On Quasi Generalized Topological Simple Groups

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Abstract: In this paper we introduce the concept of quasi $\mathcal{G}$-topological simple group. Also some basic properties, theorems and examples of a quasi $\mathcal{G}$-topological simple groups are investigated. Moreover we studied the important result, If the mapping between two quasi $\mathcal{G}$-topological simple groups is $\mathcal{G}$-contious at the identity element, then $f$ is $\mathcal{G}$-continuous.

Keywords: Quasi topological group, $\mathcal{G}$-open set, $\mathcal{G}$-continuous, Quasi $\mathcal{G}$-topological simple group.

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I. Introduction

Csaszar[6], Introduced the notion of generalized neighbourhood system and generalized topological space. Also Csaszar[6], Investigated the generalized continous mappings. In this paper we introduce the new concept of quasi $\mathcal{G}$-topological simple group. Quasi $\mathcal{G}$-topological simple group have both topological and algebraic structures such that the translation mappings and the inversion mapping are $\mathcal{G}$-continuous with respect to the generalized topology. Also some basic results studied and discussed.

II. Preliminaries

Definition 2.1 [3] Let $X$ be any set and let $\mathcal{G} \subseteq P(X)$ be a subfamily of power set of $X$. Then $\mathcal{G}$ is called a generalized topology if $\emptyset \in \mathcal{G}$ and for any index set $I, \bigcup_{i \in I} O_i \in \mathcal{G}, O_i \in \mathcal{G}, i \in I$.

Definition 2.2 [3] The elements of $\mathcal{G}$ are called $\mathcal{G}$-open sets. Similarly, generalized closed set (or) $\mathcal{G}$-closed, is defined as complement of a $\mathcal{G}$-open set.

Definition 2.3 [3] Let $X$ and $Y$ be two $\mathcal{G}$-topological space. A mapping $f: X \to Y$ is called a $\mathcal{G}$-contious on $X$ if for any $\mathcal{G}$-open set $O$ in $Y$, $f^{-1}(O)$ is $\mathcal{G}$-open in $X$.

Definition : 2.4 [3] The bijective mapping $f$ is called a $\mathcal{G}$-homeomorphism from $X$ to $Y$ if both $f$ and $f^{-1}$ are $\mathcal{G}$-continuous. If there is a $\mathcal{G}$-homeomorphism between $X$ and $Y$, then they are said to be $\mathcal{G}$-homeomorphic. It is denoted by $X \cong_\mathcal{G} Y$.

Definition : 2.5 [3] Collection of all $\mathcal{G}$-interior points of $A \subseteq X$ is called $\mathcal{G}$-interior of $A$. It denoted by $\text{Int}_\mathcal{G}(A)$. By definition it obvious that $\text{Int}_\mathcal{G}(A) \subseteq A$.

Note: 2.6 [3] (i) $\mathcal{G}$-interior of $A$, $\text{Int}_\mathcal{G}(A)$ is equal to union of all $\mathcal{G}$-open sets contained in $A$.

(ii) $\mathcal{G}$-closure of $A$ as intersection of all $\mathcal{G}$-closed sets containing $A$. It is denoted by $\text{Cl}_\mathcal{G}(A)$.

Definition: 2.7 [3] Let $(G, \ast)$ is a group and given $x \in G, L_x: G \to G$ defined by $L_x(y) = x \ast y$ and $R_x: G \to G$ defined by $R_x(y) = y \ast x$, denote left and right translation by $x$, respectively.

Definition: 2.8 [1] A quasi topological group $G$, is a group which is also a topological space if the following conditions are satisfied,
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(i). Left translation $L_x: G \to G$, $x \in G$ and right translation $R_x: G \to G$, $x \in G$ are continuous and
(ii). The inverse mapping $i: G \to G$ defined by $i(x) = x^{-1}, x \in G$ is continuous.

**Definition:** 2.9 [20] A group $G$ is called a simple group if it has no nontrivial normal subgroup of $G$.

**III. Quasi Generalized Topological Simple Groups**

**Definition:** 3.1 A quasi $G$-topological simple group $G$, is a simple group which is also a $G$-topological space if the following conditions are satisfied,

(i). Left translation $L_x: G \to G$, $x \in G$ and Right translation $R_x: G \to G$, $x \in G$ are $G$-continuous and
(ii). The inverse mapping $i: G \to G$ defined by $i(x) = x^{-1}, x \in G$ is $G$-continuous.

**Example:** 3.2 Any group of prime order with indiscrete or discrete $G$-topology is a quasi $G$-topological simple group.

**Example:** 3.3 Let $G = \{(0, 0, 0)\}$ be a trivial simple group under addition and we define a generalized topology on $G$ by $G = \{(0, 0, 0)\}$. Clearly $(G, +, G)$ quasi $G$-topological simple group.

**Example:** 3.4 $G = \{1, w, w^2\}$, where $w^3 = 1$, is a simple group under multiplication. Now we define a generalized on $G$ by $G = \{\phi, G, \{0\}\}$. Then the inverse mapping $i$ is $G$-continuous at the points $1, w^2$ and not $G$-continuous at the point $w$. In right translation mapping, $R_1$ is $G$-continuous at each point of $G, R_w$ is $G$-continuous at the points $w, w^2$ and not $G$-continuous at the point $1$ and $R_{w^{-1}}$ is $G$-continuous at the point $1, w$ and not $G$-continuous at the point $w^2$. Similarly we can prove left translation($L_x$).

**Theorem:** 3.5 Let $(G, \ast, G)$ be a quasi $G$-topological simple group and $\beta_e$ be the collection of all $G$-open neighbourhood at identity $e$ of $G$. Then

(i). For every $U \in \beta_e$, there is an element $V \in \beta_e$ such that $V^{-1} \subseteq U$.

(ii). For every $U \in \beta_e$, there is an element $V \in \beta_e$ such that $V \ast x \subseteq U$ and $x \ast V \subseteq U$, for each $x \in U$.

**Proof:** (i). Since $(G, \ast, G)$ is a quasi $G$-topological simple group. Therefore, for every $U \in \beta_e$, there exists $V \in \beta_e$ such that $i(V) = V^{-1} \subseteq U$, because the inverse mapping $i: G \to G$ is $G$-continuous.

(ii). Since $(G, \ast, G)$ is a quasi $G$-topological simple group. Thus for each $G$-open set $U$ containing $x$, there exists $V \in \beta_e$ such that $R_x(V) = V \ast x \subseteq U$. Similarly, $L_x(V) = x \ast V \subseteq U$.

**Theorem:** 3.6 Let $G$ be a quasi $G$-topological simple group and $g$ be any element of $G$. Then the right translation($R_g$) and left translation($L_g$) of $G$ by $g$ is a $G$-homeomorphism of the space $G$ onto itself.

**Proof:** First we prove that $R_g$ is surjective. Assume that $y \in G$, then the element $yg^{-1}$ maps to $y$. Therefore $R_g$ is surjective.

Assume that $R_g(x) = R_g(y).

\Rightarrow xg = yg.

\Rightarrow x = y$. Hence $R_g$ is 1-1. Since $G$ is a quasi $G$-topological simple group, $R_g$ is $G$-continuous.

Consider $R_g^{-1}$ which maps $xg$ to $x$, this is equivalent to the map from $x$ to $xg^{-1}$. Therefore $R_g^{-1}(x) = R_g^{-1}(x)$. Since $R_g^{-1}(x)$ is $G$-continuous, $R_g^{-1}(x)$ is $G$-continuous. Similarly we will prove that the left translation ($L_g$). Hence the theorem.

**Theorem:** 3.7 Let $G$ be a quasi $G$-topological simple group and $U$ be any $G$-open set in $G$. Then

(i). $a \ast U$ and $U \ast a$ is $G$-open in $G$ for all $a \in G$.

(ii). For any subset $A$ of $G$, the sets $U \ast A$ and $A \ast U$ are $G$-open in $G$.

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Proof: Let \( x \in U \ast a \). We want to show that \( x \) is a \( G \)-interior point of \( U \ast a \). Let \( x = u \ast a \) for some \( u \in U = U \ast a \ast a^{-1} \). Then \( u = x \ast a^{-1} \). We know that \( R_{a^{-1}}: G \to G \) is \( G \)-continuous. Then for every \( G \)-open set containing \( R_{a^{-1}}(x) = x \ast a^{-1} = u \), there exists a \( G \)-open set \( M_x \) containing \( x \) such that \( R_{a^{-1}}(M_x) \subseteq U \).
\[
\Rightarrow M_x \ast a^{-1} \subseteq U.
\]
\[
\Rightarrow x \in M_x \subseteq U \ast a.
\]
\[
\Rightarrow x \text{ is a } G\text{-interior point of } U \ast a. \text{ Therefore } U \ast a \text{ is } G\text{-open in } G. \text{ Similarly we can prove that } a \ast U \text{ is } G\text{-open in } G.
\]

(ii). By above result, \( U \ast a \) is \( G \)-open, for all \( a \in G \). Then \( U \ast A = \bigcup_{a \in A} U \ast a \) also \( G \)-open in \( G \). Similarly we can prove that \( A \ast U \) is \( G \)-open in \( G \).

Theorem 3.8 Suppose that a subgroup \( H \) of a quasi \( G \)-topological simple group \( G \) contains a non-empty \( G \)-open subset of \( G \). Then \( H \) is \( G \)-open in \( G \).

Proof: Let \( U \) be a non-empty \( G \)-open subset of \( G \) with \( U \subseteq H \). For every \( g \in H \), the set \( L_g(U) = U \ast g \) is \( G \)-open in \( G \), then \( H = \bigcup_{g \in H} U \ast g \) is \( G \)-open in \( G \).

Theorem 3.9 Every quasi \( G \)-topological simple group \( G \) has \( G \)-open neighbourhood at the identity element \( e \) consisting of symmetric \( G \)-neighbourhoods.

Proof: For an arbitrary \( G \)-open neighbourhood \( U \) of the identity \( e \), if \( V = U \cap U^{-1} \), then \( V = V^{-1} \), the set \( V \) is an \( G \)-open neighbourhood of \( e \), which implies that \( V \) is a symmetric \( G \)-neighbourhood and \( V \subseteq U \).

Theorem 3.10 Let \( f: G \to H \) be a homomorphism of quasi \( G \)-topological simple groups. If \( f \) is \( G \)-continuous at the neutral element \( e_G \) of \( G \), then \( f \) is \( G \)-continuous.

Proof: Let \( x \in G \) be arbitrary and suppose that \( W \) is an \( G \)-open neighbourhood of \( y = f(x) \) in \( H \). Since the left translation \( L_y \) in \( H \) is a \( G \)-continous mapping, there exists an \( G \)-open neighbourhood \( V \) of the neutral element \( e_H \) in \( H \) such that \( L_y(V) = yV \subseteq W \). Since \( f \) is \( G \)-continuous at \( e_G \) of \( G \), then \( f(U) \subseteq V \), for some \( G \)-open neighbourhood \( U \) of \( e_G \) in \( G \). Since \( L_y: G \to G \) is \( G \)-continuous, then \( xU \) is an \( G \)-open neighbourhood of \( x \) in \( G \).

Now we have \( f(xU) = f(x)f(U) = yf(U) \subseteq yV \subseteq W \). Hence \( f \) is \( G \)-continuous at the point \( x \in G \).

Theorem 3.11 Suppose that \( G, H \) and \( K \) are quasi \( G \)-topological simple groups and that \( \phi: G \to H \) and \( \psi: G \to K \) are homomorphisms. Such that \( \psi(G) = K \) and \( \text{Ker } \psi \subseteq \text{Ker } \phi \). Then there exists homomorphism \( f: K \to H \) such that \( \phi = f \circ \psi \). In addition, for each \( G \)-neighbourhood \( U \) of the identity element \( e_H \) in \( H \), there exists a \( G \)-neighbourhood \( V \) of the identity element \( e_K \) in \( K \) such that \( \psi^{-1}(V) \subseteq \phi^{-1}(U) \), then \( f \) is \( G \)-continuous.

Proof: Algebraic part of the theorem is well known. Suppose \( U \) is a \( G \)-neighbourhood of \( e_H \) in \( H \). By assumption, there exists a \( G \)-neighbourhood \( V \) of the identity element \( e_K \) in \( K \) such that, \( W = \psi^{-1}(V) \subseteq \phi^{-1}(U) \).
\[
\Rightarrow \phi(W) = \phi(\psi^{-1}(V)) \subseteq \phi(\phi^{-1}(U))
\]
\[
\Rightarrow \phi(W) = f(V) \subseteq U. \text{ Hence } f \text{ is } G\text{-continuous at the identity element of } K. \text{ Therefore by above theorem, } f \text{ is } G\text{-continuous.}
\]

Corollary 3.12 Let \( \phi: G \to H \) and \( \psi: G \to K \) be \( G \)-continuous homomorphism of a quasi \( G \)-topological simple groups \( G, H \) and \( K \). Such that \( \psi(G) = K \) and \( \text{Ker } \psi \subseteq \text{Ker } \phi \). If the homomorphism \( \psi \) is \( G \)-open, then there exists a \( G \)-continuous homomorphism, \( f: K \to H \) such that \( \phi = f \circ \psi \).

Proof: The existence of a homomorphism \( f: K \to H \) such that \( \phi = f \circ \psi \). Take an arbitrary \( G \)-open set \( V \) in \( H \). Then \( f^{-1}(V) = \psi(\phi^{-1}(V)) \). Since \( \phi \) is \( G \)-continuous and \( \psi \) is an \( G \)-open map, \( f^{-1}(V) \) is \( G \)-open in \( K \). Therefore \( f \) is \( G \)-continuous.

Theorem 3.13 Let \( G \) be a quasi \( G \)-topological simple group and \( H \) is a normal subgroup of \( G \). Then \( H \) also a normal subgroup of \( G \).

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Proof: Now we have to prove that \( ghg^{-1} \in H \ \forall \ g \in G \).

Since \( H \) is a normal subgroup of \( G \), \( ghg^{-1} \in H \ \forall \ g \in G \).

Now \( ghg^{-1} \subseteq H \ \forall \ g \in G \).

\[ \Rightarrow ghg^{-1} \subseteq H \ \forall \ g \in G. \]

\[ \Rightarrow ghg^{-1} \in H, \forall g \in G. \] Therefore \( H \) is a normal subgroup of \( G \).

Corollary: 3.14 Let \( G \) be a quasi \( G \)-topological simple group and \( Z(G) \) be the centre of \( G \). Then \( Z(G) \) is a normal subgroup of \( G \).

Proof: proof follows from the above theorem.

Corollary: 3.15 Let \( G \) and \( H \) be a quasi \( G \)-topological simple groups. If \( f: G \rightarrow H \) is a homomorphism mapping \( x \), then \( ker f \) is a normal subgroup of \( G \).

Theorem: 3.16 Let \( G \) and \( H \) be quasi \( G \)-topological simple groups with neutral elements \( e_G \) and \( e_H \), respectively, and let \( p \) be a \( G \)-continuous homomorphism of \( G \) onto \( H \) such that, for some non-empty subset \( U \) of \( G \), the set \( p(U) \) is \( G \)-open in \( H \) and the restriction of \( p \) to \( U \) is an \( G \)-open mapping of \( U \) onto \( p(U) \). Then the homomorphism \( p \) is \( G \)-open.

Proof: It suffices to show that \( x \in G \), where \( W \) is an \( G \)-open neighbourhood of \( x \) in \( G \), then \( P(W) \) is a \( G \)-open neighbourhood of \( p(x) \) in \( H \). Fix a point \( y \) in \( U \), and let \( L \) be the left translation of \( G \) by \( xy^{-1} \). Then \( L \) is a \( G \)-homeomorphism of \( G \) onto itself such that,

\[
L_{xy^{-1}}(x) = yx^{-1}x = y.
\]

So \( V = U \cap L(W) \) is an \( G \)-open neighbourhood of \( y \) in \( U \). Then \( p(V) \) is \( G \)-open subset of \( H \). Consider the left translation \( h \) of \( H \) by the inverse to \( p(xy^{-1}) \).

Now clearly, \((h o p o l)(x) = h(p(l(x))) = h(p(y)) = p(xy^{-1})p(y) = p(xy^{-1}y) = p(y) = p(x)\).

Hence \( h(p(l(W))) = p(W) \). Clearly \( h \) is a \( G \)-homeomorphism of \( H \) onto itself. Since \( p(V) \) is \( G \)-open in \( H \), \( h(p(V)) \) is \( G \)-open in \( H \). Therefore \( p(W) \) contains the \( G \)-neighbourhood \( h(p(V)) \) of \( p(x) \) in \( H \). Hence \( p(W) \) is a \( G \)-open neighbourhood of \( p(x) \) in \( H \).

Definition: 3.17 Let \( H \) be a subgroup of quasi \( G \)-topological simple group \( G \). Then \( H \) is called neutral in \( G \) if every \( G \)-neighbourhood \( U \) of the identity \( e_G \) in \( G \), there exists a \( G \)-neighbourhood \( V \) of \( e_G \) such that \( VH \subset HU \).

Theorem: 3.18 Let \( H \) be a subgroup of quasi \( G \)-topological simple group \( G \). Suppose that, for every \( G \)-open neighbourhood \( U \) of the identity \( e_G \) in \( G \), there exists an \( G \)-open neighbourhood \( V \) of \( e_G \) in \( G \) such that \( xVx^{-1} \subseteq U \) whenever \( x \in G \). Then \( H \) is neutral in \( G \).

Proof: Given a \( G \)-neighbourhood \( U \) of \( e_G \) in \( G \). Take an \( G \)-open neighbourhood \( V \) of \( e_G \) satisfying,

\[
xVx^{-1} \subseteq U, \forall \ x \in G
\]

\[
\Rightarrow xV \subseteq Ux, \forall \ x \in G
\]

\[
\Rightarrow HV \subseteq UH, \forall \ x \in G. \] Then \( H \) is neutral in \( G \).

References


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