A note on nilpotency in a Left Goldie near-ring

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Abstract: In this paper we present an important result that a nil subnear-ring of a semiprime strictly left Goldie near-ring is nilpotent. It is to be noted that the essentiality of left near-ring subgroup here, arises as crucial from its fleeble nature. In contrast to such a result in ring theory, the crucial role played by substructures already mentioned appears here with very fascinating distinctiveness.

Keywords: near-ring, semiprime, nil subring, nilpotent subring, sequentially nilpotent, Goldie near-ring

MR 2010 subject classification: 16Y30,16Pxx, 16P60,16U20

I. Introduction

Chowdhury et al [1] introduced the notions of a Goldie near-ring as well as that of a Goldie module [2,3] as two way generalizations of so-called Goldie ring - an exposition of A.W. Goldie through his classics, - a part of his thorough study of the structure of prime rings under ascending chain conditions [10] and semiprime rings with maximum conditions [11]. We discussed various aspects of a Goldie near-ring and of a Goldie module including the near-ring of quotients and its possible descending chain condition and decomposition of the zero of a Goldie module [2,3], an analogous of Artin-Rees theorem [2]. Also we delve into Some Aspects of Artinian (Noetherian) Part of a Goldie Ring and its Topological Relevance (8) as well as Wreath Sum of Near-rings and Near-ring Groups with Goldie structures (9). It is easy to see that a nilpotent subring of a ring is necessarily nil. But converse is not true, however, we see that [12] Goldie character in a ring draw attention in its favor. We here prove this interesting standard problem in a near-ring with Goldie characteristics taking into consideration various aspects of nilpotent, Goldie near-ring Goldie near-ring Groups with its free Goldie Goldie module...Goldie near-ring etc.

II. Preliminaries

For the sake of completeness we would like to begin our discussion with the definition of a right near-ring $(N,+)$ - an algebraic structure consisting of a non-empty set $N$ equipped with two binary operations viz., addition $(+)$ and multiplication $(.)$, where the first one makes $N$ a group (not necessarily abelian) and the second one a semigroup with the one-way distributive law, viz. $(a+b)c=ac+bc$ , for $a,b,c \in N$

For other relevant information regarding near-ring preliminaries we would like to refer Pilz [13]. Throughout this paper $N$ will mean a right near-ring with unity (zero symmetric) unless otherwise specified.

2.1 Definitions:

2.1.1 An element $a \in N$ is nilpotent if there is a positive integer $t$ such that $a^t=0$, $a^{t+1} \neq 0$.

2.1.2. A subnear-ring of a nil near-ring of the corresponding set is nilpotent.

2.1.3. A subnear-ring $I$ is nilpotent if there is a positive integer $t$ such that $I^t=0$, $I^{t+1} \neq 0$, (in the sense $i_1i_2...i_t=0$, for $i_i \in I$ and $i_1i_2...i_t \neq 0$, for some $i_i \in I$).

Clearly, a nilpotent subnear-ring is nil but the converse is not true. For the converse, that is a nilpotent subring is nilpotent, we’ll deal with so called sequentially nilpotent (or s-nilpotent) notion.

We note the following: the above situation is dealt with the following definition that would lead us to our expected goal.

2.1.4. An element $a \in I$ is sequentially nilpotent (s-nilpotent) if for some positive integer $k$, we have $a_k \in I$ and $a_1a_2...a_k = 0$ (for $a_k=0$) . So if an element $a \in (a_1=a)$ is s-nilpotent, then for some $(a=a_1,a_2,...,a_k \in I , a_1a_2...a_k = 0 \Rightarrow (xa_1)a_2...a_k = 0$ and so $xa_1$ is s-nilpotent , i.e. any left multiple of a is also s-nilpotent.

And hence we
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Note: $a(\varepsilon I)$ would be not s-nilpotent if for any sequence of the type $< a_i >, a_i \in I$, with $(a=a_i)$ we have

$a_i a_2 \ldots a_k \neq 0(\neq \prod_{i=1}^{k} a_i)$ whatever be the positive integer $k$

2.1.5. A sub near-ring $I$ of $N$ is sequentially nilpotent (s-nilpotent) if for each sequence $< a_i >, a_i \in I$ there is a positive integer $k$ such that $a_i a_2 \ldots a_k = 0 (\neq \prod_{i=1}^{k} a_i)$.

Note:
(i) for an s-nilpotent sub near-ring $I$ of $N$, each element of $I$ is s-nilpotent.
(ii) if $I$ is not s-nilpotent, then there is a sequence $< a_i >, a_i \in I$, for each $k$, $a_i a_2 \ldots a_k \neq 0 (\neq \prod_{i=1}^{k} a_i)$ and

2.1.6. $a_i(\varepsilon I)$ has an infinite sequence if there is a sequence $< a_i >, a_i \in I$ such that for each $k$,

$a_i a_2 \ldots a_k \neq 0 (\neq \prod_{i=1}^{k} a_i)$.

Note:
$I$ is not s-nilpotent, then there is an $a_i (\varepsilon I)$ such that $a_i$ has an infinite sequence.

2.1.7. For $x \in N$ the set $I(x) = \{ n \in N | nx = 0 \}$ is the left annihilator of $x$ in $N$.

And this a left ideal of $N$.

2.1.7(a) A near-ring is left Goldie if it satisfies the a.c.c. (ascending chain condition) on its left annihilators and it has no infinite direct sum of left ideals.

2.1.7(b) $N$ is strictly left Goldie if it satisfies the a.c.c. on its left annihilators and it has no infinite independent family of left $N$-subgroups.

Example 1: $N = \{ 0, a, b, c \}$ is a near-ring under the operations defined by the following tables.

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(i)  
(ii)

Here we note that $A = \{ 0, a \}$, $B = \{ 0, a, b \}$ and $C = \{ 0, a, c \}$ are subsets of $N$ and $BN \subseteq B$, $CN \subseteq C$ whereas $NA \subseteq A$ and $AN \subseteq A$. Thus, we define the following

2.1.8 Definitions: A non-empty subset $S$ of a near-ring $N$ is
(i) a right $N$-subset of $N$ if $SN \subseteq S$
(ii) a left $N$-subset of $N$ if $NS \subseteq S$
(iii) an invariant subset of $N$ if $NS \subseteq S$, $SN \subseteq S$.

It is clear that an invariant subset of a near-ring $N$ is a left as well as right $N$-subset of $N$. Moreover, every left (right) $N$-subset contains the zero element of $N$.

2.1.9 (i) An ideal $I$ of $N$ is strongly prime if for two non-zero invariant subsets $A$ and $B$, $AB \subseteq I \Rightarrow A \subseteq I$, or $B \subseteq I$.
(ii) A near-ring is strongly prime if $(0)$ is strongly prime.

2.1.10. Definition: If $N$ is a near-ring then the group $(E, +)$ is an $N$-group (near-ring group) NE when there exists a map $N \times E \rightarrow E$, $(n, e) \rightarrow ne$ such that
(i) $(n1 + n2)e = n1e + n2e$

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(ii) \((n_1n_2)e = n_1(n_2e)\)

(iii) \(e = e, \) for all \(n_1, n_2 \in N, \) \(e \in E.\)

In what follows, \(E\) will stand for the near-ring group \(NE.\)

Clearly near-ring \(N\) can always be considered as an \(N\)-group. We shall write \(NN\) to denote \(N\) as an \(N\)-group.

**Example 2** (Ex.1.18(c) [11]) : Let \(G\) be an additive group and \(M(G)\) be a (right) near-ring(of all maps from \(G\) to \(G)\) then \(G\) is an \(M(G)–\)group when \(M(G) \times G \rightarrow G\) such that

\[
(f, x) \rightarrow f(x), \text{ for } x \in G, f \in M(G).
\]

**Example 3** : Every left module \(M\) over a ring \(R\) is an \(R\)-group over the near-ring \(R.\)

2.1.11. **Properties** : If \(E\) is an \(N\)-group then

(i) \(e = 0\) (the first 0 is the zero element of \(N\) and the second 0 is the zero element of \(E).\)

(ii) \((n1n2)e = n1(n2e)\)

(iii) \((n–n1)e = n(e – n1e),\) for all \(e \in E; n, n1 \in N\)

2.1.12. **Definitions** : An \(N\)-group \(E\) is said to be an \(N\)-group with \(acc\) on annihilators if any ascending chain \(Ann(M1) \subset Ann(M2) \subset Ann(M3)\) \(\ldots\) of annihilators of \(subsets M1, M2, M3, \ldots \) of \(E\) stops after a finite steps.

Similarly, we define an \(N\)-group \(E\) with \(dcc\) on annihilators for any descending chain of the type \(Ann(M1) \supset Ann(M2) \supset Ann(M3)\) \(\ldots\).

2.2. **Essential ideals and essential \(N\)-subgroups.**

2.2.1. **Definitions** : Let \(A\) and \(B\) be two \(N\)-subgroups of \(E\) such that \(A \subseteq B\) then \(A\) is an essential \(N\)-subgroup of \(B\)

(denoted \(A \subseteq B\)) if any \(N\)-subgroup \(C(\neq 0)\) of \(B\) has non-zero intersection with \(A\). when \(A \subseteq B\), we say \(B\) is an essential extension of \(A in E.\) Here an essential left \(N\)-subgroup \(A\) of \(N\) will mean an essential \(N\)-subgroup of \(NN.\)

An ideal \(M\) of \(E\) is an essential ideal of \(E\) (denoted \(M \subseteq e E\)) if for any ideal \(C(\neq 0)\) of \(E,\)

\(M \cap C \neq 0).\) If a left ideal \(A\) of \(N\) is an essential ideal of \(NN\) then \(A\) is an essential left ideal of \(N.\)

A left \(N\)-subgroup of \(N\) is weakly essential if for any non zero left ideal \(I\) of \(N, A \cap I \neq 0\)

An essential left ideal \(I\) is weakly essential as a left \(N\)-subgroup. It is to be noted that an essential left \(N\)-subgroup \(A\) of \(N\) is also weakly essential. That the converse is not true is shown in example below.

**Example 4.** (H(37), Page 341-342 [11]) : Consider the near-ring \(S3 = \{0, a, b, c, x, y\}\) with operation addition [defined in table 1.3 (i)] and multiplication defined by the following table.

\[
N = \{0, a, b, c, x, y\}\) is a near–ring under the operations defined by the following tables.

Here non-zero left \(S3\)-subgroups are \(\{0, a\}, \{0, b\}, \{0, c\}, \{0, x, y\}\) and \(S3, \{0, x, y\}\) and \(S3\) are the only non-zero left ideals. This shows that the \(S_{2}\)-subgroup \(\{0, x, y\}\) is weekly essential but not an essential left \(S3\)-subgroup.

However, the following example is sufficient to show the existence of near-ring where every weakly essential left \(N\)-subgroup is also essential.

3.2.16. **Example** (J(91), Page 343[11]) :

\(N = \{0, 1, 2, 3, 4, 5, 6, 7\}\) is a near-ring under addition modulo 8 and multiplication defined by the following table

\[
\begin{array}{cccccc}
+ & 0 & a & b & c & x \\
0 & 0 & a & b & c & x \\
a & a & 0 & y & x & c \\
b & b & y & 0 & a & c \\
c & c & x & 0 & b & a \\
x & x & c & a & 0 & y \\
y & y & b & c & a & 0
\end{array}
\]

\[
\begin{array}{cccccc}
\circ & 0 & a & b & c & x \\
0 & 0 & 0 & 0 & 0 & 0 \\
a & 0 & a & b & c & 0 \\
b & 0 & a & b & c & 0 \\
c & 0 & a & b & c & 0 \\
x & 0 & 0 & 0 & 0 & 0 \\
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Here \{0, 4\} and \{0, 2, 4, 6\} are the left \(N\)-subgroup of \(N\) whereas the second one is the only non-zero proper left ideal of \(N\). Thus each of them is weakly essential and they are essential too.

**Example 5.** (J (22), Page - 342 - 343 [11]):

The group \(N = \{0,1,2,3,4,5,6,7\}\) under addition modulo 8 is an \(N\)-group w.r.t. the multiplication defined by the following table

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\(N\)-group \(NN\) has non-trivial \(N\)-subgroups \{0, 4\} and \{0,2,4,6\}. Hence each of them has non-zero intersection with other \(N\)-subgroups of \(NN\) and so each of them is an essential \(N\)-subgroup of \(NN\). Also \(0, 4 \subseteq e \{0, 2, 4, 6\}\) which shows the validity of the following lemma 2.2.2.

**2.2.2 Lemma:** If \(A, B, C\) are \(N\)-subgroup of \(E\) such that \(A \subseteq B \subseteq C\) then \(A \subseteq e B \subseteq e C\) if and only if \(A \subseteq e C\).

**Proof:** Let \(P\) be a non-zero \(N\)-subgroup of \(E\) such that \(P \subseteq C\). Since \(B \subseteq e C\), \(B \cap P \neq (0)\).

Also, \(B \cap P \subseteq B\) and \(A \subseteq e B\). So \((B \cap P) \cap A \neq (0)\).

Therefore, \(P \cap A \subseteq (B \cap P) \cap A \neq (0)\).

Hence \(A \subseteq e C\).

Conversely, let \(A \subseteq e C\). Then \(A \cap B \neq (0)\), (for \(B \subseteq C\)).

If \(M\) is a non-zero \(N\)-subgroup of \(E\) such that \(M \subseteq B \subseteq C\) then, \(M\) is a non zero \(N\)-subgroup of \(C\). Since \(A \subseteq e C\), it follows that \(A \cap M \neq (0)\) which gives \(A \subseteq e B\).

Again, if \(H\) is any non-zero \(N\)-subgroup of \(E\) with \(H \subseteq C \subseteq E\) then \(A \cap H \neq (0)\), (for \(A \subseteq e C\)).

So, \(A \subseteq B \Rightarrow (0) \neq A \cap H \subseteq B \cap H\).

Thus, \(B \subseteq e C\).

**2.2.3 Lemma:** Let \(A\) and \(B\) be two \(N\)-subgroups of \(E\) such that \(B \subseteq e A\). If \(a (\neq 0) \in A\) then there exists an essential \(N\)-subgroup \(L\) of \(NN\) such that \(La \neq (0)\).

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Proof: Write \( L = \{ n \in N \mid na \in B \} \). Clearly, \( La \subseteq B \subseteq A \) and \( Na \subseteq A \) as \( A \) is an \( N \)-subgroup of \( E \), \( a \in A \).

Since \( 1 \in N \), \( Na \neq (0) \). Again, \( B \subseteq A \) gives \( B \cap Na \neq (0) \).

Let \( (0 \neq) b \in B \cap Na \). Then \( B = na \) (say) for \( n \in N \). Thus \( b = na \in B \) which gives \( n \in L \). Hence \( b = na \in La \).

Therefore, \( La \neq (0) \) (for \( b \neq 0 \)).

Now, let \( x, y \in L \) then \( xa, ya \in B \).

So, \( (x - y) a = xa - ya \in B \).

\[ \Rightarrow x - y \in L \]  \( \ldots (i) \)

Also, since \( B \) is an \( N \)-group of \( E \), for \( n \in N \), \( (nx) a = n(xa) \in B \) (for \( xa \in B \))

Therefore, \( nx \in L \).  \( \ldots (ii) \)

Thus \( L \) is an \( N \)-subgroup of \( NN \).

Again, for an \( N \)-subgroup \( I \) ( \( \neq 0 \) ) of \( NN \),

\[ Ia = (0) \]

\[ \Rightarrow Ia \subseteq B \]

\[ \Rightarrow I \subseteq L \]

\[ \Rightarrow L \cap I = I \neq (0) \]

and, \( Ia \neq (0) \)

\[ \Rightarrow B \cap Ia \neq (0), \) (for \( Ia \) is an \( N \)-subgroup of \( A \) and \( B \subseteq A \)).

Now, let \( (\neq) x \in B \cap Ia \) then \( x = b = \alpha a \) for \( b \in B, \alpha \in I \).

Then \( \alpha a \in B \)

\[ \Rightarrow \alpha \in L, \) (by choice of \( L ) \]

\[ \Rightarrow \alpha \in L \cap I \]

Now, \( \alpha = 0 \Rightarrow x = 0 \), a contradiction.

So, \( L \cap I \neq (0) \).

Therefore, \( L \) is an essential \( N \)-subgroup of \( NN \) such that \( La \subseteq B \) and \( La \neq (0) \).//

In an \( N \)-group \( E \), the singular \( N \)-subset of \( E \) viz., the subset \( Z1(E) = \{ u \in E \mid Lu = (0) \} \), for some essential \( N \)-subgroup \( L \) of \( NN \) plays an important role in our discussion.

\( N \)-group \( E \) is \( N \)-non-singular if \( Z1(E) = 0 \) and \( N \) is \( left \) non-singular if \( Z1(N) = 0 \). It is to be noted that \( Z1(E) \) is an \( N \)-subset of \( E \) and \( Z1(N) \) is an invariant subset of \( N \).

2.2.4. Lemma: For an \( x \in E \), \( Ann(x) \) is an essential \( N \)-subgroup of \( NN \) if and only if \( x \in Z1(E) \). [easy]

2.2.5. Lemma: If \( I \) is an \( N \)-subgroup of \( NN \) and for \( B \subseteq E \), \( Ann(B) \subseteq I \) and \( Z1(E) = (0) \) then \( Ann(B) = I \).

Proof: Let \( (\neq) x \in I \) then by 2.2.3, there exists an essential \( N \)-subgroup \( L \) of \( NN \) such that \( Lx \neq (0) \), \( Lx \subseteq Ann(B) \).

So, \( (Lx) r_E(Ann(B)) \subseteq Ann(B) \)

\[ \Rightarrow L(x r_E(Ann(B)) = (0) \Rightarrow (x r_E(Ann(B)) = (0) \) [for \( Z1(E) = (0) \)]

\[ \Rightarrow x \in Ann(r_E(Ann(B))) = Ann(B) \]

\[ \Rightarrow L \subseteq Ann(B) \]

Now considering the hypothesis, we get \( Ann(B) = I \).//

2.2.6. Lemma: Let \( E \) be with acc on annihilators such that \( E \) is \( N \)-non-singular (i.e. \( Z1(E) = (0) \)). If \( N \) has no infinite direct sum of left ideals and every essential left ideal of \( N \) is an essential \( N \)-subset of \( NN \) then \( N \) satisfies the dcc on annihilators of subsets of \( E \).

Proof: Let \( X \) and \( Y \) be subsets of \( E \) such that \( B = Ann(X) \) and \( C = Ann(Y) \). Thus, \( B, C \) are \( N \)-subgroups of \( NN \).

Now, if \( B \subseteq C \) and \( B \) is an essential \( N \)-subgroup of \( C \) then by 2.2.5, \( B = C = Ann(X) \). Hence \( B \) is not an essential \( N \)-subgroup of \( C \). So, there exists an \( N \)-subgroup \( D(\neq 0) \) of \( NN \) such that \( D \subseteq C \), \( B \cap D = (0) \).
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Let \( A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \) be a strictly descending chain of annihilators of subsets of \( E \). Since \( A_1 \supseteq A_{i+1} \), by the above argument, there exists an \( N \)-subgroup \( P_i (\neq 0) \) of \( NN \) such that \( P_i \subseteq A_i \) and \( A_{i+1} \cap P_i = (0) \) .......... (i)

Consider \( M = \{ X_m \} \), the family of all left ideals of \( N \) such that \( A_{i+1} \cap X_m = (0) \). The union of each chain of \( M \) is again a left ideal in \( M \) and satisfies the condition \( A_{i+1} \cap X_m = (0) \). Thus, by Zorn’s Lemma, \( M \) has a maximal element \( X_i \) (say) such that \( A_{i+1} \cap X_i = (0) \) .......... (ii)

Again, \( A_{i+1} \) and \( X_i \) being left ideals of \( N \), \( A_{i+1} + X_i \) is also a left ideal of \( N \).

Now, let \( V \) be a left ideal of \( N \) such that \( (A_{i+1} + X_i) \cap V = (0) \).

Now, \( a_{i+1} = x + v \), for some \( a_{i+1} \in A_{i+1}, x \in X_i, v \in V \).

\[ \Rightarrow v = -x + a_{i+1} \in X_i + A_{i+1} \subseteq A_{i+1} + x \]

\[ \Rightarrow v \in (A_{i+1} + X_i) \cap V = (0) \]

\[ \Rightarrow a_{i+1} = x \in A_{i+1} \cap X_i = (0) \]

\[ \Rightarrow A_{i+1} \cap (X_i + V) = (0) \]

Since \( X_i \) is maximal with condition \( A_{i+1} \cap X_i = (0) \), it follows that \( X_i + V = X_i \) as \( X_i \subseteq X_i + V \). This gives \( V \subseteq X_i \) and so \( V = V \cap X_i \subseteq V \cap (A_{i+1} + X_i) = (0) \). 

Thus, \( A_{i+1} + X_i \) is an essential left ideal of \( N \) such that \( A_{i+1} \cap X_i = (0) \) and the assumed hypothesis gives that \( A_{i+1} + X_i \) is an essential \( N \)-subgroup of \( NN \). And so for \( P_i \), chosen above, \( P_i \cap (A_{i+1} + X_i) = (0) \).

Suppose, \( \alpha \in (P_i) = a_{i+1} + x_i \), for \( \alpha \in P_i, a_{i+1} \in A_{i+1}, x_i \in X_i \).

Then, \( x_i = -a_{i+1} + P_i \subseteq A_{i+1} + P_i \subseteq A_i + P_i \), for \( A_i \subseteq A \). So, \( x_i \in A_i \) (for \( P_i \subseteq A_i \)) which gives \( x_i \in A_i \cap X_i \).

Now, if \( x = 0 \) then \( P_i \subseteq A_{i+1} \) which gives \( P_i \subseteq A_{i+1} \). \( P_i = (0) \). So, \( P_i = 0 \).

Therefore, \( P_i \cap (A_{i+1} + X_i) = (0) \) and this is a contradiction. Hence \( x \neq 0 \) and therefore \( A_{i+1} \cap X_i \neq (0) \).

Let \( C_i = A_i \cap X_i \), a non-zero left ideal of \( N \).

Then, \( C_i \cap A_{i+1} = (A_i \cap X_i) \cap A_{i+1} \)

\[ = (A_{i+1} \cap A_i) \cap X_i \]

\[ = A_{i+1} \cap X_i, \text{ (as } A_1 \supseteq A_{i+1} \text{)} \]

\[ = (0), \text{ [by (ii)]} \]

Therefore, when \( A_i \supseteq A_{i+1} \), we get a non-zero ideal \( C_i = A_i \cap X_i \) such that \( C_i \cap A_{i+1} = (0) \) ..........(iii)

Now, for different values of \( i \), we get an infinite family \( \{ C_1, C_2, C_3, \ldots \} \) of non-zero left ideals of \( N \) such that (iii) holds.

Also, \( C_i = A_i \cap X_i \subseteq A_i \) ..........(iv)

Therefore, \( C_1 \cap C_2 \subseteq C_1 \cap A_2 = (0), \text{ [by (iii) and (iv)]} \]

Again, \( C_1 \cap (C_2 + C_3) \subseteq C_1 \cap (A_2 + A_3), \text{ [by (iv)]} \)

\[ \subseteq C_1 \cap A_2, \text{ as } A_2 \supseteq A_3 \]

\[ = (0), \text{ [by (iii)]} \]

\[ \Rightarrow C_1 \cap (C_2 + C_3) = (0) \] ..........(v)

And if \( x \in C_2 \cap (C_1 + C_3) \) then

\[ x = c_2 = c_1 + c_3, \text{ for } c_i \in C_i, i = 1,2,3. \]

\[ \Rightarrow c_1 = c_2 - c_3 \in C_2 + C_3 \]

So, \( c_1 \in C_2 \cap (C_2 + C_3) = (0), \text{ [by (v)]} \)

\[ \Rightarrow c_1 = 0 \text{ and } c_2 = c_3 \in C_3. \]

\[ \Rightarrow C_2 \subseteq C_2 \cap C_3 \subseteq C_2 \cap A_3 = (0), \text{ [by (iii) and (iv)]} \]

\[ \Rightarrow c_2 = 0 \text{ and hence } C_2 \cap (C_1 + C_3) = (0). \]

Similarly, \( C_3 \cap (C_1 + C_2) = (0). \text{ Thus } C_1 \oplus C_2 \oplus C_3 \text{ is a direct sum of non-zero left ideals of } N. \)
Proceeding in this way, we find an infinite direct sum 
\[ C_1 \oplus C_2 \oplus C_3 \oplus \ldots \] of nonzero left ideals of \( N \). This goes against our hypotheses and hence there exists a \( t \in \mathbb{Z}^+ \) such that \( At = At+1 = At+2 = \ldots \) Therefore, \( N \) satisfies the dcc on annihilators of subset of \( E \).

2.2.7. Lemma : \( Z_1(N) = \{ x \in N | Ax = (0), \text{for some essential left } N\text{-subgroup } A \text{ of } N \} \) is an invariant subset of \( N \).

Proof : Let \( x \in Z_1(N) \). Then \( Ax = (0) \), for some essential left \( N \)-subgroup \( A \) of \( N \). So, by 2.2.3, for any \( n \neq 0 \in N \), there exists an essential left \( N \)-subgroup \( L \) of \( N \) such that 
\[ L_n \subseteq A, \quad L_n \neq \{0\}. \]

This gives, 
\[ (L_n x) \subseteq A x = (0) \]
\[ (x L_n) \subseteq A x = (0) \]
\[ x \in Z_1(N) \]

2.2.8. Lemma : A strongly semiprime near-ring \( N \) with acc on left annihilators has no non-zero nil left \( N \)-subset of \( N \).

Proof : Let \( A \) be any non-zero left \( N \)-subset of \( N \). Since \( N \) satisfies the acc on left annihilators, we can choose a \( (\neq 0) \) \( E \) with \( 1(a) \) as large as possible.

Now, \( aN_a = (0) \)
\[ \Rightarrow (Na)^2 = (Na)(Na) = N(aNa) = (0) \]
And \( Na \) being a non-zero left \( N \)-subset of \( N \), \( N \{1(\neq 0) \} \), we meet a contradiction to 3.2.5.[\( N \) being strongly semi prime has no non-zero nilpotent left or right \( N \)-subset]

So, \( aN_a \neq (0) \).

Let \( x \in N \) be such that \( axa \neq 0 \)

Now, \( xa \neq 0 \) (otherwise \( axa = 0 \)
\[ x \neq 1(a) \]

Again, \( z \in 1(a) \Rightarrow za = 0 \)
\[ \Rightarrow z(aza) = (za)xa = 0 \]
\[ \Rightarrow z \in 1(aza) \]
\[ 1(\neq 0) \in 1(aza) \]

But \( 1(\neq 0) \) being maximal, \( 1(aza) = 1(a) \)

So, \( x^2 \notin 1(axa) \)
\[ \Rightarrow x(axa) = 0 \]
\[ \Rightarrow (xa)^2 = 0 \]
\[ \Rightarrow (xax)a = 0 \]
\[ \Rightarrow xax1(a) = 1(axa) \]
\[ \Rightarrow (xax)(xax) = 0 \]
\[ \Rightarrow (xa)^3 \neq 0 \] and so on.

Thus, \( (xa)^t \neq 0 \), for any \( t \in \mathbb{Z}^+ \).

Therefore, \( A \) possesses a non-zero non nilpotent element \( xa \). So \( A \) is not nil.

Hence \( N \) does not have any non-zero nil left \( N \)-subset of \( N \).

2.2.9. Lemma : If \( N \) is a strongly semiprime near-ring with acc of left annihilators then \( N \) is left non-singular.

Proof : Being \( N \) acc with acc on left annihilators, \( Z_1(N) \) is a nil invariant subset of \( N \) and by above it follows that \( Z_1(N) = 0 \). Thus the result follows.

Again \( N \) being strictly left Goldie, it is left Goldie. So it has no infinite direct sum of left ideals. And therefore as a special case of 2.2.6, we get the following ( [5], Nat, Acad,Sci. Letters.)

2.2.10. Theorem : If in a strongly semiprime strictly left Goldie near-ring \( N \), every weakly essential left \( N \)-subgroup of \( N \) is also essential, then \( N \) satisfies the dcc on left annihilators.

And now we get the following effective result for our purpose.
2.2.11 Corollary: In a strongly semiprime strictly left Goldie near-ring $N$, if every weakly essential left $N$-subgroup of $N$ is also essential, then $N$ satisfies the a.c.c. on left as well as right annihilators.

3. Main Result

3.1. Theorem

Suppose $N$ satisfies the acc on left annihilators and $I$ is a nil subring of $N$ and $I$ is not left $s$-nilpotent. Then there exists a sequence $<a_i>, a_i \in N$ such that $N a_i \neq 0$ and the family $\{N a_i\}$ is an independent family, or the sum $N a_1 + N a_2 + \cdots$ is direct.

Proof: It is assumed that $I$ is not $s$-nilpotent. Then there is an element $y \in I$ such that $y$ has an infinite chain $<y_1>, y_1 \in I$ with $(y_{y_1} y_{y_2} \cdots y_{k-1} y_k \neq 0, \forall k)$. We now consider the following

We have $y_1 \in I$ such that $y_1 y \neq 0$,

$y_2 \in I$ such that $y_1 y_2 y \neq 0$

$h_2 = y_2 \cdots$ And so on.

So we clearly have the following possibilities.

There exists $x \in I$ such that $xy \neq 0$ (for example $y_1$ is such an element, and we may have more than one such element!)

There exists $x \in I$ such that $x y \neq 0$ (for example $y_2$ is such an element, and we may have more than one such element!)

Thus it is possible to define a sequence $y_1, y_2, \ldots$ of $N$ such that

$K_1 = \{x \in I \mid y x$ has an infinite chain\}

$K_2 = \{x \in I \mid y y_1 x$ has an infinite chain\}

$K_3 = \{x \in I \mid y y_1 y_2 x$ has an infinite chain\}

In general

$K_n = \{x \in I \mid y y_1 y_2 \cdots y_{n-1} y_n x$ has an infinite chain\}

As $N$ satisfies the acc on left annihilators, now we consider the maximal element $l(y_a)$ with $y_a \in K_n$.

For each $i$, $l(y_i) = l(y_1 y_2 \cdots y_i)$ (for all $j \geq I$)

In particular note that, $l(y_i) = l(y_1 y_2 y_i)$ (i.e., $l(y_i) \subseteq l(y_1 y_2 y_i)$)

As $x \in l(y_i)$, $x y \neq 0$, clearly $x y_1 y_2 y_i = 0$ which gives

easily, $x \in l(y_1 y_2 y_i)$ i.e., $l(y_1 y_2 y_i) \subseteq l(y_1 y_2 y_i)$

Now $y_1, y_2 \in K_1$ with $l(y_1)$ maximum

$y_2 \in K_2$ as $l(y_2)$ maximum

$y_3 \in K_3$ as $l(y_3)$ maximum

so, $y y_1 y_2 y_3$ has an infinite chain. And

i.e., $xy$ has an infinite chain (here, $x \in N$)

so, $x \in K_1$ i.e., $y y_1 y_2 y \in K_1$

And therefore, $l(y_1 y_2 y_3) \subseteq l(y_i)$

[using the maximality of $l(y_i)$]

Now * gives and ** give

$l(y_1 y_2 y_3) = l(y_i)$

(***)

We now set

$a_1 = y y_1, a_2 = y y_1 y_2, a_3 = y y_1 y_2 y_3$, etc.

In general, $a_n = y y_1 \cdots y_{n-1} y_n$.

Suppose $a_1 y \neq 0$, then $a_1 y y_1 y_2 \neq 0$; for if $a_1 y y_1 y_2 = 0$, then $a_1 \in l(y_1 y_2) \Rightarrow y_1 a_1 = 0$ [as $l(y_1) = l(y_1 y_2)$], hence, $a_1 y_1 = 0$.

Similarly, we have, $a_1 y y_2 y_3 \neq 0$; for if $a_1 y y_2 y_3 = 0 \Rightarrow a_1 \in l(y_1 y_2 y_3) = l(y_1)$

Claim $a_n y_1 = 0$

Suppose $a_n y_1 = 0$, for some $n$.

we now consider the case for any $k$. 

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\[ a_n \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k \neq 0, \text{ for } l(y_1) = l(y_1 \gamma_1 \gamma_2 \ldots \gamma_k) \]
\[ a_n \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k = 0 \Rightarrow a_n \in l(y_1 \gamma_1 \gamma_2 \ldots \gamma_k) = l(y_1) \]
\[ \Rightarrow a_n \gamma_k = 0, \text{ a contradiction} \]

so, \( a_n \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k = 0 \), i.e. \( (y_1 \gamma_1 \gamma_2 \ldots \gamma_k) (y_1 \gamma_1 \gamma_2 \ldots \gamma_k) \neq 0 \)

\( \Rightarrow (y_1 \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k y_1) = 0 \)

And this gives \( y_1 \gamma_1 \ldots \gamma_k \ldots \gamma_k \gamma_k \) forms a chain for \( y_1 \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k = y \gamma_1 \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k \), \( y \in \mathbb{N} \)

Moreover, \( \mathbb{N} \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k \gamma_k \gamma_k \) is such a family \( \mathbb{N} \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k \gamma_k \gamma_k \) is an independent family.

Theorem

Now, \( a_n \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k \neq 0 \), for all \( n \in \mathbb{N} \)

(iii) \( \text{ to show that the sum } N_a + N_b = \ldots \) is direct or the family \( N_a, N_b, \ldots \) is an independent family.

We first show that \( N_a \cap N_b = 0 \).

That is if \( n_1 a_1 = n_2 a_2 \) for some \( n_1, n_2 \in \mathbb{N} \), then \( n_1 a_1 = n_2 a_2 = 0 \)

Now we note that \( y \in I \) is such that \( y \) has an infinite sequence and choose \( l(y) \) to be maximum. And \( y_1 \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k \) is such that \( y_1 \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k \gamma_k \) is an infinite chain with \( l(y_1) \) is maximum.

And therefore, \( l(y) = l(y_1 \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k) \ldots \)

But clearly we have, \( l(y) \subseteq l(y_1 \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k) \ldots \)

Therefore, \( l(y) = l(y_1 \gamma_1 \gamma_2 \ldots \gamma_k \gamma_k) \).

\[ n_1 a_1 = n_2 a_2 \Rightarrow n_1 a_1 \gamma_1 \gamma_2 = n_2 a_2 \gamma_1 \gamma_2 = 0 \] (as \( a_2 \gamma_1 \gamma_2 = 0 \))

\[ \Rightarrow n_1 a_1 \gamma_1 \gamma_2 = 0 \Rightarrow n_2 a_2 \gamma_1 \gamma_2 = 0 \Rightarrow n_1 a_1 = n_2 a_2 = 0 \]

Thus \( N_a \cap N_b = 0 \), i.e. \( N_a + N_b = 0 \), and \( N_a + N_b = 0 \) is direct.

Similarly, the sum of all \( N_a \) is direct.

Now we’ll show that

3.2. Theorem

If \( \{ S_i = a_j \mid i \geq j \} \) then \( r(S_i) = 1,2 \ldots \) form a strictly ascending chain of right annihilators.

Proof: Here, \( S_1 = \{ a_j \mid a_j \geq 1 \} = \{ a_1, a_2, a_3, \ldots \} \), \( S_2 = \{ a_j \mid a_j \geq 2 \} = \{ a_2, a_3, a_4, \ldots \} \), \( S_3 = \{ a_j \mid a_j \geq 3 \} = \{ a_3, a_4, a_5, \ldots \} \), \ldots

\[ S_i = \{ a_j \mid a_j \geq i \} = \{ a_i, a_{i+1}, a_{i+2}, \ldots \} \]

Now, \( S_1 x = 0 \Rightarrow a_1 x = a_2 x = \ldots = 0 \) and this \( \Rightarrow a_1 x = a_2 x = \ldots = 0 \)

\( S_2 x = 0 \Rightarrow r(S_1) \subseteq r(S_2) \),

similarly, \( r(S_1) \subseteq r(S_2) \subseteq r(S_3) \subseteq r(S_4) \ldots \)

Again, \( a_2 y = 0 \) and \( a_2 y_1 = a_4 y_2 = \ldots = 0 \) but \( a_1 y_1 \neq 0 \) (for \( y_1 y_2 \neq 0 \)).
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Hence, \( y_2 \in \tau(S_2) \) and \( y_2 \notin \tau(S_1) \). And therefore, \( r(S_1) \subseteq r(S_2) \), similarly, \( r(S_2) \subseteq r(S_3) \), i.e.

\[
\begin{align*}
r(S_1) & \subseteq r(S_2) \subseteq r(S_3) \subseteq \ldots \\
\end{align*}
\]

is a strictly ascending infinite chain of left annihilators.

**III. Main result**

Now we prove the main results that we are aiming for.

**3.3 Theorem:** If \( I \) is not nilpotent, then \( I \) is not s-nilpotent.

**[Note:** so, if \( I \) is s-nilpotent the \( I \) is nilpotent, and if \( I \) is nil then it is s-nilpotent]**

**Proof:** We consider \( I, I^2, I^3, \ldots \) and clearly, we have

\[
I \supseteq I^2 \supseteq I^3 \supseteq \ldots \text{ and therefore,}
\]

\[
l(I) \subseteq l(I^2) \subseteq l(I^3) \subseteq \ldots
\]

by acc on left annihilators, we have an integer , say \( k \), such that

\[
l(I^k) = l(I^k)
\]

for all \( s \geq k \).

So if we set \( K=I^k \), then \( l(K) = l(K^2) \). And \( K^2 \neq 0 \) (for if \( K^2 = 0 \), then \( I \) appears as nilpotent, which is not true.

And thus \( K^2 \neq 0 \), that gives an \( x_i \in K \) such that \( x_iK \neq 0 \).

And this gives \( x_iK^2 \neq 0 \). For if \( x_iK^2 = 0 \),

then \( x_iK \neq 0 \), a contradiction.

Now, \( x_iK \neq 0 \) \( \Rightarrow x_i \in K \) such that \( x_i, x_iK \neq 0 \). And so on.

Thus we get \( x_1, x_2, \ldots \) are such that each of \( x_1, x_1x_2, x_1x_2x_3, \ldots \) is non zero.

Therefore, the sequence \( < x_n > \) is such that

each \( x_n \in I \) and \( \ldots x_kx_{k+1} \neq 0 \).

Hence, \( I \) is not s-nilpotent.

**3.4 Theorem:** If \( N \) is a strongly semiprime strictly left Goldie near-ring where every weakly essential left \( N \)-subgroup of \( N \) is also essential, then the each nil-subring of \( N \) is nilpotent.

**Proof:** Let \( I \) be a nil subnear-ring of \( N \). (to prove that \( I \) is nilpotent!).

Suppose, \( I \) is not nilpotent. Then by above \( I \) is not s-nilpotent.

Then we have an infinite sequence \( a_1,a_2,a_3,\ldots \) in \( N \) such that each \( Na_i \) is non zero and their sum is direct, and the chain

\[
r(S_1) \subseteq r(S_2) \subseteq r(S_3) \subseteq \ldots
\]

is a strictly infinite ascending chain of right annihilators. Now as \( N \) is with the acc on right annihilators (2.2.11 Corollary) such a sequence is not possible. Thus, \( I \) must be nilpotent.

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