The Sum Span of Sum Span and the Product Span of Product Span in an Artex Space over a Bi-Monoid

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ABSTRACT: Sum Combination of elements of an Artex Space over a bi-monoid and the sum span of a subset of a completely bounded Artex space over a bi-monoid induced to define the sum span of sum span of a subset of a completely bounded Artex space over a bi-monoid. It is proved that the sum span of sum span of a subset of a completely bounded Artex space over a bi-monoid is the sum span. Product Combination of elements of an Artex Space over a bi-monoid and the product span of a subset of a completely bounded Artex space over a bi-monoid induced to define the product span of product span of a subset of a completely bounded Artex space over a bi-monoid. It is proved that the product span of product span of a subset of a completely bounded Artex space over a bi-monoid is the product span.

Keywords: Bi-monoids, Artex Spaces over bi-monoids, Completely Bounded Artex Spaces over bi-monoids, Sum Combination, Sum span the sum span of sum span the product span of product span.

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I. Introduction

The algebraic system Bi-semi-group is more general to the algebraic system ring or an associative ring. Artex Spaces over Bi-monoids were introduced. As a development of Artex Spaces over Bi-monoids, SubArtex spaces of Artex spaces over bi-monoids were introduced. From the definition of a SubArtex space, it is clear that not every subset of an Artex space over a bi-monoid is a SubArtex space. We found and proved some propositions which qualify subsets to become SubArtex Spaces. Completely Bounded Artex Spaces over bi-monoids were introduced. It contains the least and greatest elements namely 0 and 1. These elements play a good role in our study. Sum Combination of elements of an Artex Space over a bi-monoid and the sum span of a subset of a completely bounded Artex space over a bi-monoid induced to define the sum span of sum span of a subset of a completely bounded Artex space over a bi-monoid. Product Combination of elements of an Artex Space over a bi-monoid and the product span of a subset of a completely bounded Artex space over a bi-monoid induced to define the product span of product span of a subset of a completely bounded Artex space over a bi-monoid.. As the theory of Artex spaces over bi-monoids is developed from lattice theory, this theory will play a good role in many fields especially in science and engineering and in computer fields. In Discrete Mathematics this theory will create a new dimension.

II. Artex Spaces Over Bi-Monoids

2.1 Artex Space Over a Bi-monoid: Let ( M , + , · ) be a bi-monoid with the identity elements 0 and 1 with respect to + and · respectively. A non-empty set A together with two binary operations ^ and v is said to be an Artex Space Over the Bi-monoid ( M , + , · ) if

1. (A , ^ , v) is a lattice and
2. for each m ∈ M , m0 and acA, there exists an element ma ∈ A satisfying the following conditions:
   (i) m(a · b) = ma · mb
   (ii) m(a v b) = ma v mb
   (iii) ma ≤ (m + n)a and ma v na ≤ (m + n)a
   (iv) (mn)a = m(ra), for all m, n ∈ M , m0, n0, and a ∈ A
   (v) 1 · a = a, for all a ∈ A.

Here, ≤ is the partial order relation corresponding to the lattice (A , ^ , v). The multiplication ma is called a bi-monoid multiplication with an artex element or simply bi-monoid multiplication in A.

Example 2.2.1. Let W = {0,1,2,3,…}.
Then ( W , + , · ) is a bi-monoid , where + and · are the usual addition and multiplication respectively.
Let \( Z \) be the set of all integers
Then \( (Z, \leq) \) is a lattice in which \( \Lambda \) and \( V \) are defined by \( a \land b = \min \{a, b\} \) and \( a \lor b = \max \{a, b\} \) for all \( a, b \in Z \).
Clearly for each \( m, n \in W, m \neq 0 \), and for each \( a, b \in Z \).
Also,
(i) \( m(a \land b) = ma \land mb \)
(ii) \( m(a \lor b) = ma \lor mb \)
(iii) \( ma \land na \leq (m + n)a \) and \( ma \lor na \leq (m + n)a \)
(iv) \( (mn)a = m(na) \)
(v) \( 1.a = a \), for all \( m, n \in W, m \neq 0, n \neq 0 \) and \( a, b \in Z \).
Therefore, \( Z \) is an Artex Space Over the Bi-monoid \( (W, +, \cdot) \).

\[ \text{Example 2.2.2} \] As defined in Example 2.2.1, \( Q \), the set of all rational numbers is an Artex space over \( W \).
\[ \text{Example 2.2.3} \] As defined in Example 2.2.1, \( R \), the set of all real numbers is an Artex space over \( W \).

\[ \text{Example 2.2.4} : \] Let \( Q^* = Q' \cup \{0\} \), where \( Q' \) is the set of all positive rational numbers.
Then \( (Q^*, +, \cdot) \) is a bi-monoid. Now as defined in Example 2.2.1, \( Q \), the set of all rational numbers is an Artex space over \( Q' \).

\[ \text{Example 2.2.5} : \] \( R^* = R' \cup \{0\} \), where \( R' \) is the set of all positive real numbers. Then \( (R^*, +, \cdot) \) is a bi-monoid.
As defined in Example 2.2.1, \( R \), the set of all real numbers is an Artex space over \( R' \).

### 2.3 Properties

#### Properties 2.3.1:
We have the following properties in a lattice \((L, \land, \lor)\):
- \(1.a \land a = a\)
- \(2.a \land b = b \land a\)
- \(3.(a \land b) \land c = a \land (b \land c)\)
- \(4.a \land (a \lor b) = a\), for all \(a, b, c \in L\).

Therefore, we have the following properties in an Artex Space \(A\) over a bi-monoid \(M\).
- \(i). m(a \land b) = ma \land mb\)
- \(ii). m(a \lor b) = ma \lor mb\)
- \(iii). m((a \land b) \land c) = m(a \land (b \land c))\)
- \(iv). m((a \land b) \lor c) = m(a \land (b \lor c))\)
- \(v). m((a \lor b) \land c) = m(a \lor (b \land c))\)
- \(vi). m((a \lor b) \lor c) = m(a \lor (b \lor c))\)

#### 2.4 SubArtex Space:
Let \((A, \land, \lor)\) be an Artex space over a bi-monoid \((M, +, \cdot)\). Let \(S\) be a nonempty subset of \(A\). Then \(S\) is said to be a SubArtex Space of \(A\) if \((S, \land, \lor)\) itself is an Artex Space over \(M\).

#### 2.5 Complete Artex Space over a bi-monoid:
An Artex space \(A\) over a bi-monoid \(M\) is said to be a Complete Artex Space over \(M\) if as a lattice, \(A\) is a complete lattice, that is each nonempty subset of \(A\) has a least upper bound and a greatest lower bound.

#### 2.5.1 Remark:
Every Complete Artex space must have a least element and a greatest element.

#### 2.6 Lower Bounded Artex Space over a bi-monoid:
An Artex space \(A\) over a bi-monoid \(M\) is said to be a Lower Bounded Artex Space over \(M\) if as a lattice, \(A\) has the least element \(0\).

#### 2.6.1 Example:
Let \(A\) be the set of all constant sequences \(x_n\) in \([0, \infty)\)
Let \(W = \{0, 1, 2, 3, \ldots\}\).
Define \(\leq\), an order relation, on \(A\) by for \((x_n), (y_n)\) in \(A\), \((x_n) \leq (y_n)\) means \(x_n \leq y_n\), for each \(n\)
where \(\leq\) is the usual relation “less than or equal to”
Therefore, \(A\) is an Artex space over \(W\).
The sequence \((0_n)\), where \(0_n\) is 0 for all \(n\), is a constant sequence belonging to \(A\)
Also \((0_n) \leq (x_n)\), for all the sequences \((x_n)\) belonging to in \(A\)
Therefore, \((0_n)\) is the least element of \(A\).
That is, the sequence \(0,0,0,\ldots\) is the least element of \(A\).
Hence \(A\) is a Lower Bounded Artex space over \(W\).

#### 2.7 Upper Bounded Artex Space over a bi-monoid:
An Artex space \(A\) over a bi-monoid \(M\) is said to be an Upper Bounded Artex Space over \(M\) if as a lattice, \(A\) has the greatest element \(1\).

#### 2.7.1 Example:
Let \(A\) be the set of all constant sequences \(x_n\) in \((-\infty, 0]\) and let \(W = \{0, 1, 2, 3, \ldots\}\).
Define \(\leq\), an order relation, on \(A\) by for \((x_n), (y_n)\) in \(A\), \((x_n) \leq (y_n)\) means \(x_n \leq y_n\), for \(n = 1, 2, 3, \ldots\), where \(\leq\) is the usual relation “less than or equal to”
A is an Artex space over \(W\).
Now, the sequence \((1_n)\), where \(1_n\) is 0, for all \(n\), is a constant sequence belonging to \(A\)
Also \((x_n) \leq (1_n)\), for all the sequences \((x_n)\) in \(A\)
Therefore, \((1_n)\) is the greatest element of \(A\).
That is, the sequence 0,0,0,……, is the greatest element of \(A\).
Hence \(A\) is an Upper Bounded Artex Space over \(W\).

2.8 Bounded Artex Space over a bi-monoid: An Artex space \(A\) over a bi-monoid \(M\) is said to be a Bounded Artex Space over \(M\) if \(A\) is both a Lower bounded Artex Space over \(M\) and an Upper bounded Artex Space over \(M\).

2.9 Completely Bounded Artex Space over a bi-monoid: A Bounded Artex Space \(A\) over a bi-monoid \(M\) is said to be a Completely Bounded Artex Space over \(M\) if \((i)\) \(0 = a\), for all \(a \in A\) \((ii)\) \(m.0 = 0\), for all \(m \in M\).

2.9.1 Note: While the least and the greatest elements of the Complemented Artex Space is denoted by \(0\) and \(1\), the identity elements of the bi-monoid \((M, +, \cdot)\) with respect to addition and multiplication are, if no confusion arises, also denoted by 0 and 1 respectively.

III. The Sum Span Of A Sub Set Of An Artex Space Over A Bi-Monoid

3.1 Sum Combination: Let \((A, \Lambda, \nu)\) be an Artex Space over a bi-monoid \((M, +, \cdot)\). Let \(a_1, a_2, a_3, \ldots, a_n \in A\). Then any element of the form \(m_a \nu a_2 \nu a_3 \nu \ldots \nu a_n\), where \(m \in M\), is called a Finite Sum Combination or Finite Join Combination of \(a_1, a_2, a_3, \ldots, a_n\).

3.2 The Sum Span of a Sub set of an Artex Space over a Bi-Monoid: Let \((A, \Lambda, \nu)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, \cdot)\) and \(W\) be a nonempty subset of \(A\). Then the Sum Span of \(W\) or Join Span of \(W\) denoted by \(\text{S}[W]\) is defined to be the set of all finite sum combinations of elements of \(W\).
That is, \(\text{S}[W] = \{m_a \nu a_2 \nu a_3 \nu \ldots \nu a_n\} m, a \in \text{M} \text{ and } a \in W\).

3.3 PROPOSITIONS

Proposition 3.3.1: Let \((A, \Lambda, \nu)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, \cdot)\) and \(W\) be a nonempty subset of \(A\). Then \(W \subseteq S[W]\)

Proposition 3.3.2: Let \((A, \Lambda, \nu)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, \cdot)\) and \(W\) be a nonempty subset of \(A\). Then \(W \subseteq V\) implies \(S[W] \subseteq S[V]\).

Proposition 3.3.3: Let \((A, \Lambda, \nu)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, \cdot)\). Let \(W\) and \(V\) be any two nonempty subsets of \(A\). Then \(S[W \cup V] = S[W] \cup S[V]\).

IV. The Sum Span Of A Sub Set Of An Artex Space Over A Bi-Monoid

4.1 The Sum Span of a Sub set of an Artex Space over a bi-monoid: Let \((A, \Lambda, \nu)\) be a Completely Bounded Artex Space over a bi-monoid \((M, +, \cdot)\) and \(W\) be a nonempty subset of \(A\). Then the Sum Span of \(W\) or Join Span of \(W\) denoted by \(S[W]\) is defined to be the set of all finite sum combinations of elements of \(W\).

Note: \(S[\nu W] = S[S[S[S[S[S[S[S[S[S[S[S[W]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]]}
Each $m \alpha_i$ is a finite sum combination of elements of $W$
Therefore, each $m \alpha_i \in S[W]$
Since $1 \epsilon M$, $x = 1.(m_1 \alpha_1) \lor 1.(m_2 \alpha_2) \lor 1.(m_3 \alpha_3) \lor \ldots \lor 1.(m_n \alpha_n) \epsilon S[S[W]]$
Therefore, $S[W] \subseteq S[S[W]]$
Hence $S[S[W]] = S[W]$

Corollary 4.1.2: Let $(A, \Lambda, \nu)$ be a Completely Bounded Artex Space over a bi-monoid $(M, \cdot, +, \cdot)$ and $W$ be a nonempty subset of $A$. Then $S[S[W]] = S[W]$

Corollary 4.1.3: Let $(A, \Lambda, \nu)$ be a Completely Bounded Artex Space over a bi-monoid $(M, \cdot, +, \cdot)$ and $W$ be a nonempty subset of $A$. Then $S[W] = S[W]$

V. The Product Span Of A Sub Set Of An Artex Space Over A Bi-Monoid

5.1 Product Combination: Let $(A, \Lambda, \nu)$ be an Artex Space over a bi-monoid $(M, \cdot, +, \cdot)$. Let $a_1, a_2, a_3, \ldots, a_n \epsilon A$. Then any element of the form $m_a \alpha_1 A m_2 \alpha_2 A m_3 \alpha_3 A \ldots A m_n \alpha_n$, where $m \epsilon M$, is called a Finite Product Combination or Finite Meet Combination of $a_1, a_2, a_3, \ldots, a_n$.

5.2 The Product Span of a sub set of an Artex Space over a Bi-Monoid: Let $(A, \Lambda, \nu)$ be a Completely Bounded Artex Space over a bi-monoid $(M, \cdot, +, \cdot)$ and $W$ be a nonempty subset of $A$. Then the Product Span of $W$ in $W$ and $A$ is defined by $P[W] = \{m_1 \alpha_1 A m_2 \alpha_2 A m_3 \alpha_3 A \ldots A m_n \alpha_n | m \epsilon M \text{ and } a_1, a_2, \ldots, a_n \epsilon W\}$.

Note: $P^n[W] = \{P[P[P[\ldots P[W]]\ldots]] = P[W]\}$ (Product Span taken n times)

Proposition 5.1.1: Let $(A, \Lambda, \nu)$ be an Artex Space over a bi-monoid $(M, \cdot, +, \cdot)$ and $W$ be a nonempty subset of $A$. Then $P[W] = P[W]$.

Proof: Let $(A, \Lambda, \nu)$ be a Completely Bounded Artex Space over a bi-monoid $(M, \cdot, +, \cdot)$.

Let $W$ be a nonempty subset of $A$.

Let $U = P[W]$

Let $x \epsilon P[W] = P[U]$

Then, by the definition of $P[U]$, $x = m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n$, where $m \epsilon M$ and $n \epsilon U = P[W]$

Now, by the definition of $P[W]$, $s_i = m_i \alpha_i A m_i \alpha_2 A \ldots A m_i \alpha_n$, where $m \epsilon M$ and $a_i \epsilon W$

Therefore, $x = m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n = m_1(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n) A m_2(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n) A \ldots A m_n(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n)$

$= (m_1(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n)) A (m_2(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n)) A \ldots A (m_n(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n))$

$= (m_1(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n)) A (m_2(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n)) A \ldots A (m_n(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n))$

$= (m_1(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n)) A (m_2(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n)) A \ldots A (m_n(m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n))$

$= \ldots$

It is a finite product combination of elements of $W$.

Therefore, $x \epsilon P[W]$

$P[P[W]] \subseteq P[W]$

Conversely, suppose $x \epsilon P[W]$

Then $x = m_1 \alpha_1 A m_2 \alpha_2 A \ldots A m_n \alpha_n$, where $m \epsilon M$ and $a_i \epsilon W$

$= 1.(m_1 \alpha_1) A (m_2 \alpha_2) A (m_3 \alpha_3) A \ldots A (m_n \alpha_n)$

Therefore, $P[W] \subseteq P[P[W]]$

Hence $P[P[W]] = P[W]$

Corollary 5.2.2: Let $(A, \Lambda, \nu)$ be a Completely Bounded Artex Space over a bi-monoid $(M, \cdot, +, \cdot)$ and $W$ be a nonempty subset of $A$. Then $P[P[W]] = P[W]$

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Corollary 5.3.3: Let (A, +, V) be a Completely Bounded Artex Space over a bi-monoid (M, +, .) and W be a nonempty subset of A. Then P[W] = P[W]

VI. Conclusion

The Finite Product Combination and the Finite Sum Combination of elements of a subset of an Artex Space over a Bi-monoid, the Product Span of Product Span of a sub set of an Artex Space over a Bi-monoid and the Sum Span of Sum Span of a sub set of an Artex Space over a Bi-monoid will create a dimension in the theory of Artex spaces over bi-monoids. Interested researcher can do further research in this area of research.

References