \alpha_n \mu_{n,3} (\mu_{n,3} - 1) \right] |y_n - x^*| \]

\[ + \alpha_n \eta_{n,1} + \alpha_n \eta_{n,2} + \alpha_n \eta_{n,3} \]

Hence

\[ y_{n,j} - x^* \leq (1 + \sum_{t=0}^{j-1} \alpha_n t + 1 \Pi_{s=0}^{l-1} \mu_{n,j-s} (\mu_{n,j-t} - 1)) |y_n - x^*| \]

\[ + \sum_{t=0}^{j-1} \alpha_n t + 1 \Pi_{s=0}^{l-1} \mu_{n,j-s} \eta_{n,j-t} \]

\[ |x_{n+1} - x^*| \leq (1 + \sum_{t=0}^{m-1} \alpha_n t + 1 \Pi_{s=0}^{l-1} \mu_{n,m-s} (\mu_{n,m-t} - 1)) |y_n - x^*| \]
Thus by our hypothesis and lemma **Error! Reference source not found.**, we have that \( \lim_{n \to \infty} |x_n - x^*| \) exist, this completes the proof.

**Theorem 3.2**  Let \( E \) be a uniformly convex Banach space and \( K \) a nonempty closed convex subset of \( E \), let \( \{T_i^m\}_{i=1}^m \) be a finite family of uniformly continuous asymptotically nonexpansive in the intermediate sense maps form \( K \) to itself with sequences \( \{\mu_n\}_{n \geq 1}, \{\eta_n\}_{n \geq 1} \in (0, +\infty) \) such that \( \lim_{n \to \infty} \mu_n = 0 = \lim_{n \to \infty} \eta_n \) and Let \( \mu_n = \max_{i \in I} \{\mu_n\} \) and \( \eta_n = \max_{i \in I} \{\eta_n\} \) such that \( \lim_{n \to \infty} \mu_n = 0 = \lim_{n \to \infty} \mu_n = 0, \sum_{n=0}^{\infty} (\mu_n - \eta_n) < \infty, \sum_{n=0}^{\infty} \eta_n < \infty \)

Suppose that \( F = \bigcap_{i=1}^N F(T_i) \) is not empty and let \( \{x_n\}_{n \geq 1} \) be a sequence generated iteratively by (4) where \( \{\alpha_n\}_{n \geq 1} \) is sequences in \((0,1)\) satisfying the following conditions:

\[
\sum_{n=1}^{\infty} \alpha_n < \infty, 0 < \zeta < \alpha_n < \varepsilon < 1 \quad \forall \ n \geq 1
\]

\[
\forall j \in \{1, 2, \ldots, m\}, \lim_{n \to \infty} |x_n - T_j x_n| = 0
\]

**Proof.** Let \( x^* \in F \)

\[
|y_{n,j} - x^*|^2 \leq (1 - \alpha_n)|x_n - x^*|^2 + \alpha_n |T_j^n y_{n,j-1} - x^*| - \alpha_n (1 - \alpha_n) \eta_n |x_n - T_j^n y_{n,j-1}|
\]

\[
\leq (1 - \alpha_n)|x_n - x^*|^2 + \alpha_n |y_{n,j} - x^*| + |\eta_{n,j}^2|
\]

\[
- \alpha_n (1 - \alpha_n) \eta_n |y_{n,j} - T_j^n y_{n,j-1}|
\]

\[
\leq (1 - \alpha_n)|x_n - x^*|^2 + \alpha_n |y_{n,j} - x^*|^2 + \alpha_n |y_{n,j} - x^*| + \eta_{n,j} |y_{n,j} - (1 - \alpha_n) \eta_n |x_n - T_j^n y_{n,j-1}|
\]

So

\[
|y_{n,1} - x^*|^2 \leq (1 - \alpha_n) (\mu_{n,1} - 1)|x_n - x^*|^2 + (2 \mu_{n,1} |x_n - x^*|^2) + \eta_{n,1} |n_{n,1}|
\]

\[
- \alpha_n (1 - \alpha_n) \eta_n |x_n - T_{1}^n x_n|
\]

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So

\[
|y_{n,2} - x^*|^2 \leq (1 - \alpha_n) (\mu_{n,2} - 1) + (2 \mu_{n,2} |x_n - x^*|^2) + \eta_{n,2} |n_{n,2}|
\]

\[
+ \alpha_n (2 \mu_{n,2} |y_{n,1} - x^*| + \eta_{n,2}) |n_{n,2} + \alpha_n |n_{n,2} (2 \mu_{n,1} |x_n - x^*| + \eta_{n,1}) |n_{n,1}|
\]

\[
- \alpha_n (1 - \alpha_n) \eta_n |x_n - T_{2}^n x_n| - \alpha_n |n_{n,2} (2 \mu_{n,1} |x_n - x^*| + \eta_{n,1}) |n_{n,1}|
\]

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\[
|y_{n,j} - x|^{2} \leq (1 + \sum_{t=0}^{j-1} \alpha_{n} \Pi_{t=0}^{j-1}(\mu_{n,j-s})(\eta_{n,j,t-1})|y_{n,j} - x|^{2}
\]

\[
+ \sum_{t=0}^{j-1} \alpha_{n} (2\mu_{n,j-s}|y_{n,j-t-1} - x|^{2} + |\eta_{n,j-t}|)\eta_{n,j-t}
\]

\[-\alpha_{n} (1-\alpha_{n}) \sum_{t=0}^{j-1} \Pi_{t=0}^{j-1}(\mu_{n,m-t})(\eta_{n,m-t})|y_{n,m-t} - x|^{2}
\]

Hence,

\[
|x_{n+1} - x|^{2} \leq (1 + \sum_{t=0}^{m-1} \alpha_{n} \Pi_{t=0}^{m-1}(\mu_{n,m-s})(\eta_{n,m-t})|y_{n,m-t} - x|^{2}
\]

\[
+ \sum_{t=0}^{m-1} \alpha_{n} (2\mu_{n,m-s}|y_{n,m-t-1} - x|^{2} + |\eta_{n,m-t}|)\eta_{n,m-t}
\]

\[-\alpha_{n} (1-\alpha_{n}) \sum_{t=0}^{m-1} \Pi_{t=0}^{m-1}(\mu_{n,m-s})(\eta_{n,m-s})g(|y_{n,m-t} - x|^{2})
\]

\[\leq (1 + \sum_{j=1}^{2m} \sum_{j=1}^{m} \frac{2}{(n_{j-1})}|y_{n,j-1} - x|^{2} + |\eta_{n,j}|)|y_{n,j} - x|^2
\]

\[-\alpha_{n} (1-\alpha_{n}) \sum_{j=1}^{m} \Pi_{j=1}^{m}(\mu_{n,j-1})(\eta_{n,j-1})|y_{n,j-1} - x|^{2}
\]

\[\leq \prod_{j=1}^{m+1} (1-b) \sum_{j=1}^{m} g(|y_{n,j} - x|^{2})
\]

So, \( \lim_{n \to \infty} |y_{n,j} - x|^{2} = 0 \) thus \( \lim_{n \to \infty} |x_{n+1} - x|^{2} = 0 \) \( \forall j = 1, \ldots, m \).

\[
|x_{n+1} - x|^{2} \leq |x_{n+1} - x|^{2} + |T_{n,j}^{n} y_{n,j-1} - x|^{2} + |y_{n,j-1} - x|^{2}
\]

\[
\leq |x_{n+1} - x|^{2} + |y_{n,j-1} - x|^{2} + |y_{n,j} - x|^{2}
\]

\[= |x_{n+1} - x|^{2} + |y_{n,j} - x|^{2} + |y_{n,j-1} - x|^{2}
\]

\[= |x_{n+1} - x|^{2} + |y_{n,j} - x|^{2} + |y_{n,j-1} - x|^{2}
\]

\[\leq (1 + \mu_{n-1,j})|x_{n+1} - x|^{2} + |y_{n,j} - x|^{2} + |y_{n,j-1} - x|^{2}
\]

\[\leq (1 + \mu_{n-1,j})|x_{n+1} - x|^{2} + |y_{n,j} - x|^{2} + |y_{n,j-1} - x|^{2}
\]
hence, $\lim_{n\to \infty} |x_n - T_{n+1}^j x_n| = 0$ so that by uniform continuity of $T_j \forall j \lim_{n\to \infty} |T_j^n x_n - T_j^j x_n| = 0$ and hence $\lim_{n\to \infty} |x_n - T_j^n x_n| = 0 \forall j=1,...,m$. Thus, completes the proof.

**Theorem 3.3** Let $E, K, T, F(T), \{x_n\}$ be as in Theorem 3.2 Then, $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $\lim \inf_{n\to \infty} d(x_n, F) = 0$.

(where $F = F(T)$) **Proof** Let $\tau_n = \sqrt{2^m b} \sum_{j=1}^m (\mu_j, j - 1)_n, \nu_n = \sqrt{2^m b} \sum_{j=1}^m \eta_j$. Hence,

$$|x_{n+1} - x*| \leq \sum_{j=1}^m (\mu_{j,n}) |x_n - x*| + \sqrt{2^m b} \sum_{j=1}^m \eta_j.$$

So, $d(x_{n+1}, F) \leq (1 + \tau) d(x_n, F) + \nu_n$, hence, by $\lim d(x_n, F)$ exists, thus, $\lim \inf d(x_n, F) = 0$ implies $\lim d(x_n, F) = 0$.

Now,

$$|x_{n+k} - x*| \leq \sum_{j=1}^m (\mu_{j,n}) |x_n - x*| + \sqrt{2^m b} \sum_{j=1}^m \eta_j.$$

So that given any $\varepsilon > 0 \exists n_0 > 0$, integer, such that $\forall n \geq n_0$, $d(x_n, F) < \frac{\varepsilon}{4(m + 1)}$ and $n_0^{n_0+j} < \frac{\varepsilon}{4(m + 1)} \forall j = 1, 2,..., m$. So $\exists x \in F$ such that $d(x_n, F) < \frac{\varepsilon}{4(m + 1)}$. Thus,

$$|x_n - x*| < \frac{\varepsilon}{4(m + 1)}.$$

Hence, $\lim_{n\to \infty} |x_n - T_{n+1}^j x_n| = 0$ as $n \to \infty \forall i$ and $T_i$ is continuous $\forall i$. Hence, $\lim_{n\to \infty} |x_n - T_{n+1}^j x_n| = 0$ as $n \to \infty$. This completes the proof.

**Theorem 3.4** Let $E, K, T, F(T), \{\alpha_{n,j}\}, \{x_n\}$ be as in Theorem 3.2 Then, $\{x_n\}$ converges strongly to some point of $F$ if $T_j$ for some $j = 1, 2,..., m$ satisfy condition B.

**Proof** Let $T_j$ satisfy condition B. $\exists \beta: (0, +\infty) \to (0, +\infty)$ such that $f(0) = 0$ and $f(r) = 0 \forall r > 0$ such that

$$\beta(t) = \frac{\varepsilon}{2^{j+1}}.$$
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\[
f(d(x_n,F)) \leq |x_n - T_j x_j|\]

Hence

\[
\lim_{n \to \infty} f(d(x_n,F)) \leq \lim_{n \to \infty} |x_n - T_j x_j| = 0
\]

and hence,

\[
\lim_{n \to \infty} d(x_n,F) = 0
\]

Thus \( \{x_n\} \) converges strongly to some point \( x \in F \).

**Remark 3.1** Observe that if \( T \) is total asymptotically nonexpansive mapping and we assume that \( \exists c>0,k>0 \) constants such that \( \phi(t) \leq t \forall t \geq k \) collapse to asymptotically nonexpansive in the intermediate sense.

Hence we have the following Corollaries:

**Corollary 3.1** Let \( E \) be a normed linear space and \( K \) be a nonempty closed convex subset of \( E \). Let \( T_i: K \to K \) where \( i \in \{1,2,\ldots,m\} \). be \( m \) uniformly continuous total asymptotically nonexpansive mapping from \( K \) into itself with sequences \( \{\mu_{in}\}_{n=1}^{\infty}, \{\eta_{in}\}_{n=1}^{\infty} \subset [0,\infty) \) such that \( \lim_{n \to \infty} \mu_{in} = 0 = \lim_{n \to \infty} \eta_{in} \) and with function \( \phi_i: [0,\infty) \to [0,\infty) \) satisfying \( \phi_i(t) \leq M_i t_0 \) \( \forall t \geq M_i \) for some constants \( M_i, M_i > 0 \). Let \( \mu_n = \max_{i \in I} \{\mu_{in}\} \) and \( \eta_n = \max_{i \in I} \{\eta_{in}\} \) and, \( \phi_i(t) = \max_{i \in I} \{\phi_i(t)\} \forall t \in [0,\infty) \) Suppose that \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \) and let \( \{x_n\}_{n=1}^{\infty} \) be a sequence generated iteratively by (4) where \( \{\alpha_n\}_{n=1}^{\infty} \) is sequences in (0,1) satisfying the following conditions:

\[
\sum_{n=1}^{\infty} \alpha_n < \infty, 0 < \zeta < \alpha_n < \epsilon_n < 1 \forall n \geq 1
\]

Let \( x \in F \) then \( \lim_{n \to \infty} |x_n - x|\) exists.

**Corollary 3.2** Let \( E \) be a uniformly convex Banach space and \( K \) a nonempty closed convex subset of \( E \). Let \( T_i: K \to K \) where \( i \in \{1,2,\ldots,m\} \). be \( m \) uniformly continuous total asymptotically nonexpansive mapping from \( K \) into itself with sequences \( \{\mu_{in}\}_{n=1}^{\infty}, \{\eta_{in}\}_{n=1}^{\infty} \subset [0,\infty) \) such that \( \lim_{n \to \infty} \mu_{in} = 0 = \lim_{n \to \infty} \eta_{in} \) and with function \( \phi_i: [0,\infty) \to [0,\infty) \) satisfying \( \phi_i(t) \leq M_i t_0 \) \( \forall t \geq M_i \) for some constants \( M_i, M_i > 0 \). Let \( \mu_n = \max_{i \in I} \{\mu_{in}\} \) and \( \eta_n = \max_{i \in I} \{\eta_{in}\} \) and, \( \phi_i(t) = \max_{i \in I} \{\phi_i(t)\} \forall t \in [0,\infty) \) Suppose that \( F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \) and let \( \{x_n\}_{n=1}^{\infty} \) be a sequence generated iteratively by (4) where \( \{\alpha_n\}_{n=1}^{\infty} \) is sequences in (0,1) satisfying the following conditions:

\[
\sum_{n=1}^{\infty} \alpha_n < \infty, 0 < \zeta < \alpha_n < \epsilon_n < 1 \forall n \geq 1
\]

Let \( x \in F \) then \( \lim_{n \to \infty} |x_n - x|\) exists.

\[
\forall j \in \{1,2,\ldots,m\}, \lim_{n \to \infty} |x_n - T_j x_j| = 0
\]

and \( \{x_n\}_{n=1}^{\infty} \) converges strongly to some point of \( F \) if and only if \( \lim_{n \to \infty} d(x_n,F) = 0 \).

**Corollary 3.3** Suppose \( E,K,T_i,F,\{x_n\}_{n=1}^{\infty} \) are as in corollary above, then, \( \{x_n\}_{n=1}^{\infty} \) converges strongly to some point of \( F \) if any of the following conditions is satisfied:

1. \( \{x_n\}_{n=1}^{\infty} \) has a convergent subsequence
2. If \( T_i \) for some \( i \in \{1,\ldots,m\} \) satisfies condition B.

**References**

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