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On One Investigating Some Quadrature Rules For The Solution Of Second Order Volterra Integro-Differential Equations

[1] Kamoh, N. M, [2] Aboiyar, T. and [3]Onah E. S
[1] Department of Mathematics/Statistics Bingham University, Km 26 Keffi-Abuja expressway, Karu, Nassarawa State, Nigeria
[2, 3] Department of Mathematics/Statistics /Computer Science University of Agriculture, Makurdi, Benue State, Nigeria

Abstract: *In this paper, a block method was constructed for the direct solution of general second order initial value problems of the Volterra type integro-differential equations. The method was investigated for the basic properties and was found to be zero stable, consistent and convergent. The region of absolute stability showed that the method is A-stable. The method was tested on some existing standard problems, the results revealed that Trapezoidal rule performed significantly better than Simpson's 1/3 and Gaussian quadrature rules as revealed by the absolute error values shown in Tables 2 and 4 signifying that the choice of quadrature rule play an important role in the determination of the solution for VIDEs.*

***Keywords**: Volterra, integro-differential equations, Continuous, Block method, Collocation, Interpolation, second order equations, Hermite, polynomials, Trapezoidal rule, Simpson's 1/3 rule, Gaussian's quadrature rule.*

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I. Introduction

Volterra integro-differential equations has a wide spread of applications, they are applied in many physical areas such as in the glass forming process, population dynamics, economics and in chemical engineering. Unfortunately many problems involving these types of equations are very difficult if not impossible to solve analytically. However, the numerical solutions of such equations have been extensively studied by many researchers. Feldstein and Sopka (1974) introduced numerical methods for nonlinear Volterra integro-differential equations. The implicit Runge-Kutta methods of optimal order for Volterra integro-differential equations were suggested by Brunner (1984). The mixed interpolation and collocation methods for first and second order Volterra integro-differential equations with periodic solution were introduced in Brunner (1996). Yalcinbas and Sezer (2000) considered the approximate solution of high order linear Volterra-Fredholm integro-differential equations in terms of Taylor polynomials. The quadrature rules to find the numerical solutions of the initial value problems to Volterra integro-differential equations of the second kind appeared in Al-Timeme (2003). Linear multistep method for Volterra integro-differential equations was constructed by Day (1967) and Linz (1969).

In this paper, we introduce a different approach which is based on the continuous linear multistep method using the Hermite polynomials as basis function and three quadrature rules each is used in evaluating the integral part of the VIDEs.

II. Derivation of the Method

The methods of solution for second order initial value problems for ordinary differential equation of the form

$$y''(x) = f\left(x, y(x), y'(x)\right), y(x_0) = y_0 \qquad (1.0)$$

As discussed by many researchers such as in Lanczos (1956), Brunner (1996), Fox and Parker (1968), Lie and Norsett (1989), Onumanyi*et al.* (1994, 1999), Onumanyi and Yusuph (2002), Sirisena *et al.* (2004), Lambert (1973), Gear (1971), Okunuga and Sofoluwe (1990), Areo *et al.* (2008) and Okunuga and Ehigie (2009) can be modified to solve systems of equations arising from the discretization of second order initial value problems of the Volterra type of the form

$$y''(x) = f(x, y(x), z(x)),\ y(x_0) = y_0 \qquad (1.1)$$

where

$$z(x) = \int_{x_0}^{x} K(x, t, y(t))dt$$

and $y(x)$ is the unknown function. The idea is to approximate the exact solution $y(x)$ of (1.1) in the partition $I_n = [a,b] = [a = x_0 < x_1 < x_2 < \dots < x_n = b]$ of the integration interval $[a,b]$ with a constant step size h by the Hermite polynomial of the form

$$y(t) = \sum_{i=0}^{m} c_i H_i(t) \qquad (1.2)$$

where $c_i \in \mathbb{R}$, $y \epsilon C^2(a,b)$ and $t = (x - x_n)$.

The second derivative of (1.2) is substituted into (1.1) to obtain a differential system of the form

$$y'' = \sum_{i=0}^{m} c_i H_i''(t) = f(x, y(x), z(x)) \qquad (1.3)$$

Interpolating (1.2) at $x_{n+r}, r = 0$ and $k-1$ and collocating (1.3) at $x_{n+r}, r = 0,1,\dots,k$, we obtain the continuous scheme of the form

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f(x_{n+j}, y(x_{n+j}), z(x_{n+j})) \qquad (1.4)$$

where

$$z_n(x) = h \sum_{j=0}^{n} \alpha_j w_{nj} K(x_n, x_j, y_j),\ n > j,\ z_0 = 0 \qquad (1.5)$$

the weights w_{nj} depends on the choice of the quadrature rule. In this paper, three quadrature rules shall be used to evaluate the integral part, namely, Trapezoidal, Simpson's 1/3 and Gaussian quadrature. Evaluating (1.4) at some desired grid points; we obtain the corresponding discrete schemes.

Considering a three step method, that is letting $k = 3$ in (1.2), we get

$$y(x) = \sum_{i=0}^{4} c_i H_i(t)$$

Collocating (1.3) at $x_{n+r}, r = 0,1,2,3$ and interpolating (1.2) at x_n and x_{n+2}, we arrived at the continuous linear multistep method of the form

$$y(x) = \sum_{j=0}^{2} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{3} \beta_j(x) f_{n+j} \qquad (1.6)$$

where

$$\begin{bmatrix} \alpha_0(x) = 1 - \frac{1}{2h} t \\ \alpha_1(x) = 0 \\ \alpha_2(x) = \frac{t}{2h} \\ \alpha_3(x) = 1 \\ \beta_0(x) = -- \frac{14}{45} ht + \frac{1}{2} t^2 + \frac{11}{3h} t^3 + \frac{1}{12h^2} t^4 - \frac{1}{120h^3} t^5 \\ \beta_1(x) = -\frac{11}{15} ht + \frac{1}{2h} t^3 - \frac{5}{24h^2} t^4 + \frac{1}{40h^3} t^5 \\ \beta_2(x) = \frac{1}{15} ht - \frac{1}{4h} t^3 + \frac{1}{6h^2} t^4 - \frac{1}{40h^3} t^5 \\ \beta_3(x) = -\frac{1}{45} ht + \frac{1}{18h} t^3 - \frac{1}{24h^2} t^4 + \frac{1}{120h^3} t^5 \end{bmatrix} \qquad (1.7)$$

Evaluating (1.6) with coefficients (1.7) at x_{n+1}, x_{n+3} and its first derivative evaluated at the points x_n, x_{n+1}, x_{n+2} and x_{n+3}, we arrive at the discrete block method:

$$\left.\begin{aligned}
&y_{n+1}-\tfrac{1}{2}y_n-\tfrac{1}{2}y_{n+2} = -\tfrac{5}{12}h^2f_{n+1}-\tfrac{1}{24}h^2f_{n+2}-\tfrac{1}{24}h^2f_n\\
&y_{n+2} = 2hy_n' + y_n+\tfrac{22}{15}h^2f_{n+1}-\tfrac{2}{15}h^2f_{n+2}+\tfrac{2}{45}h^2f_{n+3}+\tfrac{28}{45}h^2f_n\\
&y_{n+3}+\tfrac{1}{2}y_n-\tfrac{3}{2}y_{n+2} = \tfrac{1}{2}h^2f_{n+1}+\tfrac{7}{8}h^2f_{n+2}+\tfrac{1}{12}h^2f_{n+3}+\tfrac{1}{24}h^2f_n\\
&y_{n+1}'-\tfrac{1}{2h}y_{n+2}+\tfrac{1}{2h}y_n = \tfrac{23}{360}hf_n+\tfrac{7}{120}hf_{n+1}-\tfrac{17}{120}hf_{n+2}+\tfrac{7}{360}hf_{n+3}\\
&y_{n+2}'-\tfrac{1}{2h}y_{n+2}+\tfrac{1}{2h}y_n = \tfrac{1}{45}hf_n+\tfrac{3}{5}hf_{n+1}+\tfrac{2}{5}hf_{n+2}-\tfrac{1}{45}hf_{n+3}\\
&y_{n+3}'-\tfrac{1}{2h}y_{n+2}+\tfrac{1}{2h}y_n = \tfrac{23}{360}hf_n+\tfrac{47}{120}hf_{n+1}+\tfrac{143}{120}hf_{n+2}+\tfrac{127}{360}hf_{n+3}
\end{aligned}\right\} \quad (1.8)$$

The modified block formulae is obtained from (1.8) as

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2h} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2h} & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2h} & 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2h} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+1}' \\ y_{n+2}' \\ y_{n+3}' \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} \\ 1 & \frac{1}{2h} \\ 0 & \frac{1}{2h} \\ 0 & \frac{1}{2h} \\ 0 & \frac{1}{2h} \end{bmatrix} \begin{bmatrix} y_n' \\ y_n \end{bmatrix} + \begin{bmatrix} -\frac{h^2}{24} \\ \frac{h^2}{24} \\ -\frac{14h}{45} \\ \frac{23h}{360} \\ \frac{h}{45} \\ \frac{23h}{360} \end{bmatrix} [f_n] + \begin{bmatrix} -\frac{5h^2}{12} & -\frac{h^2}{24} & 0 \\ \frac{h^2}{2} & \frac{7h^2}{8} & \frac{h^2}{12} \\ -\frac{11h}{15} & \frac{h}{15} & -\frac{h}{45} \\ \frac{7h}{120} & -\frac{17h}{120} & \frac{7h}{360} \\ \frac{3h}{5} & \frac{2h}{5} & -\frac{h}{45} \\ \frac{47h}{120} & \frac{143h}{120} & \frac{127h}{360} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \quad (1.9)$$

III. Analysis of the Method

Order and error constant

Expanding the block (1.8) in Taylor's series and collecting like terms in powers of h, we obtain;

$\check{C}_0 = \check{C}_1 = \cdots = \check{C}_5 = (0,0,0,0,0,0)^T$ and $\check{C}_6 = \left(-\frac{7}{480}, -\frac{1}{30}, -\frac{9}{160}, -\frac{19}{720}, -\frac{1}{90}, -\frac{3}{80}\right)^T$.

Hence the block method has order $p = (4,4,4,4,4,4)^T$ and error constant $\check{C}_6 = \left(-\frac{7}{480}, -\frac{1}{30}, -\frac{9}{160}, -\frac{19}{720}, -\frac{1}{90}, -\frac{3}{80}\right)^T$.

Consistency

According to Lambert (1991) and Fatunla (1988), the block method (1.8) is consistent since $p = 4 > 1$

Zero stability

The block solution (1.9) is said to be zero stable if the roots z_r; $r = 1, \ldots, n$ of the first characteristic polynomial $\check{\rho}(z)$, defined by

$$\check{\rho}(z) = det|zQ - T|$$

satisfies $|z_r| \leq 1$and every root with $|z_r| = 1$ has multiplicity not exceeding two in the limit as $h \to 0$. From the block solution (1.9), we have

$$z^6 - z^4 = 0 \text{ and } z = (0,0,0,0,-1,1).$$

Hence the method is zero stable, since all roots with modulus one do not have multiplicity exceeding the order of the differential equation in the limit as $h \to 0$.

Convergence

According to Lambert (1991) and Fatunla (1988), the block method (1.8) is convergent since it is consistent and zero stable

IV. Region of absolute stability

Reformulating the block method (1.8) as a General Linear Multistep Method (GLM) containing a partition of matrices A, B, C and DI where

$$A=\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & 1 \end{bmatrix} \quad B=\begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad C=\begin{bmatrix} \frac{7}{120} & -\frac{17}{120} & \frac{7}{360} \\ \frac{3}{5} & \frac{2}{5} & -\frac{1}{45} \\ \frac{47}{120} & \frac{143}{120} & \frac{127}{360} \end{bmatrix} \quad DI=\begin{bmatrix} -\frac{5}{12} & -\frac{1}{24} & 0 \\ \frac{22}{15} & -\frac{2}{15} & \frac{2}{45} \\ \frac{1}{2} & \frac{7}{8} & \frac{1}{12} \end{bmatrix}.$$

Substituting these matrices into the stability polynomial $r\left(A - Cz - DIz^2\right) - B$ we obtain the stability matrix whose determinant and the first derivative of the determinant are respectively given as

$$r^3 - r^2 - \frac{93}{40} r^2 z - \frac{79}{24} r^2 z^2 - \frac{167}{120} r^2 z^3 - \frac{461}{480} z^4 r^2 - \frac{97}{90} r^3 z + \frac{5}{108} r^3 z^2 + \frac{17}{270} r^3 z^3 - \frac{7}{2160} z^4 r^3 - \frac{29}{360} z^5 r^3 - \frac{1}{40} z^6 r^3$$

and

$$-\frac{93}{40} r^2 - \frac{79}{12} r^2 z - \frac{167}{40} r^2 z^2 - \frac{461}{120} r^2 z^3 - \frac{97}{90} r^3 + \frac{5}{54} r^3 z + \frac{17}{90} r^3 z^2 - \frac{7}{540} r^3 z^3 - \frac{29}{72} z^4 r^3 - \frac{3}{20} z^5 r^3$$

These are then plotted using MATLAB code based on Newton's iteration method to obtain the region of absolute stability as

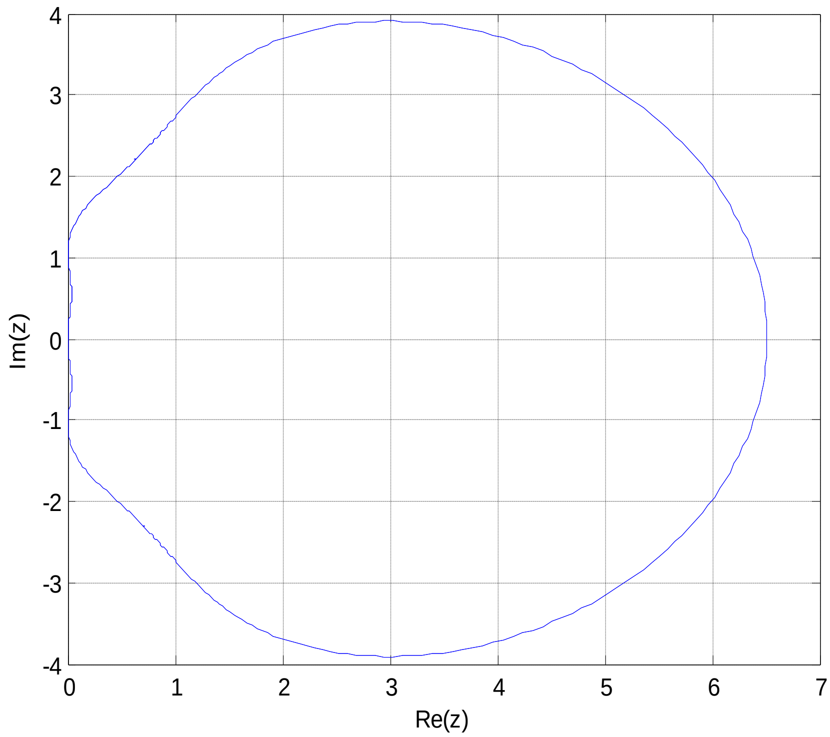

Figure 1: The region of absolute stability of method (1.8)

V. Numerical Examples

We implement our derived block method on second order initial value problems of the Volterra type integro-differential equations to support our theoretical discussion of the proposed method. The proposed method is tested on some numerical examples contained in the literature using MAPLE 18 programme.

Examples

(a) Consider a second order linear Volterra integro differential equation taken from Al-Smadi *et al*. (2013)

$$y''(x) - \int_0^x e^{-s}\sin(x)y'(s)ds + y(x) = \left(\frac{1}{2}e^{-x}\sin(x) - \sin(x)\right), 0 \leq x \leq 1$$

$$y(0) = -1, y'(0) = 1.$$

The exact solution is $y(x) = \sin(x) - \cos(x)$. The errors for $n = 100$ at some selected mesh points are displayed in Tables 1and 2

(b) Consider a second order nonlinear Volterra integro differential equation taken from Al- Smadi *et al.* (2013)

$$y''(x) + \int_0^x (y(s))^2 ds + \left(\frac{x}{2} - \sinh(x) - \frac{1}{4}\sinh(2x)\right) = 0, \quad 0 \leq x \leq 1$$

$$y(0) = 0, \qquad y'(0) = 1$$

The exact solution is $y(x) = \sinh(x)$. The errors for $n = 50$ at some selected mesh points are displayed in Tables 3 and 4

Table 1: Numerical Results of Example (a) for $n = 100$ Using the Three Quadrature Rules

x	Exact solution	AL-Smadi *et al.*, (2013)	Proposed method by Trapezoidal rule	Proposed method by Gaussian rule	Proposed method by Simpson's 1/3 rule
0.16	- 0.8279090768	- 0.8279086005	- 0.8279090779	- 0.8278965649	- 0.8279124903
0.32	- 0.6346688575	- 0.6346681575	- 0.6346688694	- 0.6343396681	- 0.6346953134
0.48	- 0.4252157473	- 0.4252150356	- 0.4252158013	- 0.4231297407	- 0.4253011813
0.64	- 0.2049003165	- 0.2049975120	- 0.2049004678	- 0.1971913811	- 0.2050927333
0.80	0.0206493816	0.0206497070	0.0206490561	0.0418185868	0.0202969851
0.96	0.2456715822	0.2456716440	0.2456709902	0.2937648271	0.2451064621

Table 2: The Absolute Error Values Example (a) for $n = 100$ Using the Three Quadrature Rules

AL-Smadi *et al.*, (2013) Absolute error	Trapezoidal rule Absolute error	Gaussian rule Absolute error	Simpson's 1/3 rule Absolute error
4.7628600 x10^{-7}	1.9000000 x10^{-9}	1.2511900 x10^{-5}	3.4135000 x10^{-6}
7.0000300 x10^{-7}	1.3900000 x10^{-8}	3.2918940 x10^{-4}	2.6455900 x10^{-5}
7.1159900 x10^{-7}	5.6600000 x10^{-8}	2.0860066 x10^{-3}	8.5434000 x10^{-5}
5.6533500 x10^{-7}	1.5330000 x10^{-7}	7.7089354 x10^{-3}	1.9241680 x10^{-4}
3.2537700 x10^{-7}	3.2662000 x10^{-7}	2.1169205 x10^{-2}	3.5239653 x10^{-4}
6.1419500 x10^{-8}	5.9210000 x10^{-7}	4.8093244 x10^{-2}	5.6512010 x10^{-4}

Table 3: Numerical Results of Example (b) for $n = 50$ Using the Three Quadrature Rules

X	Exact solution	AL-Smadi *et al.*, (2013)	Proposed method by Trapezoidal rule	Proposed method by Gaussian rule	Proposed method by Simpson's 1/3 rule
0.16	0.1606835410	0.1606828700	0.1606834953	0.1606820972	0.1607702018
0.32	0.3254893636	0.3254880336	0.3254889922	0.3254593282	0.3256425884
0.48	0.4986455052	0.4986436927	0.4986442194	0.4984168617	0.4984167169
0.64	0.6845942276	0.6845922814	0.6845910722	0.6835871768	0.6840816703
0.80	0.8881059822	0.8881044338	0.8880995443	0.8848541304	0.8876640323
0.96	1.1144017940	1.1144013728	1.1143900770	1.1057473440	1.1146683800

Table 4: The Absolute Error Values of Example (b) for $n = 50$ Using the Three Quadrature Rules

AL-Smadi *et al.*, (2013) Absolute error	Trapezoidal rule Absolute error	Gaussian rule Absolute error	Simpson's 1/3 rule Absolute error
6.70931 x10^{-7}	4.5700 x10^{-8}	1.443800 x10^{-6}	8.6660800 x10^{-5}
1.33004 x10^{-6}	3.7140 x10^{-7}	3.003540 x10^{-5}	1.5322480 x10^{-4}
1.81246 x10^{-6}	1.2858 x10^{-6}	2.286435 x10^{-4}	2.2878830 x10^{-4}
1.92460 x10^{-6}	3.1554 x10^{-6}	1.007051 x10^{-3}	5.1255730 x10^{-4}
1.54832 x10^{-6}	6.4379 x10^{-6}	3.251852 x10^{-3}	4.4194990 x10^{-4}
4.20849 x10^{-7}	1.1717 x10^{-5}	8.654450 x10^{-3}	2.6658800 x10^{-4}

VI. Conclusion

In this paper, some information about solving Volterra intgro-differential equations is provided. We have presented and illustrated the collocation approximation method using the Hermite polynomial as basis function to investigate solving an initial value problem in the class of Volterra integro-differential equations which are very difficult if not impossible to solve analytically. With the block approach, the non self starting nature associated with the predictor corrector method has been eliminated. Unlike the approach in predictor corrector method where additional equations were supplied from different formulation, all our additional equations are from the same continuous formulation. However, the absolute stability region showed that the method is A-stable and the application of this method to practical problems revealed that the method compared favorably using Trapezoidal quadrature rule with existing standard problems than Gaussian and Simpson's 1/3 quadrature rules. This investigation further revealed that the choice of quadrature rule plays a vital role in solving Volterra intgro-differential equations (VIDEs).

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