“An Inventory Model of Repairable Items with Exponential Deterioration and Linear Demand Rate”

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Abstract: In this paper an inventory model for deteriorating and repairable items is developed with linear demand. An Exponential distribution is used to represent the distribution of time for deterioration. In the model considered here, shortages are allowed to occur and defective items can be repaired. The model is solved analytically to obtain the optimal solution of the problem. The derived model is illustrated by a numerical example and its sensitivity analysis is carried out.

I. Introduction

Deterioration means damage, spoilage, dryness, vaporization, etc. It is defined as decay or damage such that the item cannot be used for its original purposes. The effect of deterioration is very important in many inventory systems. The pioneering work of Harris [7] inventory models are being treated by mathematical techniques. He developed the simplest inventory model, the Economic Order Quantity (EOQ) model which was later popularized by Wilson [24]. Relaxation of some assumptions in the formulation of the EOQ model led to the development of other inventory models that effectively tackle several other inventory problems occurring in day-to-day life. Inventory of deteriorating items was first studied by Whitin [25] where he considered the deterioration of fashion goods at the end of prescribed storage period. Ghare and Schrader [5] extended the classical EOQ formula to include exponential decay, wherein a constant fraction of on-hand inventory is assumed to be lost due to deterioration.


Manna and Chiang [13] developed an EPQ model for deteriorating items with ramp type demand. Teng and Chang [22] considered the economic production quantity model for deteriorating items with stock level and selling price dependent demand. Jain et al. [8] developed an economic production quantity model with shortages by incorporating the deterioration effect and stock dependent demand rate. Roy and Chaudhary [17] developed two production rates inventory model for deteriorating items when the demand rate was assumed to be stock dependent. In the research of Sana et al. [18] shortages are allowed to occur at the end of a cycle. With the consideration of time varying demand and constant deteriorating rate, the optimal production inventory policy was studied. Raman Patel [16] developed a production inventory model for deteriorating items following Weibull distribution with price and quantity dependent demand and varying holding cost with shortages.

Both Skouri and Papachristos [20] and Chen et al. [2] developed a production inventory model in which the shortages are allowed at the beginning of a cycle. In contrast, Manna and Chaudhari [12] have allowed shortages to occur at the end of each cycle. Goyal [6] deals with production inventory problem of a product with time varying demand, production and deterioration rates in which the shortages occur at the beginning of the cycle.

There are four synonyms of reuse according to Thierry et al. [23]. They are: Direct Reuse, Repair, Recycling and Remanufacturing. Schrady [19] was the first to consider reuse in a deterministic model. Recently, Mabini et al. [11] extended Schrady’s model to consider stock-out service level constraints and multi-item system sharing the same repair facility. In the policy, expressions for the optimal control parameter values were derived. Koh [10] developed a joint EOQ and EPQ model in which the stationary demand can be satisfied by recycled products and newly purchased products. The model assumes a fixed proportion of the used products that are collected from the customers.

Recently, Bhojak and Gothi [1] have developed inventory models for ameliorating and deteriorating items with variable demand and HIC. Kirtan Parmar and Gothi [9] have developed an EPQ model of deteriorating items using three parameter Weibull distribution with constant production rate and time varying

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holding cost. Devyani Chatterji and Gothi [3] have developed three-parametric Weibull deteriorated EOQ model with price dependent demand and shortages under fully backlogged condition.

R. K. Yadav and Rajeev Kumar [26] developed an inventory model for deteriorating and repairable items with linear demand. Here, we have tried to redevelop the same model and corrected most of the results.

II. Notations

The mathematical model in this paper is developed using the following notations:

1. \( k \) : Production rate (units/unit time).
2. \( D \) : Demand rate.
3. \( \theta(t) \) : The deterioration rate (units/unit time).
4. \( b \) : The decreasing rate of the demand (units/unit time).
5. \( c \) : Fraction of defective product.
6. \( r \) : Fraction of stock-out demand sales lost due to some stock-out demands. \( 0 < r < 1 \)
7. \( Q(t) \) : The instantaneous state of the inventory level at any time \( t \) \( (0 \leq t \leq T) \).
8. \( Q_1 \) : The maximum inventory level of the product.
9. \( Q_2 \) : The maximum inventory level during shortage period.
10. \( A \) : Ordering cost per order.
11. \( C_h \) : Inventory holding cost per unit per unit time.
12. \( C_d \) : Deterioration cost per unit per unit time.
13. \( C_s \) : Shortage cost per unit.
14. \( P_c \) : Purchase cost per unit.
15. \( TC \) : The average total cost for the time period \( [0, T] \).

III. Basic Assumptions

The model is derived under the following assumptions:

1. The inventory system deals with single item.
2. The production rate is finite and constant, which is larger than the demand rate and is unaffected by the lot size.
3. Once a unit of the product is produced, it is available to meet the demand.
4. Once the production is started, the product starts being deteriorated.
5. The annual demand rate is a linear function of time and it is \( D = \alpha + \beta t \). \( (\alpha, \beta > 0) \).
6. Shortages are allowed and completely backlogged.
7. Replenishment rate is infinite and instantaneous.
8. All defective products can be repaired and reused.
9. The second and higher powers of \( \theta, b \), and care neglected in the analysis of the derived model.
10. Total inventory cost is a real, continuous function which is convex to the origin.

IV. Mathematical Model And Analysis

Here, we consider a single commodity deterministic production inventory model with a time dependent linear demand rate. The distribution of the time to deteriorate is random variable following the exponential distribution.

The probability density function for exponential distribution is given by

\[
f(t) = \theta e^{-\theta t} \quad ; \quad (t > 0 \text{ and } 0 < \theta < 1)
\]

The instantaneous rate of deterioration \( \theta(t) \) of the non-degraded inventory at time \( t \) can be obtained from

\[
\theta(t) = \frac{f(t)}{1 - F(t)} \quad \text{where} \quad F(t) = 1 - e^{-\theta t}
\]

Thus, the instantaneous rate of deterioration of the on-hand inventory is \( \dot{\theta}(t) = \dot{\theta} \). The probability density function represents the distribution of the time to deteriorate which may have a decreasing, constant or increasing rate of deterioration.

Initially, inventory level is zero. At time \( t = 0 \), the production starts and simultaneously supply also begins and the production stops at \( t = t_1 \) when the maximum inventory level \( Q_1 \) is reached. In the interval \([0, t_1]\), before the production stops, the inventory is built up at a rate \( k - D \) and is depleted at the rate \( (\theta - b + c) \). In the interval \([t_1, t_2]\) the inventory is depleted at the rate \( D \) and rate \( (\theta - b) \). The inventory is infinitely decreasing in the time interval \([t_1, t_2]\) until inventory level reaches zero. It is decided to backlog the demands up to \( Q_2 \) level which occurs during stock-out time. Thereafter, shortages can occur during the time interval \([t_2, t_3]\), and all of the demand during the period \([t_2, t_3]\) is completely backlogged. Thereafter, production is started at a rate \((1 - r)(k - D)\) so as to clear the backlog, and the inventory level reaches to \( 0 \) (i.e. the backlog is cleared) at \( t = T \).

The pictorial presentation is shown in the Figure – 1.
The differential equations which govern the instantaneous state of $Q(t)$ over the time intervals $[0, t_1]$, $[t_1, t_2]$, $[t_2, t_3]$ and $[t_3, T]$ are given by

$$\frac{dQ(t)}{dt} + (\theta - b + c)Q(t) = k - (\alpha + \beta t), \quad (0 \leq t \leq t_1) \quad (1)$$

$$\frac{dQ(t)}{dt} + (\theta - b)Q(t) = -\alpha t, \quad (t_1 \leq t \leq t_2) \quad (2)$$

$$\frac{dQ(t)}{dt} = -(1 - r)(\alpha + \beta t), \quad (t_2 \leq t \leq t_3) \quad (3)$$

$$\frac{dQ(t)}{dt} + (1 - r)c \cdot Q(t) = (1 - r)\{k - (\alpha + \beta t)\}, \quad (t_3 \leq t \leq T) \quad (4)$$

Under the boundary conditions $Q(0) = 0$, $Q(t_1) = Q_1$, $Q(t_2) = 0$, $Q(t_3) = -Q_2$ & $Q(T) = 0$, the solutions of equations (1) to (4) are given by

$$Q(t) = (k - \alpha) \cdot t \quad (0 \leq t \leq t_1) \quad (5)$$

$$Q(t) = -\alpha t + \left[\alpha(1 - t_1^2)\right] t_2 + \left[\beta(1 - t_1^2)\right] t_2^2 \quad (t_1 \leq t \leq t_2) \quad (6)$$

$$Q(t) = (1 - r)\left(\alpha(t_2 - t) + \frac{1}{2} \beta(t_2^2 - t^2)\right) \quad (t_2 \leq t \leq t_3) \quad (7)$$

$$Q(t) = (1 - r)\left\{\left[\frac{k - \alpha}{\mu} - \beta\left(\frac{t}{\mu - \frac{1}{\mu^2}}\right)\right] + (1 + \mu T)(1 - \mu T)\left[\frac{k - \alpha}{\mu} + \beta\left(\frac{T}{\mu - \frac{1}{\mu^2}}\right)\right]\right\} \quad (t_3 \leq t \leq T) \quad (8)$$

Where $\delta = \theta - b + c$, $\xi = \theta - b$ and $\mu = (1 - r)\cdot c$

From (5), $Q(t_1) = Q_1 = (k - \alpha) \cdot t_1$ \quad (9)

and from (6), $Q(t_1) = Q_1 = -\alpha t_1 + \left[\alpha(1 - \xi t_1)\right] t_2 + \left[\beta(1 - \xi t_1)\right] t_2^2$ \quad (10)

Eliminating $Q(t_1)$ from equations (9) and (10), we get

$$t_1 = \frac{(\alpha + \beta t_2)t_2}{\alpha \xi t_2 + \beta \xi t_2^2 + k} \quad (11)$$

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Thus, $t_1$ can be written in terms of $t_2$ and so $t_1$ is not a decision variable.

From (7),

$$Q(t_3) = -Q_2 = (1 - r) \left( \alpha (t_2 - t_3) + \frac{1}{2} \beta (t_2^2 - t_3^2) \right)$$  \tag{12}$$

and from (8),

$$Q(t_3) = -Q_2 = (1 - r) \left[ \left( \frac{k - \alpha}{\mu} - \beta \left( \frac{t_2}{\mu} - \frac{1}{\mu^2} \right) \right) \right. \left. + (1 + \mu T) (1 - \mu t_3) \left[ - \frac{k - \alpha}{\mu} + \beta \left( \frac{T}{\mu} - \frac{1}{\mu^2} \right) \right] \right]$$  \tag{13}$$

Eliminating $Q(t_3)$ from equations (12) and (13), we get

$$t_2 = \frac{1}{\beta} \left( -\alpha + \sqrt{\alpha^2 + 2\beta k t_3 + \beta^2 t_3^2 + \left[ (2T \alpha \beta - 2T \beta k + 2T^2 \beta^2) (1 - \mu t_3) \right] \right)$$  \tag{14}$$

Thus, $t_2$ can be written in terms of $t_3$ and so $t_2$ is also not a decision variable.

**Cost Components:**

The total cost per replenishment cycle consists of the following cost components:

1) **Ordering Cost (OC)**

The ordering cost OC over the period $[0, T]$ is

$$OC = A \text{ (Fixed)}$$  \tag{15}$$

2) **Deterioration Cost (DC)**

The deterioration cost DC over the period $[0, t_1]$ and $[t_0, T]$ is

$$DC = C_d \cdot c \left[ \int_0^{t_1} Q(t) \, dt + (1 - r) \int_{t_2}^T Q(t) \, dt \right]$$

$$\Rightarrow DC = C_d \cdot c \left[ \frac{1}{2} (k - \alpha) t_1^2 + (1 - r) \left( \frac{k - \alpha}{\mu} + \beta \left( T - \frac{1}{\mu^2} \right) \right) T - t_2 \right]$$  \tag{16}$$

3) **Inventory Holding Cost (IHC)**

The inventory holding cost IHC over the period $[0, t_2]$ is

$$IHC = C_h \left[ \int_0^{t_1} Q(t) \, dt + \int_{t_2}^{T} Q(t) \, dt \right]$$

$$\Rightarrow IHC = C_h \left[ \frac{1}{2} (k - \alpha) t_1^2 - \frac{1}{2} \beta t_2^2 + \alpha \xi t_1^2 + \alpha (t_2^2 - t_1^2) + \beta t_2^2 (t_2 - t_1) + \alpha t_x (t_2 - t_1) \right]$$  \tag{17}$$

4) **Shortage Cost (SC)**

Demand during the time $[t_0, T]$ is satisfied at a time as the production has already started at time $t = t_0$ and so shortage cost during this interval is not taken into account.

The shortage cost SC over the period $[t_2, t_1]$ is

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\[ SC = -C_s \int_{t_2}^{t_3} Q(t) \, dt \]

\[ \Rightarrow SC = -C_s (1-r) \left[ -\frac{1}{6} \beta (t_3^3 - t_2^3) - \frac{1}{2} \alpha (t_3^2 - t_2^2) + \left( \alpha t_2 + \frac{1}{2} \beta t_2^2 \right) (t_3 - t_2) \right] \] (18)

5) Lost Sale Cost (LSC)

The lost sale cost LSC over the period \([t_2, t_3]\) is

\[ LSC = C_p \cdot r \int_{t_2}^{t_3} (\alpha + \beta t) \, dt \]

\[ \Rightarrow LSC = C_p r \left( \alpha (t_3 - t_2) + \frac{1}{2} \beta \left( t_3^2 - t_2^2 \right) \right) \] (19)

6) Purchase Cost (PC)

The purchase cost PC over the period \([0, T]\) is

\[ PC = P_c \cdot k (t_1 + t_3 - t_2) \]

\[ PC = P_c \cdot k \left[ \frac{(\alpha + \beta t_3) t_2}{\alpha \zeta t_2 + \beta \zeta t_2 + k} + t_3 - \frac{1}{\beta} \left( -\alpha + \sqrt{\alpha^2 + 2\beta \zeta t_3 + \beta^2 t_3^2 + \left( \frac{2T \alpha \beta - 2T \beta k + 2T^2 \beta^2}{\zeta^2} \right) (1 - \mu t_3)} \right) \right] \] (20)

Hence, the total cost per unit time for the time period \([0, T]\) is given by

\[ TC = \frac{1}{T} (OC + DC + IHC + SC + LSC + PC) \] (21)

Now, our objective is to determine optimum values \(t_3^*\) and \(T^*\) of \(t_3\) and \(T\) respectively to minimize the total cost \(TC\). Using mathematical software, the optimal values \(t_3^*\) and \(T^*\) can be obtained by solving \( \frac{\partial TC}{\partial t_3} = 0 \) and \( \frac{\partial TC}{\partial T} = 0 \) which can satisfy the following sufficient conditions:

\[ \left[ \begin{array}{c} \left( \frac{\partial^2 TC}{\partial t_3^2} \right) - \left( \frac{\partial^2 TC}{\partial t_3 \partial T} \right)^2 \\ \left( \frac{\partial^2 TC}{\partial T^2} \right) \end{array} \right]_{t_3 = t_3^*, T = T^*} > 0 \] (22)
To illustrate the proposed model, an inventory system with the following hypothetical values is considered. By taking $\alpha = 40, \beta = 0.001, C_a = 2.8, C_o = 2, C_p = 2, \theta = 0.01, b = 0.005, c = 0.06, r = 0.09, k = 60$ and $A = 300$(with appropriate units), optimal values of $t_1$ and $T$ are $t_1^* = 8.664682800, T^* = 25.29411443$ units and the optimal total cost per unit time $TC = 126.5618193$ units.

### VI. Sensitivity Analysis

Sensitivity analysis depicts the extent to which the optimal solution of the model is affected by the changes in its input parameter values. Here, we study the sensitivity for the cycle length $T$ and total cost per unit time $TC$ with respect to the changes in the values of the parameters $\alpha, \beta, C_a, C_o, C_p, \theta, b, c, r, k$ and $A$.

The sensitivity analysis is performed by considering different values in each one of the above parameters keeping all other parameters as fixed. The results are presented in the Table.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% change in $T$</th>
<th>$T$</th>
<th>TC</th>
<th>% change in TC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$-10$</td>
<td>9.314694607</td>
<td>24.27554607</td>
<td>129.6847228</td>
</tr>
<tr>
<td></td>
<td>$-5$</td>
<td>9.060134527</td>
<td>24.87171030</td>
<td>126.0346419</td>
</tr>
<tr>
<td></td>
<td>$+5$</td>
<td>9.575320249</td>
<td>22.83983430</td>
<td>129.1252431</td>
</tr>
<tr>
<td></td>
<td>$+10$</td>
<td>6.795258895</td>
<td>24.21483912</td>
<td>118.1465409</td>
</tr>
</tbody>
</table>

| $\beta$  | $-10$ | 8.665447815 | 25.29738321 | 126.5682307 | 0.0051 |
|          | $-5$  | 8.665114980 | 25.29896904 | 126.5379849 | 0.0025 |
|          | $+5$  | 8.664278055 | 25.29435706 | 126.5866179 | -0.0025 |
|          | $+10$ | 8.663737760 | 25.29458876 | 126.5554025 | -0.0051 |

| $C_a$    | $-10$ | 8.826234244 | 25.52664234 | 126.6352724 | -0.72 |
|          | $-5$  | 8.746522743 | 25.41424588 | 126.102482 | -0.36 |
|          | $+5$  | 8.594913624 | 25.20153588 | 126.9561036 | 0.31 |
|          | $+10$ | 8.526715876 | 25.10923951 | 127.3363820 | 0.61 |

| $C_o$    | $-10$ | 8.978771732 | 26.59655084 | 124.3731096 | -1.73 |
|          | $-5$  | 8.818701486 | 25.92256960 | 125.5093222 | -0.84 |
|          | $+5$  | 8.516208777 | 24.70602428 | 127.5620582 | 0.79 |
|          | $+10$ | 8.372836499 | 24.15393468 | 128.5069043 | 1.54 |

| $C_s$    | $-10$ | 8.480961457 | 24.32549411 | 118.6665614 | -6.24 |
|          | $-5$  | 8.578163592 | 24.86306834 | 122.630315 | -3.11 |
|          | $+5$  | 8.785634529 | 25.70576324 | 131.7549398 | 4.10 |
|          | $+10$ | 9.095227459 | 26.12744231 | 135.8244732 | 7.32 |

| $\theta$ | $-10$ | 8.59404426 | 25.19849664 | 126.8592190 | 0.23 |
|          | $-5$  | 8.631745684 | 25.24597991 | 126.7118618 | 0.12 |
|          | $+5$  | 8.698299772 | 25.34296367 | 126.4091007 | -0.12 |
|          | $+10$ | 8.73242178 | 25.39235043 | 126.2553609 | -0.24 |

| $b$      | $-10$ | 8.681382244 | 25.31842297 | 126.485711 | -0.06 |
|          | $+5$  | 8.648138296 | 25.26966772 | 126.6371560 | 0.06 |
|          | $+10$ | 8.631745684 | 25.24597991 | 126.7118618 | 0.12 |

| $c$      | $-10$ | 9.662043332 | 27.92445113 | 126.9326540 | 0.29 |
|          | $-5$  | 9.136300992 | 26.53887670 | 126.7116551 | 0.12 |
|          | $+5$  | 8.239138350 | 24.16944489 | 126.473083 | -0.07 |
|          | $+10$ | 7.853216402 | 23.14806998 | 126.4381012 | -0.10 |

| $r$      | $-10$ | 8.54402187 | 23.92635188 | 126.6369017 | 0.06 |
|          | $-5$  | 8.604072005 | 25.10915505 | 126.5995069 | 0.03 |
|          | $+5$  | 8.725865710 | 25.48123189 | 126.5237261 | -0.03 |
|          | $+10$ | 8.787630053 | 25.67054617 | 126.4853834 | -0.06 |

| $k$      | $-10$ | 8.702424294 | 24.40578093 | 103.6354453 | -18.11 |
|          | $-5$  | 9.227624646 | 24.54624572 | 116.6792374 | -0.78 |
|          | $+5$  | 9.017541849 | 24.77252051 | 134.3868513 | 0.18 |
|          | $+10$ | 9.22762496 | 24.16722090 | 141.2883776 | 11.64 |

| $A$      | $-10$ | 8.559347452 | 24.90282840 | 126.5105357 | -0.04 |
|          | $-5$  | 8.615342424 | 25.1135351 | 126.0267357 | -0.42 |
|          | $+5$  | 8.698041189 | 25.43623237 | 127.1532333 | 0.47 |
|          | $+10$ | 8.730462016 | 25.5752735 | 127.7412747 | 0.93 |
VII. Graphical Presentation

Figure – 2

Figure – 3

VIII. Conclusions

➢ From the Table we can conclude that TC is highly sensitive to the change in $C_p$ and $k$, moderately sensitive to the change in $\alpha$ and $C_d$ and less sensitive to change in $\beta$, $C_h$, $C_s$, $\theta$, $b$, $c$, $A$ and $r$.
➢ It is observed from Figure – 2 that the effect of increase or decrease in the values of $\alpha$ does not affect the total cost TC much.
➢ It is observed from Figure – 2 that when the values of $A$, $C_h$, $C_d$, $C_p$, $C_s$ and $k$ increase simultaneously the average total cost TC also increases.
➢ It is observed from Figure – 3 that when the values of $b$ increase then TC also increases and when the values of $\beta$, $\theta$, $c$ and $r$ increase, TC decreases.

References


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