Laplace Homotopy Analysis Method for Solving Fractional Order Partial Differential Equations

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Abstract: In this paper, we considered a non-linear system of fractional partial differential equations. They have been solved by a computational method which is so-called a modified Laplace Homotopy Analysis method. The fractional derivatives are described in the Caputo sense. The proposed technique is only a simple modification of the Homotopy Analysis Method. The method was applied for some illustrative examples to solve non-linear systems of fractional partial differential equations. From the result of the illustrative examples we conclude that the method is computationally efficient.

Keywords: Fractional calculus, system of fractional order partial differential equations, Laplace transform, Homotopy Analysis method.

I. Introduction

Fractional order partial differential equations are popularizations of classical partial differential equations. These have been of large attention in the recent literatures.

These topics have received a mighty deal of attention especially in the fields of viscoelasticity materials, electrochemical processes, dielectric polarization, colored noise, anomalous diffusion, signal processing, control theory and others.

Increasing extent, these models are used in applications such as fluid flow, finance and others. Most nonlinear fractional differential equations do not have exact solutions, so approximation and numerical techniques must be applied. The Laplace Homotopy Analysis method (LHAM) is a combination of the Homotopy analysis method proposed by Liao in his Ph.D. Thesis [1] and the Laplace transform [2, 3].

The Homotopy analysis method has been successfully employed to solve many types of nonlinear, homogeneous or nonhomogeneous equations and systems of equations as well as problems in science and engineering [4-5].

Various authors have proposed several schemes to solve system of fractional partial differential equations with Liouville-Caputo and Caputo-Fabrizio fractional operators.

Dehghan in [6] applied the HAM to solve linear partial differential equations, in this work, fractional derivatives are described in the Liouville-Caputo sense, Xu in [7] calculated analytically the time fractional wave-like differential equation with a changeable coefficients, the author reduced the governing equation to two fractional ordinary differential equations.

Jafari in [8] exercise the HAM to obtain the solution of multi-order fractional differential equation studied by Diethelm Ford [9], Goufo et al. [10] developed a mathematical analysis of a model of rock fracture in the ecosystem and applied the Caputo-Fabrizio fractional derivative, where analytical and computational approaches are obtained. Other analytical approaches that could be of interest are presented in [12-14].

This paper is organized as follows: in section 2 we recall the definitions of fractional derivatives and fractional integration simply, section 3 describes the formulation of Laplace Homotopy analysis method for solving system of fractional order P.D.Es. Some illustrative examples are given in section 4 finally a conclusion is given in section 5.

II. Fractional Order Derivative And Integral:

In this section, we review basic definitions of fractional order differentiation and fractional order integration such as:

**Definition 2.1:** The Riemann–Liouville fractional integral of order \( \alpha > 0 \) is defined as follows:

\[
\mathcal{I}_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (t - \tau)^{\alpha - 1} f(\tau) \, d\tau,
\]

where \( \Gamma(\alpha) \) is the Gamma function.

**Definition 2.2:** The Caputo fractional derivative of order \( \alpha > 0 \) is defined as follows:
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\[ cD_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha+m-2}} d\tau & m-1 < \alpha < m \\ \frac{d^m}{dx^m} f(x) & \alpha = m \end{cases} \]

For \( \alpha > 0 \), we have the following properties of the Caputo fractional derivative:

1. \( D_x^\alpha (f(x)) = f(x) \)
2. \( D_x^\alpha \left( \frac{D_x^b f(x)}{\Gamma(b+1)} \right) = \frac{D_x^{b+\alpha} f(x)}{\Gamma(b+\alpha+1)} \)
3. \( D_x^\alpha \left( f(x)^c \right) = f(x)^c \cdot \frac{d^c}{dx^c} f(x) \)
4. \( D_x^\alpha \left( \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-b+1)} x^{\gamma-a} \right) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-a+1)} x^{\gamma-a} \cdot \frac{d^\gamma}{dx^\gamma} f(x) \)

Where \([\alpha]\) is the floor function of \( \alpha \)

III. The Approach

Let us consider the following system of non-linear fractional partial differential equations.

\[ cD_x^\alpha u_i(x, t) + R_i(u_1, u_2, ..., u_m) + N_i(u_1, u_2, ..., u_m) = g_i(x, t), \quad n - 1 < \alpha_i \leq n, \quad i = 1, 2, ..., m \] ...

With initial data

\[ u_i(x, 0) = f_i(x) \] ...

Where \( cD_x^\alpha \) are the Caputo fractional derivatives of order \( \alpha_i \), \( R_i \) and \( N_i \), \( i = 1, 2, ..., m \) are non-linear operators, respectively, and \( g_i \) are source terms.

In order to solve this system by using Laplace Homotopy Analysis method, first we employing the Laplace transform to both sides of (1) yields:

\[ \mathcal{L}[u_i(x, t)] = \frac{1}{\alpha!} \sum_{k=0}^{\alpha-1} s^{\alpha-1-k} u_i(x, 0) + \frac{1}{\alpha!} \mathcal{L}[R_i(u_1, u_2, ..., u_m) + N_i(u_1, u_2, ..., u_m)] + \frac{1}{\alpha!} \mathcal{L}[g_i(x, t)] \]

The so-called zero-order deformation equation of the Laplace equation (3) has

\[ (1 - q)\mathcal{L}_0 u_i(x, t) - \mathcal{L}u_{i0}(x, t) = \mathcal{L}_0 [R_i(u_1, u_2, ..., u_m, t) + N_i(u_1, u_2, ..., u_m)] + \frac{1}{\alpha!} \mathcal{L}[g_i(x, t)] \] ...

Subject to the initial conditions:

\[ \mathcal{L}_0 u_i(x, 0) = f_i(x), \quad i = 1, 2, ..., m \] ...

Where \( \mathcal{L}_0 \) is an embedding parameter when \( q=0 \) we have \( \mathcal{L}_0 u_i(x, t; 0) = \mathcal{L}u_{i0}(x, t) \)

And when \( q = 1, h = -1 \) the zero-order deformation eq (4) and (5) equivalent to (3) and (2), respectively, provides

\[ \mathcal{L}_0 u_i(x, t) = \mathcal{L}u_{i0}(x, t) \]

Thus as \( q \) increasing from 0 to 1, \( \mathcal{L}_0 u_i(x, t, q) \) varies from \( \mathcal{L}u_{i0}(x, t) \) to \( \mathcal{L}u_i(x, t) \).

Expanding \( \mathcal{L}_0 u_i(x, t; q) \) in Taylor series with respect to \( q \), one has

\[ \mathcal{L}_0 u_i(x, t; q) = \mathcal{L}u_{i0}(x, t) + \sum_{m=1}^{\infty} \mathcal{L}u_{im}(x, t) q^m, \quad i = 1, 2, ..., m \]

Where

\[ \mathcal{L}u_{im}(x, t) = \frac{1}{m!} \frac{d^m}{dx^m} \mathcal{L}u_1(x, t; q) \bigg|_{q = 0} \]

Define the vectors

\[ \mathcal{L}u_{im}(x, t) = \{ \mathcal{L}u_{i0}(x, t), \mathcal{L}u_{i1}(x, t), \mathcal{L}u_{i2}(x, t), ..., \mathcal{L}u_{im}(x, t) \}, \quad i = 1, 2, ..., m \]

Differentiating equation (4) \( m \) times with respect to the embedding parameter \( q \), and setting \( q=0 \), \( h=-1 \) and finally dividing them by \( m! \), we have the so-called \( m^{th} \) order deformation equation for \( i = 1, 2, ..., n \).

\[ \mathcal{L}u_{im}(x, t) = x^m \mathcal{L}u_{im-1}(x, t) - R_{im} (\mathcal{L}u_{im-1}(x, t)) \]

Where

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\[
R_m(\mathcal{L} u_{m-1}(x, t)) = \mathcal{L} u_{m-1}(x, t) + \frac{1}{s^m} \left( \frac{1}{(m-1)!} \int_0^t (t^s - x^s)^{m-1} \mathcal{L}[R_i(\varphi_1(x, t, q), \varphi_2(x, t, q), \ldots, \varphi_m(x, t, q))] \right)
\]
\[
N_i((\varphi_1(x, t, q), \varphi_2(x, t, q), \ldots, \varphi_m(x, t, q))] \Bigg|_{q=0} = (1 - x_m) \left( \frac{1}{s^m} \sum_{k=0}^{m-1} s^{m-1-k} f(x) + \frac{1}{s^m} \mathcal{L}[g(x, t)] \right)
\]

... (10)

And \( x_m = \begin{cases} 
0 & , \quad m \leq 1 \\
1 & , \quad m > 1 
\end{cases} \) ... (11)

Applying the inverse Laplace transform of both sides of (9), then we have a power series solution of (1) which can be expressed as:
\[
u_i(x, t) = \sum_{n=0}^{\infty} u_{in}(x, t), \quad i = 1, 2, \ldots m
\]
... (12)

**IV. Illustrative Examples**

In this section we will apply the LHAM to systems of non-linear fractional partial differential equations (FPDEs).

**Example 1:** Consider the following system of linear FPDEs:
\[
\begin{aligned}
\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial^\beta u(x, t)}{\partial x^\beta} & = v(x, t) + u(x, t) = 0 \\
\frac{\partial^\beta v(x, t)}{\partial x^\beta} - \frac{\partial^\alpha v(x, t)}{\partial t^\alpha} & = u(x, t) + v(x, t) = 0
\end{aligned}
\]
... (13)

With initial conditions as
\[
u(x, 0) = \sinh(x), \quad v(x, 0) = \cosh(x) \quad , (0 < \alpha, \beta < 1)
\]
The exact solution is given in [13] as \( u(x, t) = \sinh(x - t), v(x, t) = \cosh(x - t) \)
Taking the Laplace transform to the both sides of eq (13) and using (14), we have
\[
\begin{aligned}
\mathcal{L}[u(x, t)] - \frac{\sinh(x)}{s^\alpha} \mathcal{L}[v_x(x, t)] + \frac{1}{s^\alpha} \mathcal{L}[v(x, t) + u(x, t)] & = 0 \\
\mathcal{L}[v(x, t)] - \frac{\cosh(x)}{s^\beta} \mathcal{L}[u_x(x, t)] + \frac{1}{s^\beta} \mathcal{L}[v(x, t) - u(x, t)] & = 0
\end{aligned}
\]
... (15)

Furthermore, we can construct the Homotopy as follows
\[
\begin{aligned}
R_1(m(u_{m-1}, v_{m-1})) & = \mathcal{L} u_{m-1}(x, t) - \frac{1}{s^\alpha} \mathcal{L} v_{m-1}(x, t) + \frac{1}{s^\beta} \mathcal{L} v_{m-1}(x, t) + u_{m-1}(x, t) - (1 - x_m) \left( \frac{\sinh(x)}{s^\alpha} \right) \\
R_2(m(u_{m-1}, v_{m-1})) & = \mathcal{L} v_{m-1}(x, t) - \frac{1}{s^\beta} \mathcal{L} u_{m-1}(x, t) + \frac{1}{s^\beta} \mathcal{L} v_{m-1}(x, t) + u_{m-1}(x, t) - (1 - x_m) \left( \frac{\sinh(x)}{s^\beta} \right)
\end{aligned}
\]
... (16) ... (17)

and the \( m^{th} \) order deformation equations for \( m \geq 1 \) become
\[
\begin{aligned}
\mathcal{L} u_m(x, t) & = x_m \mathcal{L} u_{m-1}(x, t) - R_1(m(u_{m-1}, v_{m-1})) \\
\mathcal{L} v_m(x, t) & = x_m \mathcal{L} v_{m-1}(x, t) - R_2(m(u_{m-1}, v_{m-1}))
\end{aligned}
\]
... (18) ... (19)
From Eqs (14), (18) and (19) and subject to initial condition \( u_{m-1}(x, 0) = 0, v_{m-1}(x, 0) = 0, \quad m > 1 \)

We successively obtain
\[
\begin{aligned}
\mathcal{L} u_0(x, t) & = \frac{\sinh(x)}{s^\alpha + 1} \\
\mathcal{L} v_0(x, t) & = \frac{\cosh(x)}{s^\beta + 1} \\
\mathcal{L} u_1(x, t) & = -\frac{1}{s^\alpha + 1} \sinh(x) + \frac{1}{s^\beta + 1} \cosh(x) \\
\mathcal{L} v_1(x, t) & = -\frac{1}{s^\alpha + 2} \cosh(x) + \frac{1}{s^\beta + 2} \sinh(x) + \frac{1}{s^\alpha + 1} \cosh(x) \\
\mathcal{L} u_2(x, t) & = -\frac{1}{s^\alpha + 2} \cosh(x) + \frac{1}{s^\beta + 2} \sinh(x) + \frac{1}{s^\alpha + 1} \cosh(x) \\
\mathcal{L} v_2(x, t) & = -\frac{1}{s^\alpha + 2} \cosh(x) + \frac{1}{s^\beta + 2} \sinh(x) + \frac{1}{s^\alpha + 1} \cosh(x)
\end{aligned}
\]
... (20)

And so on.

upon applying the inverse Laplace transform to the above equations and by using (12), one can get:
\[
\begin{aligned}
u(x, t) & = \left( 1 - \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 2)} \right) \sinh(x) + \left( \frac{1}{\Gamma(\alpha + 1)} - \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha + 3)} + \frac{\Gamma(\alpha + 3)}{\Gamma(\alpha + 4)} + \ldots \right) \cosh(x) \\
v(x, t) & = \left( 1 + \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 2)} \right) \cosh(x) + \left( \frac{1}{\Gamma(\beta + 1)} - \frac{\Gamma(\beta + 2)}{\Gamma(\beta + 3)} + \frac{\Gamma(\beta + 3)}{\Gamma(\beta + 4)} + \ldots \right) \sinh(x)
\end{aligned}
\]
... (20)

Following figure (1) represent the approximate solution of problem (13) for different values of \( \alpha \) and \( \beta \) compared with the exact solution when \( \alpha = \beta = 1 \) at \( t=0.01, 0.05, 0.1 \)

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Example 2: Consider the nonlinear system of FPDEs:

\[ C^\alpha D_t^\alpha u(x, t) + v(x, t)u_x(x, t) + u(x, t) = 1 \]
\[ C^\beta D_t^\beta v(x, t) - u(x, t)v_x(x, t) - v(x, t) = 1 \quad , \quad 0 < \alpha, \beta \leq 1 \]  

... (21)

With initial conditions

\[ u(x, 0) = e^x, \quad v(x, 0) = e^{-x} \]  

... (22)

Taking the Laplace transform to the both sides of eq. (21) and using (22), we have

\[
\begin{align*}
\mathcal{L}u(x,t) & - \frac{s^\alpha}{s} + \frac{1}{s^\alpha} \mathcal{L}[v(x,t)u_x(x,t)] + \frac{1}{s^\alpha} \mathcal{L}[u(x,t)] - \frac{1}{s^\alpha + 1} = 0 \\
\mathcal{L}v(x,t) & - \frac{e^{-x}}{s} - \frac{1}{s^\alpha} \mathcal{L}[u(x,t)v_x(x,t)] - \frac{1}{s^\alpha} \mathcal{L}[v(x,t)] - \frac{1}{s^\alpha + 1} = 0
\end{align*}
\]  

... (23)

Furthermore, we can construct the Homotopy as follows

\[
R_{m}(u_{m-1}, v_{m-1}) = \mathcal{L}u_{m-1}(x,t) + \frac{1}{s^\alpha} \mathcal{L} \left[ v_{m-1}(x,t) \left( u_{m-1}(x,t) \right)_x + u_{m-1}(x,t) \right] - (1 - x_m) \left( \frac{e^x}{s} + \frac{1}{s^\alpha + 1} \right)
\]
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\[ R_{2m}(\overline{u}_{m-1}, \overline{v}_{m-1}) = \mathcal{L}v_{m-1}(x, t) = \frac{1}{s^a} \mathcal{L} \left[ u_{m-1}(x, t) (v_{m-1}(x, t))_x + v_{m-1}(x, t) \right] - (1 - x_m) \left( \frac{e^{-x}}{s} + \frac{1}{s^{a+1}} \right) \]

and the \( m^{th} \) order deformation equations for \( m \geq 1 \) become

\[ \mathcal{L}u_m(x, t) = x_m \mathcal{L}u_{m-1}(x, t) - R_{1m}(\overline{u}_{m-1}, \overline{v}_{m-1}) \] \hspace{1cm} (24)

\[ \mathcal{L}v_m(x, t) = x_m \mathcal{L}v_{m-1}(x, t) - R_{2m}(\overline{u}_{m-1}, \overline{v}_{m-1}) \] \hspace{1cm} (25)

From Eqs (22), (24) and (25) and subject to initial condition

\[ u_{m-1}(x, 0) = 0, v_{m-1}(x, 0) = 0, \ m \geq 1 \]

We successively obtain

\[
\begin{align*}
& L_0 u_0(x, t) = \frac{e^x}{s} , \quad L_1 u_1(x, t) = \frac{-e^x}{s^{a+1}} , \\
& L_0 v_0(x, t) = \frac{e^{-x}}{s} , \quad L_2 v_2(x, t) = \frac{e^{-x}}{s^{2a+2}} , \\
& L_2 u_2(x, t) = \frac{-\Gamma(2+1) e^{3a}}{\Gamma(2+1) s^{3a+2}} , \\
& L_2 v_2(x, t) = \frac{-\Gamma(2+1) e^{3a}}{\Gamma(2+1) s^{3a+2}} , \\
& \vdots
\end{align*}
\]

Upon applying the inverse Laplace transform to the above equations and by using (12), one can get:

\[ u(x, t) = \left( 1 - \frac{\Gamma(a+1) e^{3a}}{\Gamma(2a+2) e^{3a}} + \ldots \right) e^{x} + \left( \frac{-\Gamma(2a+2) e^{3a}}{\Gamma(2a+2) e^{3a}} + \ldots \right) e^{-x} \]

\[ v(x, t) = \left( 1 - \frac{\Gamma(a+1) e^{3a}}{\Gamma(2a+2) e^{3a}} + \ldots \right) e^{x} + \left( \frac{-\Gamma(2a+2) e^{3a}}{\Gamma(2a+2) e^{3a}} + \ldots \right) e^{-x} \]

Following fig (2) represent the approximate solution of problem (21) for different of \( \alpha \)
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Figure (2) the approximate solution of problem (21) for different values of $\alpha$

V. Conclusions

We present in this paper a computational method for solving a system of fractional order partial differential equations which is known as Laplace Homotopy analysis method. The method considered proved that it is a powerful tool which enables us to handle a wide class of non-linear fractional partial differential equations in a simple way and in order to reach the desired accuracy, all what we have to do is to increase the number of iterations. If the non-linear problems has an exact solution, then after a certain stages, every iteration leads to the same exact solution. Therefore Laplace Homotopy analysis method is adequate for both linear and non-linear problems.

References