On Quasi Generalized Topological Simple Groups

*C. Selvi, R. Selvi

Research scholar, Department of mathematics, Sriparasakthi college for women, India.
Assistant Professor, Department of mathematics, Sriparasakthi college for women, India.
Corresponding Author: C. Selvi, R. Selvi

Abstract: In this paper we introduce the concept of quasi $G$-topological simple group. Also some basic properties, theorems and examples of a quasi $G$-topological simple groups are investigated. Moreover we studied the important result, If the mapping between two quasi $G$-topological simple groups is $G$-continous at the identity element, then $f$ is $G$-continous.

Keywords: Quasi topological group, $G$-open set, $G$-continous, Quasi $G$-topological simple group.

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I. Introduction

Csaszar[6], introduced the notion of generalized neighbourhood system and generalized topological space. Also Csaszar[6], investigated the generalized continuous mappings. In this paper we introduce the new concept of quasi $G$-topological simple group. Quasi $G$-topological simple group have both topological and algebraic structures such that the translation mappings and the inversion mapping are $G$-continous with respect to the generalized topology. Also some basic results are studied and discussed.

II. Preliminaries

Definition: 2.1[3] Let $X$ be any set and let $G \subseteq P(X)$ be a subfamily of power set of $X$. Then $G$ is called a generalized topology if $\emptyset \in G$ and for any index set $I$, $\bigcup_{i \in I} G_i \in G$, $\emptyset \in G$, $i \in I$.

Definition: 2.2[3] The elements of $G$ are called $G$-open sets. Similarly, generalized closed set (or) $G$-closed, is defined as complement of a $G$-open set.

Definition: 2.3[3] Let $X$ and $Y$ be two $G$-topological space. A mapping $f: X \rightarrow Y$ is called a $G$-continous on $X$ if for any $G$-open set $O$ in $Y$, $f^{-1}(O)$ is $G$-open in $X$.

Definition: 2.4[3] The bijective mapping $f$ is called a $G$-homeomorphism from $X$ to $Y$ if both $f$ and $f^{-1}$ are $G$-continous. If there is a $G$-homeomorphism between $X$ and $Y$, then they are said to be $G$-homeomorphic. It is denoted by $X \cong_g Y$.

Definition: 2.5[3] Collection of all $G$-interior points of $A \subseteq X$ is called $G$-interior of $A$. It denoted by $Int_G(A)$. By definition it obvious that $Int_G(A) \subseteq A$.

Note: 2.6[3] (i) $G$-interior of $A$, $Int_G(A)$ is equal to union of all $G$-open sets contained in $A$.

(ii) $G$-closure of $A$ as intersection of all $G$-closed sets containing $A$. It is denoted by $Cl_G(A)$.

Definition: 2.7[3] Let $(G, \ast)$ is a group and given $x \in G, L_x: G \rightarrow G$ defined by $L_x(y) = x \ast y$ and $R_x: G \rightarrow G$ defined by $R_x(y) = y \ast x$, denote left and right translation by $x$, respectively.

Definition: 2.8[1] A quasi topological group $G$, is a group which is also a topological space if the following conditions are satisfied.

(i) Left translation $L_x: G \rightarrow G, x \in G$ and right translation $R_x: G \rightarrow G, x \in G$ are continous and

(ii) The inverse mapping $i: G \rightarrow G$ defined by $i(x) = x^{-1}, x \in G$ is continous.

Definition: 2.9[20] A group $G$ is called a simple group if it has no nontrivial normal subgroup of $G$.

III. Quasi Generalized Topological Simple Groups

Definition: 3.1 A quasi $G$-topological simple group $G$, is a simple group which is also a $G$-topological space if the following conditions are satisfied.

(i) Left translation $L_x: G \rightarrow G, x \in G$ and Right translation $R_x: G \rightarrow G, x \in G$ are $G$-continous and

(ii) The inverse mapping $i: G \rightarrow G$ defined by $i(x) = x^{-1}, x \in G$ is $G$-continous.

Example: 3.2 Any group of prime order with indiscrete or discrete $G$-topology is a quasi $G$-topological simple group.
Example: 3.3 Let $G = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ be a trivial simple group under addition and we define a generalized topology on $G$ by $G = \left\{ \phi, \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) \right\}$. Clearly $(G, +, G)$ quasi $G$-topological simple group.

Example: 3.4 $G = \{1, w, w^2\}$, where $w^3 = 1$, is a simple group under multiplication. Now we define a generalized on $G$ by $G = \{G, \{w\}\}$. Then the inverse mapping $i$ is $G$-continuous at the points $1, w^2$ and not $G$-continuous at the point $w$. In right translation mapping, $R_1$ is $G$-continuous at each point of $G, R_w$ is $G$-continuous at the points $w, w^2$ and not $G$-continuous at the point $1$ and $R_{w^2}$ is $G$-continuous at the point $1$, $w$ and not $G$-continuous at the point $w^2$. Similarly we can prove left translation($L_w$).

Theorem: 3.5 Let $(G, *, G)$ be a quasi $G$-topological simple group and $\beta_e$ be the collection of all $G$-open neighbourhood at identity $e$ of $G$. Then

(i) For every $U \in \beta_e$, there is an element $V \in \beta_e$ such that $V^{-1} \subseteq U$.

Proof: (i). Since $(G, *, G)$ is a quasi $G$-topological simple group, $G$ is $G$-continuous.

(ii) For every $U \in \beta_e$, there is an element $V \in \beta_e$ such that $V \times x \subseteq U$ and $x \times V \subseteq U$, for each $x \in U$.

Proof: (ii). Since $(G, *, G)$ is a quasi $G$-topological simple group. Therefore, for every $U \in \beta_e$, there exists $V \in \beta_e$ such that $i(V) = V^{-1} \subseteq U$, because the inverse mapping $i: G \rightarrow G$ is $G$-continuous.

Theorem: 3.6 Let $G$ be a quasi $G$-topological simple group and $g$ be any element of $G$. Then the right translation($R_g$) and left translation($L_g$) of $G$ by $g$ is a $G$-homeomorphism of the space $G$ onto itself.

Proof: First we prove that $R_g$ is a bijection. Assume that $y \in G$, then the element $yg^{-1}$ maps to $y$. Therefore $R_g$ is surjective.

Assume that $R_g(x) = R_g(y)$.

$\Rightarrow xyg = yg$.

$x = y$. Hence $R_g$ is 1-1. Since $G$ is a quasi $G$-topological simple group, $G$ is $G$-continuous.

Consider $R_g^{-1}$ which maps $xyg$ to $x$, this is equivalent to the map from $x$ to $xyg^{-1}$. Therefore $R_g^{-1}(x) = R_g^{-1}(y)$. Since $R_g^{-1}(x)$ is $G$-continuous, $R_g^{-1}(y)$ is $G$-continuous. Similarly we will prove that the left translation ($L_g$). Hence the theorem.

Theorem: 3.7 Let $G$ be a quasi $G$-topological simple group and $U$ be any $G$-open set in $G$. Then

(i) $a * U$ and $U * a$ is $G$-open in $G$ for all $a \in G$.

(ii) For any subset $A$ of $G$, the sets $U * A$ and $A * U$ are $G$-open in $G$.

Proof: Let $x \in U * a$. We want to show that $x$ is a $G$-interior point of $U * a$. Let $x = u * a$ for some $u \in U = U * a * a^{-1}$. Then $u = x * a^{-1}$. We know that $R_{a^{-1}}: G \rightarrow G$ is $G$-continuous. Then for every $G$-open set containing $R_{a^{-1}}(x) = x * a^{-1} = u$, there exists a $G$-open set $M_x$ containing $x$ such that $R_{a^{-1}}(M_x) \subseteq U$.

$\Rightarrow M_x \subseteq U * a$.

$\Rightarrow x$ is a $G$-interior point of $U * a$. Therefore $U * a$ is $G$-open in $G$.

Similarly we can prove that $a * U$ is $G$-open.

(iii) By above result, $U * a$ is $G$-open, for all $a \in G$. Then $U * A = \bigcup_{a \in A} U * a$ also $G$-open in $G$.

Theorem: 3.8 Suppose that a subgroup $H$ of a quasi $G$-topological simple group $G$ contains a non-empty $G$-open subset of $G$. Then $H$ is $G$-open in $G$.

Proof: Let $U$ be a non-empty $G$-open subset of $G$ with $U \subseteq H$. For every $g \in H$, the set $L_g(U) = U * g$ is $G$-open in $G$, then $H = \bigcup_{g \in H} U * g$ is $G$-open in $G$.

Theorem: 3.9 Every quasi $G$-topological simple group $G$ has $G$-open neighbourhood at the identity element of $G$ consisting of symmetric $G$-neighbourhoods.

Proof: For an arbitrary $G$-open neighbourhood $U$ of the identity $e$, if $V = U \cup U^{-1}$, then $V = U^{-1}$, the set $V$ is an $G$-open neighbourhood of $e$. But $V$ is a symmetric $G$-neighbourhood and $V \subseteq U$.

Theorem: 3.10 Let $f: G \rightarrow H$ be a homomorphism of quasi $G$-topological simple groups. If $f$ is $G$-continuous at the neutral element $e_G$ of $G$, then $f$ is $G$-continuous.

Proof: Let $x \in G$ be arbitrary and suppose that $W$ is an $G$-open neighbourhood of $y = f(x)$ in $H$. Since the left translation $L_x$ in $H$ is a $G$-continuous mapping, there exists an $G$-open neighbourhood of $V$ of the neutral element $e_H$ in $H$ such that $L_x(V) = xV \subseteq W$. Since $f$ is $G$-continuous at $e_G$ of $G$, then $f(V) \subseteq V$, for some $G$-open neighbourhood $U$ of $e_G$ in $G$. Since $L_x: G \rightarrow H$ is $G$-continuous, then $xU$ is an $G$-open neighbourhood of $x$ in $G$. Now we have $f(xU) = f(x)f(U) = yf(U) \subseteq yV \subseteq W$. Hence $f$ is $G$-continuous at the point $x \in G$. DOI: 10.9790/5728-1304035760 www.iosrjournals.org 58 | Page
Theorem: 3.11 Suppose that \( G, H \) and \( K \) are quasi \( G \)-topological simple groups and that \( \phi: G \to H \) and \( \psi: G \to K \) are homomorphism. Such that \( \psi(G) = K \) and \( \text{Ker } \psi \subset \text{Ker } \phi \). Then there exists homomorphism \( f: K \to H \) such that \( \phi = f \circ \psi \). In addition, for each \( G \)-neighbourhood \( U \) of the identity element \( e_H \) in \( H \), there exists a \( G \)-neighbourhood \( V \) of the identity element \( e_K \) in \( K \) such that \( \psi^{-1}(V) \subset \phi^{-1}(U) \), then \( f \) is \( G \)-continuous.

Proof: Algebraic part of the theorem is well known. Suppose \( U \) is a \( G \)-neighbourhood of \( e_H \) in \( H \). By assumption, there exists a \( G \)-neighbourhood \( V \) of the identity element \( e_K \) in \( K \) such that \( W = \psi^{-1}(V) \subset \phi^{-1}(U) \).

\[ \Rightarrow f(W) = \phi(\psi^{-1}(V)) \subset \phi(\phi^{-1}(U)) \]
\[ \Rightarrow f(W) = f(V) \subset U. \]

Hence \( f \) is \( G \)-continuous at the identity element of \( K \). Therefore by above theorem, \( f \) is \( G \)-continuous.

Corollary: 3.12 Let \( \phi: G \to H \) and \( \psi: G \to K \) be \( G \)-continous homomorphism of a quasi \( G \)-topological simple groups \( G, H \) and \( K \) such that \( \psi(G) = K \) and \( \text{Ker } \psi \subset \text{Ker } \phi \). If the homomorphism \( \psi \) is \( G \)-open, then there exists a \( G \)-continous homomorhism, \( f: K \to H \) such that \( \phi = f \circ \psi \).

Proof: The existence of a homomorphism \( f: K \to H \) such that \( \phi = f \circ \psi \). Take an arbitrary \( G \)-open set \( V \) in \( H \). Then \( f^{-1}(V) = \phi(\psi^{-1}(V)) \). Since \( \phi \) is \( G \)-continous and \( \psi \) is an \( G \)-open map, \( f^{-1}(V) \) is \( G \)-open in \( K \). Therefore \( f \) is \( G \)-continous.

Theorem: 3.13 Let \( G \) be a quasi \( G \)-topological simple group and \( H \) is a normal subgroup of \( G \). Then \( H \) also a normal subgroup of \( G \).

Proof: Now we have to prove that \( gHg^{-1} \in H \forall g \in G \).

Since \( H \) is a normal subgroup of \( G \), \( gHg^{-1} \in H \forall g \in G \).

Now \( gHg^{-1} \in H \forall g \in G \).

\[ \Rightarrow gHg^{-1} \in H, g \in G. \]
\[ \Rightarrow gHg^{-1} \in H, g \in G. \]

Therefore \( H \) is a normal subgroup of \( G \).

Corollary: 3.14 Let \( G \) be a quasi \( G \)-topological simple group and \( Z(G) \) be the centre of \( G \). Then \( Z(G) \) is a normal subgroup of \( G \).

Proof: proof follows from the above theorem.

Corollary: 3.15 Let \( G \) and \( H \) be a quasi \( G \)-topological simple groups. If \( \phi: G \to H \) is a homomorphism mapping \( . \)then \( \text{Ker } \phi \) is a normal subgroup of \( G \).

Theorem: 3.16 Let \( G \) and \( H \) be quasi \( G \)-topological simple groups with neutral elements \( e_G \) and \( e_H \), respectively, and let \( p \) be a \( G \)-continous homomorphism of \( G \) onto \( H \) such that, for some non-empty subset \( U \) of \( G \), the set \( p(U) \) is \( G \)-open in \( H \) and the restriction of \( p \) to \( U \) is an \( G \)-open mapping of \( U \) onto \( p(U) \). Then the homomorphism \( p \) is \( G \)-open.

Proof: It suffices to show that \( x \in G \), where \( W \) is an \( G \)-open neighbourhood of \( x \) in \( G \), then \( p(W) \) is a \( G \)-open neighbourhood of \( p(x) \) in \( H \). Fix a point \( y \) in \( U \), and let \( L \) be the left translation of \( G \) by \( xy^{-1} \). Then \( L \) is a \( G \)-homeomorphism of \( G \) onto itself such that \( L_{xy^{-1}}(x) = yx^{-1} \) and \( = y \).

So \( V = U \cap L(W) \) is an \( G \)-open neighbourhood of \( y \) in \( U \). Then \( p(V) \) is \( G \)-open subset of \( H \).

Consider the left translation \( h \) of \( H \) by the inverse to \( p(xy^{-1}) \).

Now clearly, \( (h \circ p \circ l) = h(p(l(x))) = h(p(y)) = p(xy^{-1})p(y) = p(xy^{-1}y) = p(x) \).

Hence \( h \left( p(l(W)) \right) = p(W) \). Clearly \( h \) is a \( G \)-homeomorphism of \( H \) onto itself. Since \( p(V) \) is \( G \)-open in \( H \), \( h(p(V)) \) is also \( G \)-open in \( H \). Therefore \( p(W) \) contains the \( G \)-open neighbourhood \( h(p(V)) \) of \( p(x) \) in \( H \). Hence \( p(W) \) is a \( G \)-open neighbourhood of \( p(x) \) in \( H \).

Definition: 3.17 Let \( H \) be a subgroup of quasi \( G \)-topological simple group \( G \). Then \( H \) is called neutral in \( G \) if every \( G \)-neighbourhood \( U \) of the identity \( e_G \) in \( G \), there exists a \( G \)-neighbourhood \( V \) of \( e_G \) such that \( VH \subset HU \).

Theorem: 3.18 Let \( H \) be a subgroup of quasi \( G \)-topological simple group \( G \). Suppose that, for every \( G \)-open neighbourhood \( U \) of the identity \( e_G \) in \( G \), there exists an \( G \)-open neighbourhood \( V \) of \( e_G \) in \( G \) such that \( xVx^{-1} \subset U \) whenever \( x \in G \). Then \( H \) is neutral in \( G \).

Proof: Given a \( G \)-neighbourhood \( U \) of \( e_G \) in \( G \). Take an \( G \)-open neighbourhood \( V \) of \( e_G \) satisfying, \( xVx^{-1} \subset U, \forall x \in G \).

\[ \Rightarrow xV \subset Ux, \forall x \in G \]
\[ \Rightarrow HV \subset UH, \forall x \in G. \]

Then \( H \) is neutral in \( G \).
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References

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