Nonlocal Boundary Value Problem for Nonlinear Impulsive $q_k$-Symmetric Integrodifference Equation

Long Cheng, Chengmin Hou and Yansheng He
(1Department of Mathematics, Yanbian University, Yanji, 133002, P.R. China)

Abstract: A first order nonlinear impulsive integrodifference equation within the frame of $q_k$-symmetric quantum calculus is investigated by applying using fixed point theorems. The conditions for existence and uniqueness of solutions are obtained.

Keywords: $q_k$-Symmetric integrodifference equation, $q_k$-symmetric derivatives, $q_k$-symmetric integrals, Boundary value problem.

I. Introduction

The $q$-calculus was initiated in twenties of the last century. However, it has gained considerable popularity and importance during the last three decades or so. Their study has not only important theoretical meaning but also wide applications in conformal quantum mechanics, high energy physics, etc. We refer the reader to recent articles[1-7]. Recently, in [8], authors research first order nonlocal boundary value problem for nonlinear impulsive $q_k$-integrodifference equation.

On this line of thought in this paper, we study the existence and uniqueness of solutions for second order nonlinear $q_k$-symmetric integrodifference equation with nonlocal boundary condition and impulses:

\[
\begin{align*}
\Delta u(t) &= I_k(u(t_k)), \quad t_k \in (0,1), k = 1,2,\ldots, p, \\
0 < q_k < 1, t \in J',
\end{align*}
\]

where $D_{q_k}, I_{q_k}$ are $q_k$-symmetric derivatives and $q_k$-symmetric integrals $(k = 0,1,\ldots, p + 1)$.

Respectively, $f, g \in C(J \times R, R), I_k, h \in C(R, R), J = [0,1] \cup \{1(q_p^{-1} + (1 - q_p^{-1})t_p\}, 0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1, J' = [0,1] \setminus \{t_1, t_2, \ldots, t_p\}, \Delta u(t_k) = u(t_k^+) - u(t_k^-).

Where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k (t = 1,2,\ldots, p)$ respectively.

II. Preliminaries

Let us set $J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{p-1} = (t_{p-1}, t_p], J_p = (t_p, 1], I = (1(q_p^{-1} + (1 - q_p^{-1})t_p\},$ and introduce the space: $PC(J, R) = \{u : J \rightarrow R | u \in C((t_k, t_{k+1}] \cup I), k = 0,1,\ldots, p \text{ and } x(t_k^+), x(t_k^-) \text{ exist with } x(t_k^+) = x(t_k^-), k = 1,2,\ldots, p\}.$ Where $J = [0,1] \cup (1(q_p^{-1} + (1 - q_p^{-1})t_p\}.$ Clearly, it is a Banach space with the norm $\|u\| = \sup_{t \in J} |u(t)|.$

Definition A function $u \in PC(J, R)$ with its derivative of second order existing on $J$ is a solution of (1) if it satisfies (2).

For convenience, let us introduce some basic concepts of $q_k$-symmetric calculus.

For $0 < q_k < 1$ and $t \in J_k$, we define the $q_k$-symmetric derivatives of a real valued continuous function $f$ as

\[
D_{q_k}f(t) = \frac{f(q_k^{-1}t + (1 - q_k^{-1})t_k) - f(q_kt + (1 - q_k)t_k)}{(q_k^{-1} - q_k)(t - t_k)},
\]

where $t_k \in (0,1), k = 1,2,\ldots, p.$

DOI: 10.9790/5728-13020195101 www.iosrjournals.org 95 | Page
Nonlocal Boundary Value Problem for Nonlinear Impulsive $q$-Symmetric Integrodifference...

$D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t)$.

Higher order $q_k$ – symmetric derivatives are given by

$$D_{q_k}^0 f(t) = f(t), D_{q_k}^n f(t) = D_{q_k} D_{q_k}^{n-1} f(t) \quad n \in \mathbb{N}, t \in J_k.$$ (4)

The $q_k$ – symmetric integral of a function $f$ is defined by

$$\int_{t_k} I_{q_k} f(t) = \int_{t_k} f(s) d_{q_k} s = (1-q_k^2) (t-t_k) \sum_{n=0}^{\infty} q_k^{2n} f(t_k + q_k^{2n+1} (t-t_k)), \quad t \in J_k.$$ (5)

Provided the series converges. If $a \in (t_k, t)$ and $f$ is defined on the interval $(t_k, t)$, then

$$\int_{a}^{t} f(s) d_{q_k} s = \int_{a}^{t} f(s) d_{q_k} s - \int_{a}^{t_k} f(s) d_{q_k} s.$$ (6)

Observe that

$$D_{q_k} I_{q_k} f(t) = D_{q_k} \int_{t_k} f(s) d_{q_k} s = f(t),$$

$$D_{q_k} I_{q_k} f(t) = D_{q_k} \int_{a}^{t} f(s) d_{q_k} s = f(t) - f(a), \quad a \in (t_k, t).$$ (7)

For $t \in J_k$, the following reversing order of $q_k$ – symmetric integration holds

$$\int_{a}^{t} f(r) d_{q_k} r d_{q_k} s = \int_{t_k}^{t} (t-r) f(r) d_{q_k} r.$$ (8)

In fact,

$$\int_{a}^{t} f(r) d_{q_k} r d_{q_k} s = (t-t_k) (1-q_k^2) \sum_{n=0}^{\infty} q_k^{2n+2} f(t_k) + q_k^{2n+1} (t_k + q_k^{2n+1} (t-t_k) - t_k))$$

$$= (t-t_k)^2 (1-q_k^2)^2 \sum_{n=0}^{\infty} q_k^{2n+2} f(t_k) + q_k^{2n+1} (t_k + q_k^{2n+1} (t-t_k) - t_k))$$

$$= (t-t_k)^2 (1-q_k^2)^2 \sum_{n=0}^{\infty} q_k^{2n+2} f(t_k) + q_k^{2n+1} (t-t_k))$$

$$= (t-t_k)^2 (1-q_k^2)^2 \sum_{n=0}^{\infty} q_k^{2n+2} (t_k + q_k^{2n+1} (t-t_k))$$

$$= (t-t_k)^2 (1-q_k^2)^2 \sum_{n=0}^{\infty} q_k^{2n+2} (t_k + q_k^{2n+1} (t-t_k))$$

$$= \int_{t_k}^{t} (t-r) f(r) d_{q_k} r.$$ (9)

Note that if $t_k = 0$ and $q_k = q$ in (3) and $D_{q_k} f = D_q f$, $I_{q_k} f = I_q f$ (5), then where $D_q$ and $I_q$ are the well-known $q$ – symmetric derivative and $q$ – symmetric integral of the function $f(t)$ defined by

$$D_q f(t) = \dfrac{f(q^{-1} t) - f(q t)}{(q^{-1} - q) t},$$

$$I_q f(t) = \int_{s}^{t} f(s) d_q s = (1-q^2) t \sum_{n=0}^{\infty} q^{2n} f(q^{2n+1} t).$$

**Lemma 1.** For given $y_{q_k} \in C(J, R)$, the function $u \in C(J, R)$ is a solution of the impulsive $q_k$ – symmetric integrodifference equation

DOI: 10.9790/5728-13020195101 www.iosrjournals.org 96 | Page
Nonlocal Boundary Value Problem for Nonlinear Impulsive \( q_k \)-Symmetric Integrodifference Eq.

\[ D_{q_k} u(t) = y_q(t), \quad 0 < q_k < 1, t \in J', \]
\[ \Delta u(t_k) = I_k(u(t_k)), \quad t_k \in (0,1), k = 1,2,\cdots, p, \quad (9) \]
\[ u(0^+) = h(u) + u_0, \quad u_0 \in \mathbb{R}. \]

If and only if \( u \) satisfies the \( q_k \)-integral equation

\[
u(t) = \begin{cases} 
\int_0^1 y_{q_k}(s) d_{q_k} s + h(u) + u_0, & t \in J_0 \\
\int_{i_k}^t y_{q_k}(s) d_{q_k} s + \sum_{i=0}^{k-1} \int_{i_k}^{i_{k+1}} y_{q_k}(s) d_{q_k} s + \sum_{i=0}^{k} I_i(u(t_i)) + h(u) + u_0, & t \in J_k,
\end{cases}
\]

(10)

Proof. Let \( u \) be a solution of \( q_k \)-symmetric difference equation (9). For \( t \in J_0 \), applying the operator \( I_{q_k} \) on both sides of \( D_{q_k} u(t) = y_q(t) \), we have \( u(t) = u(0^+) + \int_0^t y_{q_k}(s) d_{q_k} s \).

Similarly, for \( t \in J_1 \), applying the operator \( I_{q_k} \) on both sides of \( D_{q_k} u(t) = y_q(t) \), then
\[ u(t) = u(t_1^+) + \int_{t_1}^t y_{q_k}(s) d_{q_k} s. \]

In view of \( \Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k)) \), it holds
\[ u(t) = u(0^+) + \int_0^t y_{q_k}(s) d_{q_k} s + I_1(u(t_1)), t \in J_1. \]

Repeating the above process, we can get
\[ u(t) = u(0^+) + \int_0^t y_{q_k}(s) d_{q_k} s + \sum_{i=0}^{k-1} \int_{i_k}^{i_{k+1}} y_{q_k}(s) d_{q_k} s + \sum_{i=0}^{k} I_i(u(t_i)) + h(u) + u_0, t \in J_k. \]

Using the boundary value conditions given in (9), it follows
\[ u(t) = \int_{i_k}^{i_{k+1}} y_{q_k}(s) d_{q_k} s + \sum_{i=0}^{k-1} \int_{i_k}^{i_{k+1}} y_{q_k}(s) d_{q_k} s + \sum_{i=0}^{k} I_i(u(t_i)) + h(u) + u_0, t \in J_k. \]

For \( t \in J_p \cup I \), we have
\[ u(t) = \int_{t_p}^t y_{q_k}(s) d_{q_k} s + \sum_{i=0}^{p-1} \int_{t_p}^{t_{p+1}} y_{q_k}(s) d_{q_k} s + \sum_{i=0}^{p} I_i(u(t_i)) + h(u) + u_0. \]

Thus we can get (10). Conversely, assume that \( u \) satisfies the impulsive \( q_k \)-integral equation (9), applying \( D_{q_k} \) on both sides of (10) and substituting \( t = 0 \) in (10), then (9) holds. This completes the proof.

**Remark 1.** In (10) \( t \), allow belongs to \( I \). Since
\[ (d_{q_p}^{-1} t + (1 - q_p^{-1}) t_p) d_{q_p}^{2n+1} + (1 - q_p^{2n+1}) t_p < t, n \in \mathbb{N} \]
and
\[ \int_{t_p}^{t_{p+1}} y_{q_p}(s) d_{q_p} s = (1 - q_p^2) q_p^{-1} (t - t_p) \sum_{n=0}^{N} q_p^{2n} y_{q_p}(q_p^{-1} t + (1 - q_p^{-1}) t_p) d_{q_p}^{2n+1} + (1 - q_p^{2n+1}) t_p, \]
we see that \( u(q_p^{-1} t + (1 - q_p^{-1}) t_p), \) for \( t \in I \) have definition.
### III. Main Results

Letting \( y_{q_k}(t) = f(t, u(t)) + \int_{t_k}^{t} I_{q_k} g(t, u(t)) \), in view of Lemma 1, we introduce an operator \( PC \rightarrow PC \) as

\[
\begin{align*}
(Qu)(t) &= \int_{t_k}^{t} [f(s, u(s)) + \int_{s}^{t} g(r, u(r))dr]ds + \\
&\quad + \sum_{i=0}^{k-1} \left[ f(s, u(s)) + \int_{s}^{s+i} g(r, u(r))dr \right]ds + \sum_{i=0}^{k} I_{i}(u(t)) + h(u) + u_0. 
\end{align*}
\]

(11)

By (8), we obtain

\[
\begin{align*}
(Qu)(t) &= \int_{t_k}^{t} [f(s, u(s))ds + \int_{s}^{t} g(r, u(r))dr]ds + \\
&\quad + \sum_{i=0}^{k-1} \left[ f(s, u(s))ds + \int_{s}^{s+i} g(r, u(r))dr \right]ds + \sum_{i=0}^{k} I_{i}(u(t)) + h(u) + u_0. 
\end{align*}
\]

(12)

Then, the impulsive \( q_k \)-symmetric integrodifference equation (1) (2) has a solution if and only if the operator \( u = Qu \) has a fixed point .

In order to prove the existence of solutions for (1) (2), we need the following known result (J. X. Sun, 2008).

**Lemma 2.** Let \( E \) be a Banach space. Assume that \( T : E \rightarrow E \) is a completely continuous operator and the set \( V = \{ x \in E | x = \mu Tx, 0 < \mu < 1 \} \) is bounded. Then \( T \) has a fixed point in \( E \).

**Theorem 3.** Assume the following.

1. (H1) There exist nonnegative bounded function \( M_i(t) \) \( (i = 1, 2, 3, 4) \) such that

\[ |f(t, u)| \leq M_1(t) + M_2(t)|u|, |g(t, u)| \leq M_3(t) + M_4(t)|u|, \] for any \( t \in J, u \in R \).

2. denote \( \sup_{t \in J} |M_i(t)| = M_i \), \( i = 1, 2, 3, 4 \).

3. (H2) There exists positive constants \( \bar{L}, \bar{L}' \) such that

\[ |I_k(u)| \leq \bar{L}, \quad h(u) \leq \bar{L}' \]

for any \( u \in R, k = 1, 2, \ldots, p \).

Then the problem (1) (2) has at least one solution provided.

\[
\tau = \sup_{\epsilon > 0} \left\{ \left( q_p^{-1} + (1 - q_p^{-1})t_p \right)M_2(t) + M_4(t) \sum_{i=0}^{p} (t_{i+1} - t_i)^2 + M_4(t)q_p^{-2}(1 - t_p)^2 \right\} < 1.
\]

**Proof.** Firstly, similar to the proof of Theorem 3 in [8], we prove the operator \( Q \) is completely continuous. Next we define the set \( W_1 = \{ u \in PC(J, R) | u = \lambda Qu, 0 < \lambda < 1 \} \). We show \( W_1 \) is bounded. Let \( u \in W_1 \), then

\[
u = \lambda Qu, 0 < \lambda < 1.
\]

For any \( t \in J \) by conditions (H1) and (H2), we have
|u(t)| = \lambda |(Qu)(t)| \\
\leq \int_{t}^{t+1} f(s,u(s)) \, ds + \int_{t}^{t+1} (t-s) \, g(s,u(s)) \, ds \\
+ \sum_{i=0}^{k-1} \int_{t}^{t+i+1} f(s,u(s)) \, ds + \sum_{i=0}^{k-1} \int_{t}^{t+i+1} (t-s) \, g(s,u(s)) \, ds \\
+ \sum_{i=0}^{k-1} |I_i(u(t_i))| + |h(u)| + |u_0| \\
\leq (M_1 + M_2 \|u\|)(t-t_i) + (M_3 + M_4 \|u\|)q_i(t-t_i)^2 \\
+ (M_3 + M_4 \|u\|)q_i^2(1-t_p)^2 + \sum_{i=0}^{k-1} ((M_1 + M_2 \|u\|)(t_{i+1} - t_i) \\
+ q_i(t_{i+1} - t_i)^2(M_3 + M_4 \|u\|)) + \sum_{i=0}^{k-1} L + L'|u_0| \\
\leq M_1(q_t^{-1} + (1-q_p^{-1})t_t) + M_3 \sum_{i=0}^{k-1} (t_{i+1} - t_i)^2 + M_3q_t^{-1}(1-t_p)^2 + mL + |u_0| \\
+ \|u\| (M_2(q_t^{-1} + (1-q_p^{-1})t_t) + M_4 \sum_{i=0}^{k-1} (t_{i+1} - t_i)^2 + M_4q_t^{-1}(1-t_p)^2. \\
\|u\| \leq \frac{1}{1-r} [M_1(q_t^{-1} + (1-q_p^{-1})t_t) + M_3 \sum_{i=0}^{k-1} (t_{i+1} - t_i)^2 + M_3q_t^{-1}(1-t_p)^2 + mL + |u_0| = \text{constant} \\

So, the set \( W_i \) is bounded. Thus, Lemma 2 ensures the impulsive \( q_k \)-symmetric integrodifference equation (1) (2) has at least one solution.

**Corollary 4.** Assume the following.

(H_3) There exist nonnegative constants \( L_i \), \( i = 1, 2, 3, 4 \) such that

\[ |f(t,u)| \leq L_1 \quad |g(t,u)| \leq L_2 \quad |I_k(u)| \leq L_3 \quad |h(u)| \leq L_4 \]

for any \( t \in J, u \in R, k = 1, 2, \cdots, p \). Then problem (1) (2) has at least one solution.

**Theorem 5.** Assume the following.

(H_4) There exist nonnegative bounded functions \( M(t) \) and \( N(t) \) such that

\[ |f(t,u) - f(t,v)| \leq M(t)|u-v| \quad |g(t,u) - g(t,v)| \leq N(t)|u-v| \]

for \( t \in J, u, v \in R \).

(H_5) There exist positive constants \( K, L \) such that
Nonlocal Boundary Value Problem for Nonlinear Impulsive $q_k$-Symmetric Integrodifference

\[ |I_k(u) - I_k(v)| \leq K |u - v|, \quad |h(u) - h(v)| \leq G |u - v| \quad \text{for } u, v \in R \text{ and } k = 1, 2, \ldots, p. \]

(H$_6$) $K_i = \sup_{t \in J_i} \{ M(t) + mK + G + N(t) \sum_{i=0}^p (t_{i+1} - t_i)^2 + N(t)q_p^2 (1 - t_p)^2 \} < 1.$

Then problem (1) (2) has a unique solution.

Proof. Clearly $Q$ is a continuous operator. Denote $\sup_{t \in J_i} |M(t)| = M, \sup_{t \in J_i} |N(t)| = N$. For $u, v \in PC(J, R)$, by (H$_4$) and (H$_5$), we have

\[
\left| (Qu)(t) - (Qv)(t) \right| \\
\leq \int_t^\infty \left| f(s, u(s)) - f(s, v(s)) \right| \, ds + \int_t^\infty \left| g(s, u(s)) - g(s, v(s)) \right| \, ds \\
+ \sum_{i=0}^{k-1} \int_{t_i}^t \left| f(s, u(s)) - f(s, v(s)) \right| \, ds + \sum_{i=0}^{k-1} \int_{t_i}^t \left| g(s, u(s)) - g(s, v(s)) \right| \, ds \\
+ \sum_{i=0}^{k-1} |I_i(u(t_i)) - I_i(v(t_i))| + |h(u) - h(v)| \\
\leq \| u - v \| \left( \int_t^\infty M(s) \, ds \right) + \int_t^\infty g(s, u(s)) \, ds + \sum_{i=0}^{k-1} K + G \\
\leq K_1 \| u - v \|.
\]

As $K_1 < 1$ by (H$_6$). Therefore, $Q$ is a contractive map. Thus, the conclusion of the Theorem 5 follows by

**IV. Example**

Consider the following second order nonlinear $q_k$-symmetric integrodifference equation with impulses

\[
D_{1/2+k}u(t) = 8 + 3\sqrt{t} + \ln(1 + 2t^3 + \frac{t^2}{25^3} |u(t)|) \\
+ \int_{1/2+k}^t (10s + \frac{s^3}{16} \sin(u(s))) \, ds, t \in (0,1), t \neq \frac{1}{1+k},
\]

\[
\Delta u(\frac{1}{1+k}) = \cos(u(\frac{1}{1+k})), k = 1, 2, 3, 4,
\]

\[ u(0) = 5 + e^{-u^2}, \quad k = 1, 2, 3, 4. \]
Obviously, \( q_k = \frac{1}{2+k}, \ t_k = \frac{1}{1+k}, \ k = 1,2,3,4. \ f(t,u) = 8 + 3\sqrt[3]{t} + \ln (1 + 2t^3 + \frac{t^2}{25^3} |u|), \)

\[ g(t,u) = 10t + \frac{t^3}{16^t} \sin u, \ I_k(u) = \cos u, \] and \( u(0) = 5 + e^{-u}. \) By a simple calculation, we can get

\[ |f(t,u)| \leq 1 + 3\sqrt[3]{t} + 2t^3 + \frac{t^2}{25} |u|, \ |g(t,u)| \leq t + \frac{t^3}{16} |u|, \ |h(u)| \leq 1, |I_k(u)| \leq 1. \]

Take \( M_1(t) = 1 + 3\sqrt[3]{t} + 2t^3, M_2(t) = \frac{t^2}{25^t}, M_3(t) = t, M_4(t) = \frac{t^3}{16^t}, L = L' = 1. \) Then all conditions of Theorem 3, the above nonlinear impulsive \( q_k - \) symmetric integrodifference has at least one solution.

References


