On Supra Bitopological Spaces

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Abstract: The aim of this paper is to introduce the concept of supra bitopological spaces and discuss the fundamental properties of separation axioms in supra bitopological spaces.

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I. Introduction

The concept of bitopological spaces was introduced by J.C.Kelly[7]. A set equipped with two topologies is called a bitopological spaces. In 1980, R.C. Jain[6] was introduced separation axioms in bitopological spaces. The supra topological spaces have been introduced by A.S. Mashhour at[9] in 1983. In topological space the arbitrary union condition is enough to have a supra topological space. Here every topological space is a supra topological space but the converse is not always true. Then the authors S.P.Arav and T.M. Nour[1] was discussed some higher separation axioms in bitopological spaces. In this paper we introduce and study the concept of supra bitopological spaces and investigate some new separation axioms called supra pairwise T₀, supra pairwise T₁ and supra pairwise T₂ spaces. Also we study some of their basic properties in supra bitopological spaces.

II. Preliminaries

Definition 2.1 [9] \((X, \tau)\) is said to be a supra topological space if it is satisfying these conditions:
(1) \(X, \emptyset \in \tau\)
(2) The union of any number of sets in \(\tau\) belongs to \(\tau\).

Definition 2.2 [9] Each element \(A \subseteq \tau\) is called a supra open set in \((X, \tau)\), and its compliment is called a supra closed set in \((X, \tau)\).

Definition 2.3 [9] If \((X, \tau)\) is a supra topological spaces, \(A \subseteq X\), \(A \neq \emptyset\), \(\tau_A\) is the class of all intersection of \(A\) with each element in \(\tau\), then \((A, \tau_A)\) is called a supra topological subspace of \((X, \tau)\).

Definition 2.4 [9] The supra closure of the set \(A\) is denoted by supra cl\(A\) and is defined as supra cl\(A\) = \(\bigcap \{B : B \text{ is a supra closed and } A \subseteq B\}\).

Definition 2.5 [9] The supra interior of the set \(A\) is denoted by supra int\(A\) and is defined as supra int\(A\) = \(\bigcup \{B : B \text{ is a supra closed and } B \subseteq A\}\).

III. Supra Bitopological Spaces

Definition 3.1 If \(\tau_1\) and \(\tau_2\) are two supra topologies on a non-empty set \(X\), then the triplet \((X, \tau_1, \tau_2)\) is said to be a supra bitopological space.

Definition 3.2 Each element of \(\tau_i\) is called a supra \(\tau_i\)-open sets in \((X, \tau_1, \tau_2)\) for \(i=1, 2\). Then the complement of supra \(\tau_i\)-open sets are called a supra \(\tau_i\)-closed sets.
Definition 3.3 If $(X, \tau_1, \tau_2)$ is a supra bitopological space, $Y \subseteq X$, $Y \neq \emptyset$ then $(Y, \tau'_1, \tau'_2)$ is a supra bitopological subspace of $(X, \tau_1, \tau_2)$ if

\[
\tau'_1 = \{ U \cap Y \mid U \text{ is a supra } \tau_1 \text{-open in } X \} \quad \text{and} \quad \tau'_2 = \{ V \cap Y \mid V \text{ is a supra } \tau_2 \text{-open in } X \}.
\]

Definition 3.4 The supra $\tau_1$-closure of the set $A$ is denoted by supra $\tau_1$-cl$(A)$ and is defined as supra $\tau_1$-cl$(A) = \bigcap \{ B : B \text{ is a supra } \tau_1 \text{-closed and } A \subseteq B \text{ for } i = 1, 2 \}$.

Definition 3.5 The supra $\tau_1$-interior of the set $A$ is denoted by supra $\tau_1$-int$(A)$ and is defined as supra $\tau_1$-int$(A) = \bigcup \{ B : B \text{ is a supra } \tau_1 \text{-open and } B \subseteq A \text{ for } i = 1, 2 \}$.

IV. A New Classes Of Supra Pairwise Separation Axioms

Definition 4.1 A supra bitopological space $(X, \tau_1, \tau_2)$ is called a supra pairwise $T_0$ if for each pair of distinct points $x, y \in X$, $x \neq y$, there is a supra $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$, $y \notin V$ or $y \in V$, $x \notin U$.

Example 4.2 Let $X = \{ a, b, c \}$

$\tau_1 = \{ \emptyset, X, \{ a \}, \{ a \}, \{ b \}, \{ c \}, \{ a, b \}, \{ a, c \}, \{ b, c \}, \{ a, c \} \} \quad \text{and} \quad \tau_2 = \{ \emptyset, X, \{ b \}, \{ b, c \}, \{ a \} \}$

Let $a, b \in X$. Then there is a supra $\tau_1$-open set $U = \{ a, c \}$ and a supra $\tau_2$-open set $V = \{ b, c \}$ such that $a \in U$, $a \notin V$ or $b \in V$, $b \notin U$.

Let $b, c \in X$. Then there is a supra $\tau_1$-open set $U = \{ a, b \}$ and a supra $\tau_2$-open set $V = \{ a, c \}$ such that $b \in U$, $b \notin V$ or $c \in V$, $c \notin U$.

Let $a, c \in X$. Then there is a supra $\tau_1$-open set $U = \{ a, b \}$ and a supra $\tau_2$-open set $V = \{ b, c \}$ such that $a \in U$, $a \notin V$ or $c \in V$, $c \notin U$.

Therefore $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space.

Theorem 4.3 If $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space and $(Y, \tau'_1, \tau'_2)$ is a supra bitopological subspace of $(X, \tau_1, \tau_2)$ then $(Y, \tau'_1, \tau'_2)$ is also a supra pairwise $T_0$-space.

Proof: Suppose $x, y \in Y$, $x \neq y$. Since $Y \subseteq X$, $x, y \in X$. Since $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space, there exist a supra $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$, $y \notin V$ or $y \in V$, $x \notin U$. Then $U \cap Y$, $V \cap Y$ are $\tau'_1, \tau'_2$ supra open sets in $Y$ respectively such that $x \in U \cap Y$, $y \notin U \cap Y$ or $y \in U \cap Y$, $x \notin U \cap Y$.

Hence $(Y, \tau'_1, \tau'_2)$ is a supra pairwise $T_0$-space.

Theorem 4.4 If $(X, \tau_1, \tau_2)$, $(X', \tau'_1, \tau'_2)$ are two supra bitopological spaces, $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space and $f$ is a supra open function and bijective then $(X', \tau'_1, \tau'_2)$ is also a supra pairwise $T_0$-space.

Proof: Suppose that $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space. Let $x, y \in X \times X$, $x \neq y$, since $f$ is bijective function then there exist $x, y \in X$ such that $x' = f(x)$, $y' = f(y)$ and $x \neq y$. Since $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space then there exist a supra $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$, $y \notin V$ or $y \in V$, $x \notin U$.

Clearly $f(U) \subseteq X$ and $f(V) \subseteq X$ since $f$ is a supra open function, $f(U)$ is a supra $\tau_1$-open set and $f(V)$ is a supra $\tau_2$-open set in $X'$. Also $f(x) \in f(U)$, $f(y) \notin f(U)$ or $f(y) \in f(V)$, $f(x) \notin f(V)$.

Hence $(X', \tau'_1, \tau'_2)$ is a supra pairwise $T_0$-space.

Theorem 4.5 If $(X, \tau_1, \tau_2)$, $(X', \tau'_1, \tau'_2)$ are two supra bitopological spaces where $(X', \tau'_1, \tau'_2)$ is a supra pairwise $T_0$-space and $f : X \to X'$ is a bijective continuous function then $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space.

Proof: Let $x, y \in X$, $x \neq y$, since $f$ is bijective then there exist $x', y' \in X \times X', x' \neq y'$ such that $x' = f(x)$, $y' = f(y)$ since $X'$ is a supra pairwise $T_0$-space then there exist a supra $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$, $y \notin V$ or $y \in V$, $x \notin U$.

Since $f$ is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are supra $\tau_1$-open set, supra $\tau_2$-open set respectively in $X$. Also $f(x) \in U$ implies $x \notin f^{-1}(U)$, $y \notin f^{-1}(U)$ or $x \notin f^{-1}(V)$, $y \notin f^{-1}(V)$.

Hence $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space.

Remark 4.6 Every pairwise $T_0$-space is a supra pairwise $T_0$-space, but the converse is not true as shown in the following example.
Example 4.7 Let $X = \{a, b, c\}$
$\tau_1 = \{\emptyset, X, \{b\}, \{a, c\}, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}\}$
Therefore $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space but not a pairwise $T_0$-space

Definition 4.8 A supra bitopological space $(X, \tau_1, \tau_2)$ is called a supra pairwise $T_1$ if for each pair of distinct points $x, y \in X$, $x \neq y$ there is a supra $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $y \in U$, $y \notin V$ and $y \notin V$, $x \notin V$.

Example 4.9 Let $X = \{a, b, c\}$
$\tau_1 = \{\emptyset, X, \{b\}, \{a, c\}, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}\}$
Let $a, b \in X$. Then there is a supra $\tau_1$-open set $U = \{a, c\}$ and a supra $\tau_2$-open set $V = \{b, c\}$ such that $a \in U$, $b \notin U$ and $b \in V$, $a \notin V$.
Let $a, c \in X$. Then there is a supra $\tau_1$-open set $U = \{a, b\}$ and a supra $\tau_2$-open set $V = \{a, c\}$ such that $b \in U$ and $c \notin U$, $a \notin V$.
Let $b, c \in X$. Then there is a supra $\tau_1$-open set $U = \{a\}$ and a supra $\tau_2$-open set $V = \{a, c\}$ such that $b \notin U$ and $c \in V$, $b \notin V$.
Therefore $(X, \tau_1, \tau_2)$ is a supra pairwise $T_1$-space.

Theorem 4.10 If $(X, \tau_1, \tau_2)$ is a supra pairwise $T_1$-space and $(Y, \tau_1', \tau_2')$ is a supra bitopological subspace of $(X, \tau_1, \tau_2)$ then $(Y, \tau_1', \tau_2')$ is also a supra pairwise $T_1$-space.

Proof: suppose $x, y \in Y$, $x \neq y$. Since $Y \subseteq X$, $x, y \in X$. Since $(X, \tau_1, \tau_2)$ is a supra pairwise $T_1$-space, there exist two supreme $\tau_1$-open sets $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$, $y \notin U$, $y \notin V$, $x \notin V$. Then $U \cap Y, V \cap Y$ are $\tau_1'$, $\tau_2'$ supra open sets in $Y$ respectively such that $x \in U \cap Y$, $y \notin U \cap Y$ and $y \notin V \cap Y$, $x \notin V \cap Y$. Hence $(Y, \tau_1', \tau_2')$ is a supra pairwise $T_1$-space.

Theorem 4.11 If $(X, \tau_1, \tau_2)$, $(X', \tau_1', \tau_2')$ are two supra bitopological spaces, $(X, \tau_1, \tau_2)$ is a supra pairwise $T_1$-space and $f$ is a supra open function and bijective then $(X', \tau_1', \tau_2')$ is a supra pairwise $T_1$-space.

Proof: Suppose that $(X, \tau_1, \tau_2)$ is a supra pairwise $T_1$-space. Let $x', y' \in X$, $x' \neq y'$, since $f$ is bijective there exist $x, y \in X$ such that $f(x) = x'$, $f(y) = y'$ and $x \neq y$.
Since $(X, \tau_1, \tau_2)$ is a supra pairwise $T_1$-space, there exist two supreme $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Clearly $f(U) \subseteq X'$
and $f(V) \subseteq X'$ since $f$ is a supra open function, $f(U)$ is a supra $\tau_1'$-open set and $f(V)$ is a supra $\tau_2'$-open set in $X'$. Also $f(x) \in U$, $f(y) \notin U$ and $f(x) \notin V$.
Hence $(X', \tau_1', \tau_2')$ is a supra pairwise $T_1$-space.

Remark 4.12 Every supra $T_1$-space is a supra $T_0$-space but the converse is not true as shown in the following example.

Example 4.13 Let $X = \{a, b, c\}$
$\tau_1 = \{\emptyset, X, \{b\}, \{a, c\}, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{a\}, \{c\}, \{b, c\}\}$
Here $(X, \tau_1, \tau_2)$ is a supra pairwise $T_0$-space but not a supra pairwise $T_1$-space

Definition 4.14 A supra bitopological space $(X, \tau_1, \tau_2)$ is called a supra pairwise $T_2$ (or supra pairwise Hausdorff space) if for each pair of distinct points $x, y \in X$, $x \neq y$, there is a supra $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$ and $y \notin V$, $U \cap V = \emptyset$.

Example 4.15 Let $X = \{a, b, c\}$
$\tau_1 = \{\emptyset, X, \{b\}, \{a\}, \{a, b\}, \{a, c\}\}, \tau_2 = \{\emptyset, X, \{c\}, \{a, b\}, \{b, c\}\}$
Let $a, b \in X$. Then there is a supra $\tau_1$-open set $U = \{a\}$ and a supra $\tau_2$-open set $V = \{b, c\}$ such that $a \in U$ and $b \notin U$, $U \cap V = \emptyset$.
Let $b, c \in X$. Then there is a supra $\tau_1$-open set $U = \{b\}$ and a supra $\tau_2$-open set $V = \{c\}$ such that $b \notin U$ and $c \in V$, $U \cap V = \emptyset$. DOI: 10.9790/5728-1305025558 www.iorsjournals.org 57 | Page
Let $a, c \in X$. Then there is a supra $\tau_1$-open set $U = \{a\}$ and a supra $\tau_2$-open set $V = \{c\}$ such that $a \in U$ and $c \in V$. $U \cap V = \emptyset$.

Therefore $(X, \tau_1, \tau_2)$ is a supra pairwise $T_2$-space.

**Theorem 4.16** If $(X, \tau_1, \tau_2)$ is a supra pairwise $T_2$-space and $(Y, \tau_1', \tau_2')$ is a supra bitopological subspace of $(X, \tau_1, \tau_2)$ then $(Y, \tau_1', \tau_2')$ is also a supra pairwise $T_2$-space.

**Proof:** Suppose $x, y \in Y$, $x \neq y$. Since $Y \subseteq X$, $x, y \in X$. Since $(X, \tau_1, \tau_2)$ is a supra pairwise $T_2$-space, there exist a supra $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$. Then $U \cap Y, V \cap Y$ are $\tau_1'$, $\tau_2'$ supra open sets in $Y$ respectively such that $x \in U \cap Y, y \notin U \cap Y$ and $y \in V \cap Y, x \notin V \cap Y$.

$U \cap V = (U \cap Y) \cap (V \cap Y)$

$= (U \cap Y) \cap Y$

$= \emptyset \cap Y$

$= \emptyset$. Hence $(Y, \tau_1', \tau_2')$ is a supra pairwise $T_2$-space.

**Theorem 4.17** If $(X, \tau_1, \tau_2)$ and $(X', \tau_1', \tau_2')$ are two supra bitopological spaces,

$(X, \tau_1, \tau_2)$ is a supra pairwise $T_2$-space and $f$ is a supra open function and bijective then $(X', \tau_1', \tau_2')$ is a supra pairwise $T_2$-space.

**Proof:** Suppose that $(X, \tau_1, \tau_2)$ is a supra pairwise $T_2$-space. Let $x', y' \in X'$, $x' \neq y'$, since $f$ is bijective there exist $x, y \in X$ such that $f(x') = x'$, $f(y') = y'$ and $x \neq y$.

Since $(X, \tau_1, \tau_2)$ is a supra pairwise $T_2$-space, there exist a supra $\tau_1$-open set $U$ and a supra $\tau_2$-open set $V$ such that $x \in U$ and $y \notin U$, $\emptyset \neq V \cap Y = \emptyset$. Clearly $f(U) \subseteq X'$ and $f(V) \subseteq X'$, since $f$ is a supra open function, $f(U)$ is a supra $\tau_1'$-open set and $f(V)$ is a supra $\tau_2'$-open set in $X'$. Also $f(x) \notin f(U)$ and $f(y) \notin f(V)$, $f(U) \cap f(V) = \emptyset$.

Hence $(X', \tau_1', \tau_2')$ is a supra pairwise $T_2$-space.

**Remark 4.18** Every supra pairwise $T_2$-space is a supra pairwise $T_1$-space but the converse is not true such as shown in the following example.

**Example 4.19** Let $X = \{a, b, c\}$

$\tau_1 = \emptyset, X, \{a, b\}, \{a, c\}, \{b, c\}\}$. $\tau_2 = \emptyset, X, \{a\}, \{a, c\}, \{b, c\}$.

Here $(X, \tau_1, \tau_2)$ is a supra pairwise $T_1$-space but not a supra pairwise $T_2$-space.

**V. Conclusion**

In this paper, basic concepts of supra bitopological spaces are introduced and also separation axioms of supra bitopological spaces are analysed.

**References**


