A Study on the Rate of Convergence of Chlodovsky-Durrmeyer Operator and Their Béziers Variant

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Abstract: In this paper, we have studied the Bézier variant of Chlodovsky-Durrmeyer operators $D_{m,\theta}$ for function $f$ measurable and locally bounded on the interval $[0,\infty)$. In this we improved the result given by Ibikl E. And Karshi H. [14]. We estimate the rate of pointwise convergence of $(D_{m,\theta}f)(x)$ at those $x > 0$ at which the one-sided limits $f(x+), f(x-)$ exist by using the Chanturia modulus of variation. In the special case $\theta = 1$ the recent result of Ibikl E. And Karshi H. [14] concerning the Chlodovsky-Durrmeyer operators $D_m$ is essentially improved and extended to more general classes of functions.

Keywords: Rate of convergence, Chlodovsky-Durrmeyer operator, Bézier basis, Chanturia modulus of variation, $p$-th power variation.

I. Introduction

For a function the classical Bernstein-Durrmeyer operators (see [7]) $M_n$ applied to $f$ are defined as

$$(M_m f)(x) = (m + 1) \sum_{i=0}^{m} p_m(x) \int_{0}^{x} f(t) p_{m,i}(t) dt, \quad x \in [0,1]$$

where $p_{m,i}(x) = \binom{m}{i} x^i (1-x)^{m-i}$.

Several researchers have studied approximation properties of the operators $M_n$ ([8], [10]) for function of bounded variation defined on the interval $[0, 1]$. After that Zeng and Chen [22] defined the Bézier variant of Durrmeyer operators as

$$(M_{m,\theta} f)(x) = (m + 1) \sum_{i=0}^{m} Q_{m,i}^{(\theta)}(x) \int_{0}^{t} f(t) p_{m,i}(t) dt,$$

where $Q_{m,i}^{(\theta)}(x) = \frac{\theta}{\theta + 1} f(x) - \frac{1}{\theta + 1} f(x+1)$ and $f_i(x) = \sum_{j=0}^{1} p_{m,i}(x)$ for $i = 0, 1, 2, \ldots, m$.

Recently Agratini [1], Anioli and Pych-Taberska [3], Pych-Taberska [20], and Gupta [11, 12] have investigated the rate of pointwise convergence for Kantorovich and Durrmeyer Type Baskakov–Bézier and Bézier operators using a different approach. They have proved their theorems in terms of the Chanturia modulus of variation, which is a generalization of the classical Jordan variation. It is useful to point out that a deeper analysis of the Chanturia modulus of variation can be found in [6], but actually the modulus of variation was introduced for the first time by Langrange [18]. Although the Chanturia modulus of variation was defined as a
generalization of the classical variation nearly four decades years ago, it was not used to a sufficient extent to solve the problem mentioned above.

The paper is concerned with the rate of pointwise convergence of the operators (5) when \( f \) belongs to \( X_{loc}(0, \infty) \). Using the Chanturia modulus of variation defined in [6], we examine the rate of pointwise convergence of \((D_{m, \beta}(x))\) at the points of continuity and at the first kind discontinuity points of \( f \).

For some important papers on different operators related to the present study we refer the readers to Gupta et. Al. [9, 21] and Zeng and Piriou [23]. It is necessary to point out that in the present paper we extend and improve the result of Ibikili E. and Karshi H.[14] for Chlodowsky-Durrmeyer operators.

We being by giving

**Definition 1.1** Let \( f \) be a bounded function on a compact interval \( I = [a, b] \). The modulus of variation \( \mu_m(f; [a, b]) \) of a function \( f \) is defined for nonnegative integers \( m \) as

\[
\mu_m(f; [a, b]) = \sup_{x_m} \sum_{i=0}^{m-1} |f(x_{2i+1}) - f(x_{2i})|,
\]

where \( \pi_m \) is an arbitrary system of \( m \) disjoint intervals \( (x_{2i}, x_{2i+1}) \), \( i = 0, 1, ..., m-1 \), i.e.,

\[
a \leq x_0 < x_1 \leq x_2 < x_3 \leq \cdots \leq x_{2m-2} < x_{2m-1} \leq b.
\]

The modulus of variation of any function is a non-decreasing function of \( m \). Some other properties of this modulus can be found in [6].

If \( f \in BV_p(I) \), \( p \geq 1 \), i.e., if \( f \) of \( p \)-th bounded power variation on \( I \), then for every \( i \in \mathbb{N} \),

\[
\mu_i(f; I) \leq l^{1-1/p}V_p(f, I),
\]

where \( V_p(f, I) \) denotes the total \( p \)-th bounded power variation of \( f \) on \( I \), defined as the upper bound of the set of numbers \( \left\{ \sum |f(i_j) - f(l_j)|^p \right\}^{1/p} \) over all finite systems of non-overlapping intervals \( (i_j, l_j) \subset I \).

We also consider the class \( BV_{loc}^p(0, \infty) \), \( p \geq 1 \), consisting of all function of bounded \( p \)-th power variation on every compact interval \( I \subset [0, \infty) \).

In the sequel it will be always assumed that the sequence \( (a_m) \) satisfies the fundamental conditions (4). The symbol \( [a] \) will be denote the greatest integer not greater than \( a \).

**Remark.** Now, let us consider the special case \( \beta = 1, p = 1 \), and let us suppose that function \( f \) is of bounded variation in the Jorden sense on the whole interval \([0, \infty)\)

\( (f \in BV[0, \infty)) \). Then, for all integers \( m \) such that \( a_m > 2x \) and \( 4a_m \leq m \), we have the following estimation for the rate of convergence of the Chlodowsky-Durrmeyer operators (3):

\[
\left| (D_{m, \theta}(x)) - \frac{f(x) + f(x-)}{2} \right| \leq 2V \left( g; H(xa_m/m) \right) + \frac{2l^{m/a_m}x}{m} \sum_{i=0}^{m} V \left( g; H(xa_m/m) \right) + \frac{4M}{m} \sum_{i=0}^{m} V \left( g; H(xa_m/m) \right) \leq 2V \left( g; H(xa_m/m) \right),
\]

where \( M = \sup_{0 \leq x < \infty} |f(x)| \) and \( V(g; H) \) denotes the Jordan variation of \( g \) on the interval \( H \).

The above estimation is essentially better than the estimation presented in [14]. Namely, it is easy to see that the right-hand side of the main inequality given in Theorem 1.1 in [14] is not convergent to zero for all function \( f \in BV[0, \infty) \) and for all sequences \( (a_m) \) satisfying (4).

**II. Auxiliary Result**

In this section we give certain results, which are necessary to prove the main result.

For this, let us introduce the following notation. Given any \( x \in [0, a_m] \) and any non-negative integer \( q \), we write

\[
W^{q}_{m,q}(x) := (D_{m, \theta}^q(x)) \equiv \frac{m + 1}{a_m} \sum_{l=0}^{m} P_{m,l} \left( \frac{x}{a_m} \right) [ (t - x)^q dt.
\]

**Lemma 2.1** If \( m \in \mathbb{N}, x \in [0, a_m] \), then

\[
W_{m,0}(x) = 1, \quad W_{m,1}(x) = \frac{a_m - 2x}{m + 2}.
\]

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and, for \( q > 1 \),

\[
W_{m,2q}(x) = \frac{a_m}{m} \sum_{j=0}^{q} \beta_{j,q}(\frac{x - x}{a_m})^{q-j} \left( \frac{a_m}{m} \right)^j,
\]  \( (8) \)

where \( \beta_{j,q} \) are real numbers independent of \( x \) and bounded uniformly in \( m \). Moreover, for \( m \geq 2 \)

\[
W_{m,2q}(x) \leq 2 \frac{a_m}{m} \left( 1 - \frac{x}{a_m} \right) + \frac{a_m}{m}
\]

and, for \( q > 1 \),

\[
W_{m,2q}(x) \leq c_q \left( \frac{a_m}{m} \right)^q \left( 1 - \frac{x}{a_m} \right) + \frac{a_m}{m} q,
\]  \( (10) \)

where \( c_q \) is a positive constant depending only on \( q \).

**Proof.** Formulas for \( W_{m,0}, W_{m,1}, W_{m,2} \) and inequality (9) follow by simple calculation. Suppose \( q > 1 \) and put \( y := x/a_m \). Then \( y \in [0,1] \) and

\[
W_{m,2q}(x) = \frac{m+1}{a_m} \sum_{i=0}^{m} P_{m,i}(y) \int_0^{a_m} (t - ya_m)^{2q} P_{m,i} \left( \frac{t}{a_m} \right) dt
\]

\[
= (m+1)a_m^{2q} \sum_{i=0}^{m} P_{m,i}(y) \int_0^{1} (s - y)^{2q} P_{m,i}(s) ds = a_m^{2q} (M_m \psi_q^2)(y),
\]  \( (11) \)

where \( M_m \) is the classical Bernstein-Durrmeyer operator (1).

The representation formula (8) follows at once from the known identity

\[
(M_m \psi_q^2)(y) = \sum_{j=0}^{q} \beta_{j,q,m} \left( \frac{y(1-y)}{m} \right)^{q-1} m^{-2j},
\]

where \( \beta_{j,q,m} \) are real numbers independent of \( y \) and bounded uniformly in \( m \) (see [13]). Lemma 4.8 with \( c = -1 \). Now, let us observe that for \( y \in [0,1/m] \), \( y \in \left[ 1 - \frac{1}{m}, 1 \right], m \geq 2 \), one has \( y(1-y) \leq \frac{m-1}{m^2} \) and

\[
(M_m \psi_q^2)(y) = \sum_{j=0}^{q} \beta_{j,q,m} \left( \frac{m-1}{m^3} \right)^{j-1} m^{-2j} \leq \sum_{j=0}^{q} \eta_{j,q} m^{-2q},
\]

where \( \eta_{j,q} \) are non-negative numbers depending only on \( j \) and \( q \). If \( y \in \left[ \frac{1}{m}, 1 - \frac{1}{m} \right] \), then \( \frac{1}{my(1-y)} \leq \frac{m-1}{m^2} \leq 2 \) and

\[
(M_m \psi_q^2)(y) = \left( \frac{y(1-y)}{m} \right)^{q} \sum_{j=0}^{q} \beta_{j,q,m} \left( \frac{1}{my(1-y)} \right)^{j},
\]

\[
\leq \left( \frac{y(1-y)}{m} \right)^{q} \sum_{j=0}^{q} \eta_{j,q} 2^{j}.
\]

Consequently,

\[
(M_m \psi_q^2)(y) \leq C_q \left( \frac{y(1-y)}{m} + \frac{1}{m} \right)^{q} \text{ with } C_q = \sum_{j=0}^{q} \eta_{j,q} 2^{j}.
\]

Taking in (11) and in the above inequality \( y = x/a_m \) we easily obtain estimation (10).

**Lemma 2.2** Let \( x \in (0, \infty) \) and let

\[
K_{m,\varphi} \left( \frac{x}{a_m}, \frac{t}{a_m} \right) := \frac{m+1}{a_m} \sum_{i=0}^{m} P_{m,i} \left( \frac{x}{a_m} \right) \left( \frac{t}{a_m} \right)
\]

Then

\[
\int_{0}^{a_m} K_{m,\varphi} \left( \frac{x}{a_m}, \frac{u}{a_m} \right) du \leq \frac{\varphi}{(t-x)^2} W_{m,2}(x) \text{ if } x < t \]

\( (12) \)

and

\[
\int_{0}^{t} K_{m,\varphi} \left( \frac{x}{a_m}, \frac{u}{a_m} \right) du \leq \frac{\varphi}{(x-t)^2} W_{m,2}(x) \text{ if } 0 < t < x,
\]  \( (13) \)
where $W_{m,2}(x)$ is given by (7) (with $q = 2$).

**Proof.** As is known $Q_{m,j}^{(0)}(x) \leq \partial P_{m,j}(x)$ for $\partial \geq 1$. Hence, if $x < t$, then

$$
\int_{t}^{q_{m}} K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{t}{a_{m}} \right) dt \leq \frac{1}{(t-x)^{2}} \int_{t}^{q_{m}} (u-x)^{2}K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{u}{a_{m}} \right) du
$$

$$
\leq \frac{1}{(t-x)^{2}} (D_{\partial,\partial} \frac{1}{2})(x) \leq \frac{\partial}{(t-x)^{2}} W_{m,2}(x).
$$

The proof of (13) is similar.

In order to formulate the next lemma we introduce the following intervals. If $x > 0$, we write

$$
I_{x}(u) := [x + u, x] \cap [0, \infty) \text{ if } u < 0
$$

$$
I_{x}(u) := [x, x + u] \text{ if } u > 0
$$

**Lemma 2.3** Let $f \in \mathcal{A}_{1,1}[0, \infty)$ and let the one-sided limits $f(x +), f(x -)$ exist at a fixed point $x \in (0, \infty)$. Consider the function $g_{x}$ defined by (6) and write $d_{m} = \sqrt{a_{m} / m}$. If $h = -x$ or $h = x$, then for all integers $m$ such that $d_{m} \leq 1/2$ we have

$$
\left| \int_{I_{x}(t)} g_{x}(t)K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{t}{a_{m}} \right) dt \right| \leq v_{1}(g_{x}; I_{x}(h d_{m}))
$$

$$
+ 8\partial W_{m,2}(x) \sum_{j=1}^{n} \frac{v_{j}(g_{x}; I_{x}(j h d_{m}))}{n^{2}} + v_{n}(g_{x}; I_{x}(0) h d_{m})
$$

where $n = [1/d_{m}]$ and $W_{m,2}(x)$ is estimated in (9).

**Proof.** Restricting the proof to $h = -x$ we define the point $t_{j} = x + j h d_{m}$ for $j = 1, 2, 3, ..., n + 1$ and we denote $t_{n+1} = 0$. Put $T_{j} = [t_{j}, x]$ for $j = 1, 2, 3, ..., n + 1$ and we have

$$
\left| \int_{I_{x}(t)} g_{x}(t)K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{t}{a_{m}} \right) dt \leq \int_{t_{1}}^{x} (g_{x}(t) - g_{x}(x))K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{t}{a_{m}} \right) dt
$$

$$
+ \sum_{j=1}^{n} \left( g_{x}(t_{j}) - g_{x}(t_{j-1}) \right) K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{t}{a_{m}} \right)
$$

Clearly,

$$
|I_{1}(m, x)| \leq \int_{t_{1}}^{x} \left| (g_{x}(t) - g_{x}(x))K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{t}{a_{m}} \right) dt
$$

By the Abel lemma on summation by parts and by (13) we have

$$
|I_{2}(m, x)| \leq \left| g_{x}(t_{1}) \right| \int_{0}^{t_{1}} K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{t}{a_{m}} \right) dt + \sum_{j=1}^{n-1} \left| g_{x}(t_{j}) - g_{x}(t_{j-1}) \right| \int_{0}^{t_{j+1}} K_{\partial,\partial} \left( \frac{x}{a_{m}}, \frac{t}{a_{m}} \right) dt
$$

$$
\leq \frac{\partial W_{m,2}(x)}{h^{2} d_{m}^{2}} \left( \left| g_{x}(t_{1}) - g_{x}(x) \right| + \sum_{j=1}^{n-1} \left| g_{x}(t_{j+1}) - g_{x}(t_{j}) \right| \frac{1}{(j+1)^{2}} \right)
$$

$$
= \frac{\partial W_{m,2}(x)}{h^{2} d_{m}^{2}} \left( \left| g_{x}(t_{1}) - g_{x}(x) \right| + \sum_{j=1}^{n-1} \sum_{i=1}^{j} \left| g_{x}(t_{i+1}) - g_{x}(t_{i}) \right| \frac{1}{(j+1)^{2}} - \frac{1}{(j+2)^{2}} \right)
$$

$$
+ \frac{1}{n^{2}} \sum_{j=1}^{n} \left| g_{x}(t_{j+1}) - g_{x}(t_{j}) \right|
$$

$$
\leq \frac{\partial W_{m,2}(x)}{h^{2} d_{m}^{2}} \left( v_{1}(g_{x}; T_{1}) + 2 \sum_{j=1}^{n-2} v_{j+1}(g_{x}; T_{j+1}) + v_{n}(g_{x}; T_{n}) \right)
$$
Substituting this in (13) we get
\[ |I_3(m,x)| \leq \sum_{j=1}^{n} v_j(g_x;[t_{j+1},t_j]) \int_{t_{j+1}}^{t_j} K_m,\delta \left( \frac{x}{a_m}, \frac{t}{a_m} \right) dt, \]
Next, in view of (13) and the Abel transformation,
\[
|I_3(m,x)| \leq \frac{\vartheta W_{m,2}(x)}{h^2 d_m^2} \sum_{j=1}^{n} v_j(g_x;[t_{j+1},t_j]) \left[ \frac{1}{j^2 - \frac{1}{(j+1)^2}} \right].
\]

Combining the result and observing that \( T_i = I_x(jhd_m) \) we get the desired estimation for \( h = -x \). In the case of \( h = x \) the proof runs analogously; we use inequality (12) instead of (13).

III. Main Results

In this section we prove our main theorems.

**Theorem 3.1** Let \( f \in \mathcal{H}_{loc}[0,\infty) \) and let the one-sided limits \( f(x), f(x-) \) exist at a fixed point \( x \in (0,\infty) \). Then, for all integers \( m \) such that \( a_m > 2x \) and \( 4a_m \leq m \) one has
\[
\left[ (D_{m,\delta} f)(x) - \frac{f(x+)}{\vartheta + 1} + \frac{f(x-)}{\vartheta + 1} \right] \leq 2\mu_1 \left( g_x; x\sqrt{a_m/m} \right)
\]
\[
+ \frac{32\vartheta}{x^2} \left[ 1 - \frac{x}{a_m} \right] + \frac{a_m}{m} \left[ \sum_{j=1}^{n-1} \frac{\mu_j(g_x; H_x(j\sqrt{a_m/m} + \mu_j(g_x; H_x(x)) \right) \right]
\]
\[
+ \frac{2\vartheta c_q}{x^q} (a_m/m)^{q/2} \left[ 1 - \frac{x}{a_m} \right] + \frac{a_m}{m} \right)^q + \frac{2\vartheta |f(x+)| - |f(x-)|}{\sqrt{a_m(1 - x/m)}}.
\]

where \( n = \sqrt{m/a_m} H_x(u) = [x - u, x + u] \) for \( 0 \leq u \leq x, \varphi(a; f) = \sup_{0 \leq t \leq h} |f(t)| \)
\[
g_x(t) = \begin{cases} f(t) - f(x+) & \text{if } t > x, \\ 0 & \text{if } t = x, \\ f(t) - f(x-) & \text{if } 0 \leq t < x, \end{cases}
\]
\[
g_x(t) \quad (14)
\]

where \( q \) is an arbitrary positive integer and \( c_q \) a positive constant depending only on \( q \).

**Proof.** We decompose \( f(t) \) into four parts as
\[
f(t) = \frac{f(x+)}{\vartheta + 1} + \frac{f(x+) - f(x-)}{\vartheta + 1} \left[ \text{sgn}_x(t) + \frac{\vartheta - 1}{\vartheta + 1} \right] + g_x(t)
\]

where \( g_x(t) \) is defined as (14) and \( \text{sgn}_x(t) := \text{sgn}(t - x), \)
\[
g_x(t) = \begin{cases} 1, & \text{if } x = t, \\ 0, & \text{if } x \neq t. \end{cases}
\]

From (15) we have
\[
(D_{m,\delta} f)(x) = \frac{f(x+)}{\vartheta + 1} + \left( D_{m,\delta} g_x \right)(x) + \frac{f(x+) - f(x-)}{\vartheta + 1} \left[ \text{sgn}_x(t) + \frac{\vartheta - 1}{\vartheta + 1} \right] + \left[ f(x) - \frac{f(x+)}{\vartheta + 1} \right] (D_{m,\delta} \delta_x)(x).
\]

For operators \( D_{m,\delta} \) using (16) we can observe that the last term on the right hand side vanishes. In addition it is obvious that \( (D_{m,\delta} 1)(x) = 1 \). Hence we have
In order to prove our theorem we need the estimates for $(D_{m,\theta}g_\ast)(x)$ and $(D_{m,\theta}sgn_\ast)(x)+\frac{\vartheta-1}{\vartheta+1}$.

To estimate $(D_{m,\theta}g_\ast)(x)$ with the help of the fixed points $x$ and $2x$, we decompose it into three parts as follows:

$$
\left|\int_0^x g_x(t)K_{m,\theta}(\frac{x}{a_m}, \frac{t}{a_m}) dt \right| \leq \int_0^x g_x(t)K_{m,\theta}(\frac{x}{a_m}, \frac{t}{a_m}) dt + \int_{2x}^x g_x(t)K_{m,\theta}(\frac{x}{a_m}, \frac{t}{a_m}) dt + \int_{2x}^{2x} g_x(t)K_{m,\theta}(\frac{x}{a_m}, \frac{t}{a_m}) dt
$$

$$
= |E_{1,\theta}(m,x)| + |E_{2,\theta}(m,x)| + |E_{3,\theta}(m,x)|,
$$

where $K_{m,\theta}(\frac{x}{a_m}, \frac{1}{a_m})$ is defined in Lemma 2.2.

The estimations for $|E_{1,\theta}(m,x)|$ and $|E_{2,\theta}(m,x)|$ are given in Lemma 2.3 in which we put $h = -x$ and $h = x$, respectively. Using the obvious inequality

$$
\|g_x(i_x(\ast))\| \leq 2\|g_x(H_x(u))\|, \
$$

where $H_x(u) = [x - u, u + x]$, $0 \leq u \leq x$, we obtain

$$
|E_{1,\theta}(m,x)| + |E_{2,\theta}(m,x)| \leq 2\|g_x(H_x(x\sqrt{a_m/m}))\| + 16\vartheta W_{m,2}(x)m
\sum_{j=1}^{n-1} \|g_x(H_x(jx\sqrt{a_m/m}))\| + \frac{\|g_x(H_x(x))\|}{n^2}.
$$

(19)

Now, we estimate $|E_{3,\theta}(m,x)|$. Clearly, given any $q \in \mathbb{N}$, we have

$$
|E_{3,\theta}(m,x)| \leq 2\varphi(a_m;f) \frac{m+1}{a_m} \sum_{i=0}^{m} Q_{m,i}(\frac{x}{a_m}) \int_{0}^{2x} p_{m,q}(\frac{t}{a_m}) dt
$$

$$
\leq 2\varphi(a_m;f) \frac{m+1}{a_m} \sum_{i=0}^{m} Q_{m,i}(\frac{x}{a_m}) \int_{0}^{2x} (t-x)^{2q} p_{m,q}(\frac{t}{a_m}) dt
$$

$$
\leq 2\varphi(a_m;f) \frac{m+1}{a_m} \sum_{i=0}^{m} P_{m,q}(\frac{x}{a_m}) \int_{0}^{2x} (t-x)^{2q} p_{m,q}(\frac{t}{a_m}) dt
$$

$$
= 2\varphi(a_m;f) \frac{m+1}{a_m} \sum_{i=0}^{m} P_{m,q}(\frac{x}{a_m}) \int_{0}^{2x} (t-x)^{2q} p_{m,q}(\frac{t}{a_m}) dt
$$

(20)

Finally, replacing $x$ by $x/a_m$ in the result of X. M. Zeng and W. Chen [22] (sect. 3, pp. 9-11) we immediately get

$$
\left|\left(D_{m,\theta}sgn_\ast\right)(x) + \frac{\vartheta-1}{\vartheta+1}\right| \leq \frac{4\vartheta}{|x - \frac{x}{a_m}|}
$$

Putting (18), (19), (20) and (21) into (17), we get the required result. Thus the proof of Theorem 1 is complete.

From Theorem 3.1 and inequality (6) we get

**Theorem 3.2** Let $f \in BV_{loc}^p[0,\infty), p \geq 1$ and let $x \in (0,\infty)$. Then, for all integers $m$ such that $a_m > 2x$ and $4a_m \leq m$ we have

$$
\left|\left(D_{m,\theta}f\right)(x) - \frac{f(x^+) + \vartheta f(x^-)}{\vartheta+1}\right| \leq 2W_p(g_x; H_x(x\sqrt{a_m/m}))
$$

$$
+ \frac{2^{q+1/p}\vartheta}{x^{2q+1/p}} \frac{n_{q-1}^2 - 1}{m} \sum_{i=1}^{m} \left(\frac{x}{a_m} + \frac{a_m}{m}\right) \int_{0}^{2x} (t-x)^{2q} p_{m,q}(\frac{t}{a_m}) dt
$$

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\[
+ \frac{2\vartheta C_q}{x^{2q}} \varphi(a_m; f) \left( \frac{a_m}{m} \right)^q \left( x \left( 1 - \frac{x}{a_m} \right) + \frac{a_m}{m} \right)^q + \frac{2\vartheta |f(x+) - f(x-)|}{\sqrt{a_m} \left( 1 - \frac{x}{a_m} \right)}.
\]

In order to show this it is necessary to prove that the right-hand sides of the inequalities mentioned in the hypotheses of the theorems tend to zero as \( m \to \infty \). In view of (4) we have \( n = \left[ \sqrt{m/a_m} \right] \to \infty \) as \( m \to \infty \). So, in Theorem 1 it is enough to consider only the term
\[
\Lambda_n(x) = \sum_{j=1}^{n-1} \frac{\mathbb{V}_l(g_{j}; H_x((j+1)d_m))}{j^2}, \quad \text{where} \quad d_m = \sqrt{a_m/m}.
\]

Clearly,
\[
\Lambda_n(x) = \sum_{j=1}^{n-1} \frac{\mathbb{V}_l(g_{j}; H_x((j+1)d_m))}{j^2} \leq 4d_m \int_{1}^{n/\sqrt{d_m}} \frac{\mathbb{V}_l(g_{j}; H_x(x))}{x^2} \, dx \leq 4 \sum_{i=1}^{n} \frac{\mathbb{V}_l(g_{j}; H_x(x))}{x^2}.
\]

Since the function \( g_x \) is continuous at \( x \) and \( \mathbb{V}_l(g_{j}; H_x(x)) \) denotes the oscillation of \( g_x \) on the interval \( H_x(\frac{x}{\sqrt{d_m}}) \), we have
\[
\lim_{n \to \infty} \frac{\mathbb{V}_l(g_{j}; H_x(x))}{x^2} = 0
\]
and consequently,
\[
\lim_{n \to \infty} \Lambda_n(x) = 0
\]

As regards Theorem 3.2, it is easy to verify that in view of the continuity of \( g_x \) at \( x \),
\[
\lim_{n \to \infty} \frac{1}{n^{1+1/p}} \sum_{i=1}^{n^2-1} \frac{1}{(\sqrt{d_m})^{1-1/p}} \mathbb{V}_p(g_{j}; H_x(x)) = 0.
\]

Thus we get the following approximation theorem.

**Proof.** Let \( f \in BV_{10}^p(0, \infty), p \geq 1 \). In view of (6) and the notation \( d_m = \sqrt{a_m/m} \), \( n = \left[ \sqrt{m/a_m} \right] \), we have
\[
\sum_{j=1}^{n-1} \frac{\mathbb{V}_l(g_{j}; H_x((j+1)d_m))}{j^{2+1/p}} \leq \sum_{j=1}^{n-1} \mathbb{V}_p(g_{j}; H_x((j+1)d_m)) \leq (2d_m)^{2+1/p} \int_{1}^{n/\sqrt{d_m}} \frac{\mathbb{V}_p(g_{j}; H_x(x))}{x^{2+1/p}} \, dx \leq (2d_m)^{2+1/p} \int_{1}^{\sqrt{n}} \frac{\mathbb{V}_p(g_{j}; H_x((x+\sqrt{n}))}{(\sqrt{n})^{2+1/p}} \, dx \leq (2d_m)^{2+1/p} \int_{1}^{\sqrt{n}} \frac{\mathbb{V}_p(g_{j}; H_x((x+\sqrt{n})))}{(\sqrt{n})^{2+1/p}} \, dx
\]

and
\[
\mathbb{V}_n(g_{j}; H_x(x)) \leq \mathbb{V}_p(g_{j}; H_x(x)) \leq \mathbb{V}_p(g_{j}; H_x(x))
\]

and
\[
\mathbb{V}_n(g_{j}; H_x(x)) \leq \mathbb{V}_p(g_{j}; H_x(x)) \leq \mathbb{V}_p(g_{j}; H_x(x))
\]

moreover,
\[
\mathbb{V}_n(g_{j}; H_x(x)) \leq \mathbb{V}_p(g_{j}; H_x(x)) \leq \mathbb{V}_p(g_{j}; H_x(x))
\]

The estimation given in Theorem 3.2 now immediately follows from Theorem 3.1.

**Corollary.** Suppose that \( f \in X_{10}(0, \infty) \) (in particular, \( f \in BV_{10}^p(0, \infty), p \geq 1 \)) and that there exists a positive integer \( q \) such that
\[
\lim_{m \to \infty} \frac{d_m}{m}^q \varphi(a_m; f) = 0.
\]

Then at every point \( x \in [0, \infty) \) at which the limits \( f(x+), f(x-) \) exist we have
\[
\lim_{m \to \infty} (P_{m, \vartheta} f)(x) = \frac{f(x+)}{\vartheta + 1} + \frac{f(x-)}{\vartheta + 1}
\]

Obviously, the above relations hold true for every measurable function \( f \) bounded on \([0, \infty)\), in particular for every function \( f \) of bounded \( p \)-th power variation \( (p \geq 1) \) on the whole interval \([0, \infty)\).

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References


