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Rc- Closed Sets and its Topology

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Abstract. *The aim of this paper is to introduce a new collection of sets called rc- closed sets and its topology which is stronger than the collection of w-closed sets due to Arhangel'skii.*
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I. Introduction and Preliminaries

The notion of w- closedness was introduced by Arhangel'skii in [1]. A subset A of a space X is called w- closed if $Cl(B) \subset A$ for every countable subset B of A. In [2, 3] it is shown that the family of all w- open subsets of a space form a topology for it. The notion of countable tightness was introduced in [2]. A space in which the closure operator is determined by countable sets is called countably tight. A topological space X has countable tightness if every w- closed subset is closed in X [2]. It is proved that every sequential space and every hereditarily separated space has countable tightness. Especially every countable space (respectively every perfectly regular countable compact has countable tightness [3]). Ekici and Jafari [4] introduced and study a class of sets stronger than the class of w- closed sets, called w_*-closed sets. In this paper we introduce and study a new class of closed sets named by cr- closed sets.

Throughout this paper X and Y are topological spaces with no separation axioms assumed, unless otherwise stated. For a subset A of X, the closure of A and the interior of A will be denoted by *cl*(*A*) and *int*(*A*) respectively.

A subset A of a space X is said to be regular- open [5] if $A = \text{int}(cl(A))$, the complement of regular open set is called regular closed. Since the intersection of two regular open sets is regular open, the family of all regular open forms a base for a smaller topology τ_s on *X*, called the semi-regularization of τ . The union of all regular open sets of X contained in a set A is called the regular interior of A (briefly $r\,\text{int}(A)$) and the intersection of all regular closed sets of X containing a set A is called the regular closure of A (briefly $rcl(A)$). A subset A of a space X is called ω - closed if $cl(B) \subset A$ for every countable subset $B \subset A$ and the complement of ω - closed sets is called ω - open.

II. RC-Closed Sets and its Topology

Definition2.1. A subset A of a space X is called rc - closed if $rcl(B) \subset A$ for every countable subset $B \subset A$ and its complement is called rc - open. The family of all rc - open subsets of a space X is denoted by τ_{rc} .

Remark2.2. For a subset A of a space X, the following implications hold and none of these implications is reversible as shown in the following examples.

$$\begin{array}{ccc} rc-open & \Rightarrow & \omega-open \\ \Uparrow & & \Uparrow \\ r-open & \Rightarrow & open \end{array}$$

Example2.3. Consider the standard topological space (R, τ_{st}) , then the set (1, 4) is $\omega - open$ but not $rc - open$.

Example2.4. Consider the co-countable topological space (R, τ_{cc}) , then (1, 4) is $\omega - open$ but not open.

Example2.5. Consider the standard topological space (R, τ_{st}) **,** then the set N^c where *N* is the set of natural numbers is rc- open but not r-open.

Theorem2.6. Let (X,τ) be a regular space and $A \subset X$. Then,

$$rc-open \Rightarrow \omega-open \text{ in } (X,\tau) \Leftrightarrow \omega-open \text{ in } (X,\tau_s)$$

Proof. This follows from Remark 2.2 and the fact that any regular space is semi- regular.

Theorem 2.7. For a space (X,τ) and $A \subset X$. The following are equivalent:

(1) A is $rc-open$.

(2) $A \subset r\operatorname{int}(B^c)$ for any countable subset B of X such that $A \subset B^c$.

Proof. (1) $\Rightarrow$ (2): Let A be $rc-open$ set and B be a countable subset of X such that $A \subset B^c$. Now A^c is $rc-closed$ and $B \subset A^c$ and B is countable, so $rcl(B) \subset A^c$.

Hence $A \subset (rcl(B))^c = r\operatorname{int}(B)^c$.

(2) $\Rightarrow$ (1): Let $A \subset r\operatorname{int}(B^c)$ for any countable subset B where $A \subset B^c$. Then $B \subset A^c$ and $A \subset r\operatorname{int}(B^c) = (rcl(B))^c$. So $rcl(B) \subset A^c$. Thus A^c is $rc-closed$ and hence A is $rc-open$.

Corollary 2.8. Let A be a subset of a space of X. Then the following are equivalent:

(1) A is $rc-open$.

(2) $A \subset r\operatorname{int}(C)$ for any $C \in \tau_{cc}$ of X such that $A \subset C$.

Theorem 2.9. Let (X,τ) be a topological space. Then τ_{rc} is a topology for X.

Proof. It is clear that $X \in \tau_{rc}$ and $\phi \in \tau_{rc}$. Now, let $U, V \in \tau_{rc}$. Then U^c and V^c are $rc-closed$. Let B be a countable subset such that $B \subset (U \cap V)^c = U^c \cup V^c$. Then there are two sets B_1 and B_2 such that $B = B_1 \cup B_2$ and $B_1 \subset U^c$ and $B_2 \subset V^c$. Since B_1 and B_2 are countable and U^c and V^c are $rc-closed$, hence $rcl(B_1) \subset U^c$ and $rcl(B_2) \subset V^c$. Then $rcl(B) = rcl(B_1 \cup B_2) = rcl(B_1) \cup rcl(B_2) \subset U^c \cup V^c$. Therefore $U^c \cup V^c = (U \cap V)^c$ is $rc-closed$. Thus $U \cap V \in \tau_{cr}$.

Let $\{U_\alpha : \alpha \in \nabla\}$ be a family of $rc-open$ subsets of X. Then $\{(U_\alpha)^c : \alpha \in \nabla\}$ is a family of $rc-closed$ sets of X. Let B be a countable subset such that $B \subset \bigcap_{\alpha \in \nabla}\{(U_\alpha)^c\}$. Hence $B \subset (U_\alpha)^c$ for each $\alpha \in \nabla$. But $(U_\alpha)^c$ is $rc-closed$ for all $\alpha \in \nabla$. So $rcl(B) \subset (U_\alpha)^c$ for all $\alpha \in \nabla$. Hence $rcl(B) \subset \bigcap_{\alpha \in \nabla}\{(U_\alpha)^c\}$. Thus $\bigcap_{\alpha \in \nabla}\{(U_\alpha)^c\}$ is an $rc-closed$ subset of X. Therefore $\bigcup_{\alpha \in \nabla}\{U_\alpha\}$ is an $rc-open$ subset of X.

Definition 2.10. Let X be a topological space. Then

(1) $rc-closure$ (resp. $\omega-closure$ [4]) of a subset A of X is the intersection of all $rc-closed$ (resp. $\omega-closed$) sets of X containing A and is denoted by $rcCl(A)$ (resp. $\omega Cl(A)$).

(2) $rc-$ interior (resp. $\omega-$ interior[4]) of A is the union of all $rc-open$ (resp. $\omega-open$) sets of X contained in A and is denoted by $rcInt(A)$ (resp. $\omega Int(A)$).

Remark 2.11. [4] If A is open set. Then $\omega Int(A) = int(A)$ but the converse is not true.

Theorem 2.12. For a topological space X. The following hold:

(1) If A is $r-open$, then (i) A is $rc-open$ and $rcInt(A) = r\operatorname{int}(A)$ (ii) $\omega Int(A) = int(A)$.

(2) If A is $rc-open$ A. Then A is $\omega-open$ and $rcInt(A) = \omega Int(A)$.

Proof. (1) (i) Since each $r-open$ is $rc-open$ and hence $rcInt(A) = r\operatorname{int}(A)$.

(ii) This follows from the fact that every $r-open$ is open, and Remark 2.11.

(2) Since each $rc-open$ is $\omega-open$ and hence $rcInt(A) = \omega Int(A)$.

III. *Rc*- Continuous Functions

In this section, we introduce a new class of functions is called $rc-continuous$ functions and investigate some of its properties and characterizations.

Definition 3.1. Let $f:X\to Y$ be a function, then *f* is called to be $rc-continuous$ if $f^{-1}(V)$ is $rc-open$ in X for every open subset of Y.

Theorem 3.2. A function $f:(X,\tau)\to(Y,\eta)$ is $rc-continuous$ if and only if $f:(X,\tau_{rc})\to(Y,\eta)$ continuous.

Theorem 3.3. A function $f:(X,\tau)\to(Y,\eta)$ is $rc-continuous$ if and only if $f^{-1}(V)$ is $rc-closed$ in X for every closed subset of Y.

Definition 3.4. [4] A function $f:X\to Y$ is $\omega-continuous$ if $f^{-1}(V)$ is $\omega-open$ in X for every open subset of Y.

Definition 3.5.[6] A function $f:X\to Y$ is $r-continuous$ if $f^{-1}(V)$ is $r-open$ in X for every open subset of Y.

Remark 3.6. For a function $f:X\to Y$ the following implications hold:

$$\begin{array}{ccc} rc-continuous & \Rightarrow & \omega-continuous \\ \Uparrow & & \Uparrow \\ r-continuous & \Rightarrow & continuous \end{array}$$

None of the above implications is reversible as shown in the following examples:

Example 3.7. Consider the standard topological space (R,τ_{st}). Let Y= {a, b, c} σ = {Y, ϕ , {a}, {b}, {a, b}}.

Define $f:(R,\tau_{st})\to(Y,\sigma)$ as follows:

$f(x)=$ a for $x\in(1,5)$ and $f(x)=$ c for $x\notin(1,5)$, then f is $\omega-continuous$ but not $rc-continuous$.

Example 3.9. For an example of a function which is $\omega-continuous$ but not continuous see [4].

Question: Does there exist a function $f:(X,\tau)\to(Y,\eta)$ which is $rc-continuous$ but not $r-continuous$.

Definition 3.10. A function $f:X\to Y$ is called *weakly* $rc-continuous$ (resp. *weakly* $\omega-continuous$ [4]) at $x\in X$ if for each open subset V in Y containing $f(x)$, there is an $rc-open$ (resp. $\omega-open$) subset U in X containing x such that $f(U)\subset cl(V)$. f is called *weakly* $rc-continuous$ (resp. *weakly* $\omega-continuous$) f is *weakly* $rc-continuous$ (resp. *weakly* $\omega-continuous$) at every $x\in X$.

Remark 3.11. The following implications hold for a function $f:X\to Y$:

$$\begin{array}{ccc} weakly\ rc-continuous & \Rightarrow & weakly\ \omega-continuous \\ \Uparrow & & \Uparrow \\ rc-continuous & \Rightarrow & \omega-continuous \end{array}$$

None of these implications is reversible as shown in the following examples:

Example 3.12. [4] Let $f:(R,\tau_{st})\to(Y,\sigma)$ where Y= {a, b, c, d} and σ = {Y, ϕ ,{a}, {c}, {a, b},{a, c},{a, b, c}, {a, c, d}}be defined by:

$f(x)=$ a for $x\in(-\infty,0]\cup[1,\infty)$ and $f(x)=$ b for $x\notin(-\infty,0)\cup[1,\infty)$, then f is *weakly* $\omega-continuous$ but not $\omega-continuous$.

Question 3.13. Does there exist a function $f:(X,\tau)\to(Y,\eta)$ which is *weakly* $rc-continuous$ and it is not $rc-continuous$.

Question 3.14. Does there exist a function $f:(X,\tau)\to(Y,\eta)$ which is *weakly* $\omega-continuous$ and it is not *weakly* $rc-continuous$.

Theorem 3.15. For a function $f:X\to Y$ the following are equivalent:

(1) f is *weakly* $rc-continuous$.

(2) $rcCl(f^{-1}(\text{int}(cl(V)))) \subset f^{-1}(cl(V))$ for any subset V of Y.

(3) $rcCl(f^{-1}(\text{int}(V))) \subset f^{-1}(V)$ for any regular closed set V of Y.

(4) $rcCl(f^{-1}(V)) \subset f^{-1}(cl(V))$ for any open set V of Y.

(5) $f^{-1}(V) \subset rcInt(f^{-1}(cl(V)))$ for any open set V of Y.

Proof. (1) $\Rightarrow$ (2): Let V be a subset of Y and $x \in (f^{-1}(cl(V))^{c}$. Then $f(x) \in (cl(V))^{c}$. Then there is an open set U containing $f(x)$ and $U \cap V = \phi$. Then $cl(U) \cap \text{int}(cl(V)) = \phi$. Since f is *weakly* $rc-continuous$, then there is a $rc-open$ set W containing x such that $f(W) \subset cl(U)$. So $W \cap f^{-1}\text{int}(cl(V)) = \phi$. Hence $x \in (rcCl(f^{-1}(\text{int}(cl(V)))))^{c}$ and $rcCl(f^{-1}(\text{int}(cl(V)))) \subset f^{-1}(cl(V))$.

(2) $\Rightarrow$ (3) Let V be regular closed set in Y. Hence, by (3) we have $rcCl(f^{-1}(\text{int}(V))) = rcCl(f^{-1}(\text{int}(cl(\text{int}(V))))) \subset f^{-1}(cl(\text{int}(V))) = f^{-1}(V)$.

(3) $\Rightarrow$ (4): Let V be an open subset of Y. Then $cl(V)$ is regular closed in Y, hence $rcCl(f^{-1}(V)) \subset rcCl(f^{-1}(\text{int}(cl(V))) \subset f^{-1}(cl(V))$

(4) $\Rightarrow$ (5): Let V be any open set of Y. Since $(cl(V))^{c}$ is open in Y, then $(rc\ \text{int}(f^{-1}(cl(V))))^{c} = rcCl(f^{-1}(cl(v))^{c}) \subset f^{-1}(cl(cl(V))^{c}) \subset (f^{-1}(V))^{c}$. Thus, $f^{-1}(V) \subset rc\ \text{int}(f^{-1}(cl(V)))$.

(5) $\Rightarrow$ (1): Let $x \in X$ and V be any open subset of Y containing $f(x)$. Then $x \in f^{-1}(V) \subset cr\ \text{int}(f^{-1}(cl(V)))$. Put $U = rc\ \text{int}(f^{-1}(cl(V)))$. Hence $f(U) \subset cl(V)$ and f is $weakly\ rc-continuous$ at x in X.

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