Common Fixed Point Theorem under Sub Compatibility and Sub Sequentially Continuous Mappings in Menger Spaces by Using Implicit Relation

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Abstract: In this paper we establish Common fixed point theorem for six mappings in Menger spaces by using implicit relation under the notion of sub compatibility and sub sequentially continuity.

Keywords: Menger space, Compatibility, Sub compatibility, Reciprocal continuity, Sub sequentially continuity, Common fixed point.

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I. Introduction

In 1942, K. Menger [16] introduced the notion of probabilistic metric space (briefly, PM-space) as a generalization of metric space. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. The development of fixed point theory in PM-spaces was due to Schweizer and Sklar[23, 24].

In 1972, V. M. Sehgal and A. T. Bharucha-Reid [25] initiated the study of contraction mappings on probabilistic metric (briefly, PM) spaces. Since then there has been a massive growth of fixed point theorems using certain conditions on the mappings or on the space itself. Sesa [26] introduced weakly commuting maps in metric space. In 1986, Jungck [13] introduced the notion of compatible mappings in metric spaces. And this condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [14].

Pant [21] noticed these criteria for fixed points of contraction mappings and introduced a new continuity condition, known as reciprocal continuity and obtain a common fixed point theorem by using the compatibility in metric spaces. He also showed that in the setting of common fixed point theorems for compatible mappings satisfying contraction conditions, the notion of reciprocal continuity is weaker than the continuity of one of the mappings. Later on, Jungck and Rhoades [14] termed a pair of self maps to be coincidently commuting or equivalently weakly compatible if they commute at their coincidence points.

Recently, Bouhadjera and Godet-Thobie [4] introduced two new notions namely sub sequential continuity and sub compatibility which are weaker than reciprocal continuity and compatibility respectively (see also [3, 5]). Further, Imdad et al. [12] improved the results of Bouhadjera and Godet-Thobie [4] and showed that these results can easily recovered by replacing sub compatibility with compatibility or sub sequential continuity with reciprocally continuity. Several interesting and elegant results have been obtained by various authors in different settings [e.g. 5, 7, 8, 11, 12, 15, 19, 27]. Many authors [1, 2, 10] proved several fixed point theorems in Menger spaces and showed the applications of corresponding results in metric spaces. Most recently Pant and Chauhan[9] established a common fixed point theorem in Menger space using the notion of compatibility and sub sequentially continuous mapping of a pair of self maps. In this paper we generalize and extend the result of Pant and Chauhan[9] for six mappings in Menger space using the concept of sub compatibility and sub sequentially continuity by using implicit relation.
II. Preliminaries

Definition 2.1. [23] A probabilistic metric space (PM-space) is an ordered pair \((X, F)\) consisting of a non empty set \(X\) and a function \(F: X \times X \to L\), where \(L\) is the collection of all distribution functions and the value of \(F\) at \((u, v) \in X \times X\) is represented by \(F_{u,v}\). The function \(F_{u,v}\) is assumed to satisfy the following conditions:

\begin{align*}
\text{(PM-1)} & \quad F_{u,v}(x) = 1, \text{for all } x > 0 \text{ if and only if } u = v \\
\text{(PM-2)} & \quad F_{u,v}(0) = 0, \\
\text{(PM-3)} & \quad F_{u,v} = F_{v,u} \; , \\
\text{(PM-4)} & \quad F_{u,v}(x) = 1 \text{ and } F_{v,w}(y) = 1 \text{ then } \\
& \quad F_{u,w}(x + y) = 1 \text{ for all } u, v, w \in X \text{ and } x, y \geq 0.
\end{align*}

Definition 2.2. [23] A mapping \(\ast : [0, 1] \times [0, 1] \to [0, 1]\) is called a \(t\)-norm if

\begin{align*}
(a) & \quad \ast(a, 1) = a \quad \text{for all } a \in [0, 1] \\
(b) & \quad \ast(a, b) = \ast(b, a) \quad \text{(symmetric property)} \\
(c) & \quad \ast(c, d) \geq \ast(a, b) \quad \text{for } c \geq a, \; d \geq b \\
(d) & \quad \ast(\ast(a, b), c) = \ast(a, \ast(b, c)).
\end{align*}

Definition 2.3. [23] A Menger space is a triplet \((X, F, \ast)\) where \((X, F)\) is a PM-space and \(\ast\) is a \(T\)-norm such that the inequality

\[ F_{u,v}(x + y) \geq \{ F_{u,v}(x), F_{v,w}(y) \} \quad \text{for all } u, v, w \in X \text{ and } x, y > 0 \]

Definition 2.4. [18] The self maps \(A\) and \(B\) of a Menger space \((X, F, \ast)\) are said to be compatible if

\[ F_{ABx_n,Bx_n}(t) \to 1 \quad \text{for all } t > 0 \text{ whenever } \{x_n\} \text{ is a sequence in } X \text{ such that } A x_n, B x_n \to z \text{ for some } z \in X \text{ as } n \to \infty. \]

Definition 2.5. [28] Self-maps \(A\) and \(S\) of a Menger space \((X, F, \ast)\) are said to be weak compatible (or coincidentally commuting) if they commute at their coincidence points i.e. if \(Ap = Sp\) for some \(p \in N\) then \(ASP = SAp\)

Remark: Two compatible self mappings are weakly compatible, however the converse is not true in general.

Definition 2.6. [4]: A pair of self mappings \((A, S)\) defined on a Menger space \((X, F, \ast)\) is said to be sub compatible if there exists a sequence \(\{x_n\}\) such that

\[ \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = z, \]

For some \(z \in X\) and

\[ F_{ASx_n,SAx_n}(t) = 1, \text{ for all } t > 0. \]

Definition 2.7. [4]: A pair of self mappings \((A, S)\) defined on a Menger space \((X, F, \ast)\) is called subsequentially continuous if there exists a sequence \(\{x_n\}\) such that

\[ \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n = z \]

For some \(z \in X\) and

\[ \lim_{n \to \infty} AS x_n = Az \text{ and } \lim_{n \to \infty} SA x_n = Sz, \]

Implicit Relations [2, 8]: In [17], Mihet established a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. This implicit relation is similar to that in [22]. In [22] Popa used the family \(F_4\) of implicit real functions to find the fixed points of two pairs of semi compatible mapping in a \(d\)- compatible topological space. Here \(F_4\) denote the family of all real continuous functions \(F: (R^+)^4 \to R\) satisfying the following properties:
(F₁) There exists \( h \geq 1 \) such that every \( u \geq 0, v \geq 0 \) with \( F(u, v, u, v) \geq 0 \) or \( F(u, v, y, x) \geq 0 \), we have \( u \geq h v \).

(F₂) \( F(u, u, 0, 0, 0) < 0 \) for all \( u > 0 \).

In our result, we deal with the class \( \Phi \) of all real continuous functions \( \varphi : (R^+)^4 \to R \), non-decreasing in the first argument and satisfying the following conditions:

I. For \( u, v \geq 0, \varphi(u, v, u, v) \geq 0 \) or \( \varphi(u, v, y, x) \geq 0 \) implies that \( u \geq v \).

II. \( \varphi(u, u, 1, 1) \geq 0 \) for all \( u \geq 1 \).

Example: Define \( \varphi(t_1, t_2, t_3, t_4) = at_1 + bt_2 + ct_3 + dt_4, a, b, c, d \in R \) with \( a + b + c + d = 0, a > 0, a + c > 0, a + b > 0 \) and \( a + d > 0 \). Then \( \varphi \in \Phi \).

Example: Define \( \varphi(t_1, t_2, t_3, t_4) = 20t_1 - 18t_2 + 6t_3 - 8t_4 \). Then \( \varphi \in \Phi \).

The following theorem proved by Chauhan Sunny and Pant B.D.[9]

**Theorem:** Let \( A, B, S \) and \( T \) be self maps of a Menger space \( (X, F, \Delta) \), where \( \Delta \) is a continuous \( t \)-norm. If the pairs \((A, S)\) and \((B, T)\) are compatible and sub sequentially continuous, then

1. The pair \((A, S)\) has a coincident point,
2. The pair \((A, S)\) has a coincident point,
3. There exists a constant \( k \in (0, 1) \) such that
   \[
   F_{AX, BY}(kt) \geq \min\{ F_{Ax, sx}(t), F_{Ax, sy}(t), F_{By, tx}(t), F_{By, ty}(t), F_{Ax, ty}(t), F_{By, sx}(t) \}
   \]

For all \( x, y, t \in X \) and \( t > 0 \), then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

In this section we extend the theorem for six mapping by using implicit relation.

### III. Main Result

**Theorem (3.1):** Let \( A, B, P, Q, S \) and \( T \) be self maps of a Menger space \( (X, F, \ast) \), where \( \ast \) is a continuous \( t \)-norm defined by \( t \ast t \geq t \) for all \( t \in [0, 1] \). If the pairs \((P, AB)\) and \((Q, ST)\) are sub compatible and sub sequentially continuous and satisfying
Common fixed point theorem under sub compatibility and sub sequentially continuous mappings in

3.1.1 The pairs \((P, T), (AB, T), (Q, B), (ST, B)\) are commutes.

3.1.2 For some \(\varphi \in \Phi\), there exists \(k \in (0, 1)\) such that for all \(x, y \in X\) and \(t > 0\)

\[ \varphi(F_{P,xy}(kt), F_{AB,STy}(t), F_{P,ABx}(t), F_{Q,ySt}(kt)) \geq 0 \]

Then the pairs \((P, AB)\) and \((Q, ST)\) have a coincident point each. Moreover, \(A, B, P, Q, S, T\)
and have a unique common fixed point in \(X\).

Proof: Since the pair \((P, AB)\) is sub compatible and sub sequentially continuous then there exists a sequence \(\{x_n\}\) in \(X\) such that

\[ \lim_{n \to \infty} Px_n = \lim_{n \to \infty} ABx_n = z, z \in X \text{ and satisfy} \]

\[ \lim_{n \to \infty} F_{P,ABx_n} (ABx_n) (t) = \lim_{n \to \infty} F_{P,ABx_n} (t) = 1 \]

For all \(t > 0\) then \(Pz = ABz\), whereas in respect of the pair \((Q, ST)\) is sub compatible and sub sequentially continuous then there exists a sequence \(\{y_n\}\) in \(X\) such that

\[ \lim_{n \to \infty} Qy_n = \lim_{n \to \infty} STy_n = w, w \in X \text{ and satisfy} \]

\[ \lim_{n \to \infty} F_{Q,STy_n} (STy_n) (t) = \lim_{n \to \infty} F_{Q,STy_n} (w) (t) = 1 \]

For all \(t > 0\) then \(Qw = STw\). Hence \(z\) is a coincident point of the pair \((P, AB)\) and \(w\) is a coincident point of the pair \((Q, ST)\).

Now we prove that \(z = w\). By putting \(x = x_n\) and \(y = y_n\) in 3.1.2 we get

\[ \varphi(F_{P,xy}(yt), F_{AB,STy}(t), F_{P,ABx}(t), F_{Q,ySt}(kt)) \geq 0 \]

Taking the limit as \(n \to \infty\), we get

\[ \varphi(F_{z,w}(t), F_{z,w}(t), F_{z,w}(t), F_{w,w}(kt)) \geq 0 \]

\[ \varphi(F_{z,w}(t), F_{z,w}(t), 1, 1) \geq 0 \]

As \(\varphi\) is non-decreasing in the first argument, we have

\[ \varphi(F_{z,w}(t), F_{z,w}(t), 1, 1) \geq 0 \]

In view of implicit relation 2.8 we get

\[ F_{z,w}(t) \geq 1 \text{ for all } t > 0. \text{ This gives } F_{z,w}(t) = 1 \]

we get \(z = w\)

Now we prove that \(Pz = z\) then we putting \(x = z\) and \(y = y_n\) in 3.1.2 we get

\[ \varphi(F_{Pz,xy}(kt), F_{ABz,STy}(t), F_{Pz,ABz}(t), F_{Q,ySt}(kt)) \geq 0 \]

Taking the limit as \(n \to \infty\), and using \(Pz = ABz\) we get

\[ \varphi(F_{Pz,w}(kt), F_{Pz,w}(t), F_{Pz,Pz}(t), F_{w,w}(kt)) \geq 0 \]

\[ \varphi(F_{Pz,w}(kt), F_{Pz,w}(t), 1, 1) \geq 0 \]

As \(\varphi\) is non-decreasing in the first argument, we have

\[ \varphi(F_{Pz,w}(t), F_{Pz,w}(t), 1, 1) \geq 0 \]

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In view of implicit relation 2.8 we get
\[ F_{P_z,w}(t) \geq 1 \text{ for all } t > 0. \] This gives \[ F_{P_z,w}(t) = 1 \]
we get \( Pz = w = z \) Therefore \( Pz = ABz = z \). Now we assert that \( Qz = z \), then by putting \( x = x_n \) and \( y = z \) in 3.1.2 we get
\[ \varphi(F_{x_n,Qz}(kt), F_{ABx_n,STz}(t), F_{x_n,ABz}(t), F_{Qz,STz}(kt)) \geq 0 \]
Taking the limit as \( n \to \infty \), and using \( Qz = STz \) we get
\[ \varphi(F_{z,Qz}(kt), F_{z,Qz}(t), F_{z,z}(t), F_{Qz,Qz}(kt)) \geq 0 \]
\[ \varphi(F_{z,Qz}(kt), F_{z,Qz}(t), 1, 1) \geq 0 \]
As \( \varphi \) is non-decreasing in the first argument, we have
\[ \varphi(F_{z,Qz}(t), F_{z,Qz}(t), 1, 1) \geq 0 \]
In view of implicit relation 2.8 we get
\[ F_{z,Qz}(t) \geq 1 \text{ for all } t > 0. \] This gives \( F_{z,Qz}(t) = 1 \) thus \( z = Qz \)
we get, \( Qz = STz = z \) in all \( Pz = ABz = Qz = STz = z \)
Now we claim that \( Tz = z \), by putting \( x = Tz \) and \( y = z \) in 3.1.4 we get
\[ \varphi(F_{z,STz}(kt), F_{z,Tz}(t), F_{z,Tz}(t), F_{Qz,STz}(kt)) \geq 0 \]
Using 3.1.1 the pairs \((P, T)\) and \((AB, T)\) are commutes, i.e. \( PT = TP \) and \((AB)T = T(AB)\) we have
\[ \varphi(F_{TPz,Qz}(kt), F_{Tz,STz}(t), F_{TPz,Tz}(t), F_{Qz,STz}(kt)) \geq 0 \]
\[ \varphi(F_{z,Tz}(kt), F_{z,Tz}(t), F_{z,Tz}(t), F_{z,z}(kt)) \geq 0 \]
\[ \varphi(F_{z,Tz}(kt), F_{z,Tz}(t), 1, 1) \geq 0 \]
As \( \varphi \) is non-decreasing in the first argument, we have
\[ \varphi(F_{z,Tz}(t), F_{z,Tz}(t), 1, 1) \geq 0 \]
In view of implicit relation 2.8 we get
\[ F_{Tz,z}(t) \geq 1 \text{ for all } t > 0. \] This gives \( F_{Tz,z}(t) = 1 \) thus \( Tz = z \) and \( STz = z \) implies \( Sz = z \)
Now we claim that \( Bz = z \), by putting \( x = z \) and \( y = Bz \) in 3.1.2 we get
\[ \varphi(F_{z,Bz}(kt), F_{z,STz}(t), F_{z,Bz}(t), F_{Qz,STz}(kt)) \geq 0 \]
Using 3.1.1 the pairs \((Q, B)\) and \((ST, B)\) are commutes, i.e. \( QB = BQ \) and \((ST)B = B(ST)\) we have
\[ \varphi(F_{z,Bz}(kt), F_{z,STz}(t), F_{z,Bz}(t), F_{Qz,STz}(kt)) \geq 0 \]
\[ \varphi(F_{z,Bz}(kt), F_{z,Bz}(t), F_{z,z}(t), F_{z,Bz}(kt)) \geq 0 \]
\[ \varphi(F_{z,Bz}(kt), F_{z,Bz}(t), 1, 1) \geq 0 \]
As \( \varphi \) is non-decreasing in the first argument, we have
\[ \varphi(F_{z,Bz}(t), F_{z,Bz}(t), 1, 1) \geq 0 \]
In view of implicit relation 2.8 we get
\[ F_{z,z'}(t) \geq 1 \text{ for all } t > 0. \] This gives \( F_{z,z'}(t) = 1 \) Thus \( Bz = z \) and \( ABz = z \) implies \( A^2z = z \).

Thus in all \( z = Pz = Qz = Az = Bz = Sz = Tz \), i.e., \( z \) is a common fixed point of \( P, Q, A, B, S \) and \( T \).

**Uniqueness of \( z \):** Let \( z' (z \neq z') \) be another common fixed point of \( P, Q, A, B, S \) and \( T \); then \( z' = Pz' = Qz' = Az' = Bz' = Sz' = Tz' \).

Putting \( x = z \) and \( y = z' \) in \( 3.1.2 \) we get
\[
\varphi(F_{Pz,Qz}(kt), F_{AZ,STz'}(t), F_{Pz,ABz}(t), F_{Qz,STz'}(kt)) \geq 0
\]
\[
\varphi(F_{z,z'}(kt), F_{z,z}(t), F_{z,z'}(t), F_{z,z'}(kt)) \geq 0
\]
\[
\varphi(F_{z,z}(kt), F_{z,z}(t), 1, 1) \geq 0
\]

As \( \varphi \) is non-decreasing in the first argument, we have
\[
\varphi(F_{z,z}(t), F_{z,z}(t), 1, 1) \geq 0
\]

In view of implicit relation 2.8 we get
\[ F_{z,z}(t) \geq 1 \text{ for all } t > 0. \] This gives \( F_{z,z}(t) = 1 \) we get \( z = z' \).

Thus \( z \) is a unique common fixed point of \( P, Q, A, B, S \) and \( T \). This completes the proof of the theorem.

**Corollary (3.2):** Let \( A, P, Q \) and \( S \) be self maps of a Menger space \( (X, F, *) \), where \( * \) is a continuous \( t \)-norm defined by \( t * t \geq t \) for all \( t \in [0, 1] \). If the pairs \( (P, A) \) and \( (Q, S) \) are sub compatible and sub sequentially continuous then

1. \( P \) and \( A \) have a coincidence point,
2. \( Q \) and \( S \) have a coincidence point,
3. For some \( \varphi \in \Phi \), there exists \( k \in (0, 1) \) such that for all \( x, y \in X \) and \( t > 0 \)
\[
\varphi(F_{Pxy}(kt), F_{Ax,yx}(t), F_{Pz,Ax}(t), F_{Qyx}(kt)) \geq 0
\]

Then \( A, P, Q \) and \( S \) have a unique common fixed point in \( X \).

**Corollary (3.3):** Let \( A, P \) and \( S \) be self maps of a Menger space \( (X, F, *) \), where \( * \) is a continuous \( t \)-norm defined by \( t * t \geq t \) for all \( t \in [0, 1] \). If the pairs \( (P, A) \) and \( (P, S) \) are sub compatible and sub sequentially continuous then

1. \( P \) and \( A \) have a coincidence point,
2. \( P \) and \( S \) have a coincidence point,
3. For some \( \varphi \in \Phi \), there exists \( k \in (0, 1) \) such that for all \( x, y \in X \) and \( t > 0 \)
\[
\varphi(F_{Pxy}(kt), F_{Ax,yx}(t), F_{Pz,As}(t), F_{Py,yx}(kt)) \geq 0
\]

Then \( A, P \) and \( S \) have a unique common fixed point in \( X \).
Corollary (3.4): Let \( P \) and \( S \) be self maps of a Menger space \((X, F, \cdot\), where \( \cdot\) is a continuous \( \tau \)-norm defined by \( \tau \cdot \tau \geq \tau \) for all \( \tau \in [0, 1] \). If the pairs \((P, S)\) is sub compatible and sub sequentially continuous then

1. \( P \) and \( S \) have a coincidence point,
2. For some \( \varphi \in \Phi \), there exists \( k \in (0, 1) \) such that for all \( x, y \in X \) and \( \tau > 0 \)

\[
\varphi(F_{P\tau, P\tau}(kt), F_{S\tau, S\tau}(t), F_{P\tau, P\tau}(t), F_{P\tau, S\tau}(kt)) \geq 0
\]

Then \( P \) and \( S \) have a unique common fixed point in \( X \). This completes the proof of the theorem.

Example 3.5. Let \( X = [0, \infty) \) and \( d \) be the usual metric on \( X \) and for each \( \tau \in [0, 1] \) define

\[
F_{x,y}(t) = \begin{cases} 
\frac{\tau}{t+|x-y|} & \text{if } t > 0 \\
0 & \text{if } t = 0
\end{cases}
\]

For all \( x, y \in X \). Clearly \((X, F, \cdot)\) is a Menger space. Now we define a self maps \( P \) and \( S \) on \( X \) by

\[
P(x) = \begin{cases} 
\frac{x}{5} & \text{if } 0 \leq x \leq 3 \\
\frac{4x-9}{3} & \text{if } 3 < x < \infty
\end{cases}
\]

\[
S(x) = \begin{cases} 
\frac{x}{6} & \text{if } 0 \leq x \leq 3 \\
3x-6 & \text{if } 3 < x < \infty
\end{cases}
\]

Consider a sequence \( x_n = \frac{1}{n} \) in \( X \).

We have \( \lim_{n \to \infty} P(x_n) = \lim_{n \to \infty} \frac{1}{5n} = 0 = \lim_{n \to \infty} \frac{1}{6n} = \lim_{n \to \infty} S(x_n) \)

Next \( \lim_{n \to \infty} PS(x_n) = \lim_{n \to \infty} P\left(\frac{1}{6n}\right) = \lim_{n \to \infty} \left(\frac{1}{30n}\right) = 0 = P(0) \)

\[
\lim_{n \to \infty} SP(x_n) = \lim_{n \to \infty} \left(\frac{1}{5n}\right) = \lim_{n \to \infty} \left(\frac{1}{30n}\right) = 0 = S(0)
\]

And \( \lim_{n \to \infty} F_{PSx_n, SPx_n}(t) = 1 \), for all \( t > 0 \).

Consider another sequence \( x_n = 3 + \frac{1}{n} \). Then

\[
\lim_{n \to \infty} P(x_n) = \lim_{n \to \infty} P\left(3 + \frac{1}{n}\right) = \lim_{n \to \infty} \left\{4 \left(3 + \frac{1}{n}\right) - 9\right\} = 3
\]

\[
\lim_{n \to \infty} S(x_n) = \lim_{n \to \infty} S\left(2 + \frac{1}{n}\right) = \lim_{n \to \infty} \left\{3 \left(3 + \frac{1}{n}\right) - 6\right\} = 3
\]

Also

\[
\lim_{n \to \infty} PS(x_n) = \lim_{n \to \infty} P\left(3 + \frac{1}{n}\right) = \lim_{n \to \infty} \left\{4 \left(3 + \frac{3}{n}\right) - 9\right\} = 3 \neq P(3)
\]

\[
\lim_{n \to \infty} SP(x_n) = \lim_{n \to \infty} S\left(2 + \frac{1}{n}\right) = \lim_{n \to \infty} \left\{3 \left(3 + \frac{4}{n}\right) - 6\right\} = 3 \neq S(3)
\]

But \( \lim_{n \to \infty} F_{PSx_n, SPx_n}(t) = 1 \). Thus the pair \((P, S)\) is compatible as well as subsequential continuous, but not reciprocally continuous. Therefore all the conditions of Corollary 3.4 are satisfied for some \( k \in (0, 1) \). Here, \( 0 \) is a coincidence as well as unique common fixed point of the pair \((P, S)\).
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