Connected and Distance in \( G \bigotimes_2 H \)

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**Abstract**: The tensor product \( G \bigotimes H \) of two graphs \( G \) and \( H \) is well-known graph product and studied in detail in the literature. This concept has been extended by introducing 2-tensor product \( G \bigotimes_2 H \) and it has been discussed for special graphs like \( P_n \) and \( C_n \) [5]. In this paper, we discuss \( G \bigotimes_2 H \), where \( G \) and \( H \) are connected graphs. Mainly, we discuss connectedness of \( G \bigotimes_2 H \) and obtained distance between two vertices in it.

**Keywords**: Bipartite graph, Connected graph, Non-bipartite graph, 2-tensor product of graphs.

I. Introduction

The tensor product \( G \bigotimes H \) of two graphs \( G \) and \( H \) is very well-known and studied in detail ([1], [2], [3], [4]). This concept has been extended by introducing 2-tensor product \( G \bigotimes_2 H \) of \( G \) and \( H \) and studied for special graphs [5]. In this paper, we discuss connectedness of \( G \bigotimes_2 H \) for any connected graphs \( G \) and \( H \). We also obtained the results for the distance between two vertices in \( G \bigotimes_2 H \).

If \( G = (V(G), E(G)) \) is finite, simple and connected graph, then \( d_G(u, u') \) is the length of the shortest path between \( u \) and \( u' \) in \( G \). For a graph \( G \), a maximal connected subgraph is a component of \( G \). For the basic terminology, concepts and results of graph theory, we refer to ([1], [5], [6], [7]).

We recall the definition of 2-tensor product of graphs.

**Definition 1.1** [5] Let \( G \) and \( H \) be two connected graphs. The 2-tensor product of \( G \) and \( H \) is the graph with vertex set \( \{(u, v) : u \in V(G), v \in V(H)\} \) and two vertices \((u, v)\) and \((u', v')\) are adjacent in 2-tensor product if \( d_G(u, u') = 2 \) and \( d_H(v, v') = 2 \). It is denoted by \( G \bigotimes_2 H \).

Note that \( G \bigotimes_2 H \) is a null graph, if the diameter \( D(G) < 2 \) or \( D(H) < 2 \). So, throughout this paper we assume that \( G \) and \( H \) are non-complete graphs.

II. Connectedness of \( G \bigotimes_2 H \)

In this section, first we consider the graphs \( G \) and \( H \), both connected and bipartite with \( N^2(w) \neq \emptyset \) \( \forall w \in V(G) \cup V(H) \), where \( N^2(u) = \{u' \in V(G) : d_G(u, u') = 2\} \).

In usual tensor product \( G \bigotimes H \), the following result is known.

**Proposition 2.1** [4] Let \( G \) and \( H \) be connected bipartite graphs. Then \( G \bigotimes H \) has two components.

Note that in case of \( G \bigotimes_2 H \), the similar result is not true. We discuss the number of components in \( G \bigotimes_2 H \) with different conditions on \( G \) and \( H \).

We fix the following notations.

Let \( V(G) = U_i \cup U_2 \) and \( V(H) = V_i \cup V_2 \) with \( U_i \) and \( V_j \), \((i, j = 1, 2)\) are partite sets of \( G \) and \( H \) respectively. Then, \( V(G \bigotimes_2 H) = W_{11} \cup W_{12} \cup W_{21} \cup W_{22} \) with \( W_i = U_i \times V_j \).

**Remark 2.2** If \((u, v)\) and \((u', v')\) are from different \( W_{ij} \), then \((u, v)\) and \((u', v')\) can not be adjacent in \( G \bigotimes_2 H \) as \( d_G(u, u') \neq 2 \) or \( d_H(v, v') \neq 2 \). So, \( G \bigotimes_2 H \) has at least four components. Suppose \((u, v)\) and \((u', v')\) are in the same \( W_{ij} \). Then \( d_G(u, u') \) and \( d_H(v, v') \) are even.

**Proposition 2.3** Let \( G \) and \( H \) be connected bipartite graphs. If \( d_G(u, u') \) and \( d_H(v, v') \) are of the same form, \( 4k \) or \( 4k + 2 \), \((k \in \mathbb{N} \cup \{0\})\) then \((u, v)\) and \((u', v')\) are in the same component of \( G \bigotimes_2 H \).

**Proof.** Let \((u, v)\) \& \((u', v') \in U_i \times V_j \). Suppose, \( P_i : u = u_0 \rightarrow u_1 \rightarrow \ldots \rightarrow u_m = u' \) and \( P_j : v = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_n = v' \) are paths between \( u, u' \) and \( v, v' \) respectively.

Suppose \( l(P_i) = 4k / 4k + 2 \) and \( l(P_j) = 4t / 4t + 2 \) with \( k \leq t \). First assume that \( k \neq 0 \neq t \), then there is a path \( P \) or \( P' \) between \((u, v)\) and \((u', v')\) in \( G \bigotimes_2 H \) as follows:
Proposition 2.4

[i] Suppose \((u,v)\) and \((u',v')\) are in same \(W_j\). But if \(d_\theta(u,u')\) and \(d_\theta(v,v')\) are not of the same form, then \((u,v)\) and \((u',v')\) may be in different components. So, \(U_i\times V_i\) give at most two components. Thus \(G\otimes_2 H\) has at most eight components.

[ii] Suppose \(\Delta(G)\leq 2\) and \(\Delta(H)\leq 2\). Then \(G\otimes_2 H\) has at most eight components.

Next, we discuss the conditions on \(G\) and \(H\) under which \(G\otimes_2 H\) has 4, 5 or 6 components.

Proposition 2.5 Let \(G\) and \(H\) be connected bipartite graphs and at least one of the graphs contains a cycle \(C_{4i+2}\) \((i \in \mathbb{N})\). Then \(G\otimes_2 H\) has exactly four components.

Proof. Let \((u,v)\) and \((u',v')\) be in \(U_i\times V_i\). As we have seen in Proposition 2.3, if \(d_\theta(u,u')\) and \(d_\theta(v,v')\) are of the same form \(4k\) or \(4k+2\), then \(U_i\times V_i\) gives connected component.

Let \(P_1\) and \(P_2\) be two paths between \(u-u'\) and \(v-v'\) in \(G\) and \(H\) respectively, as we have considered in Proposition 2.3. Suppose \(l(P_1)\) and \(l(P_2)\) are of the different form.

Suppose \(H\) contains a cycle \(C_{4i+2}\) with \(V(C_{4i+2}) = \{x_1, x_2, \ldots, x_{4i+2}\}\). Then select a vertex from \(C_{4i+2}\), which is nearest to \(v_n = v'\) and also it is in \(V_i\). Suppose this vertex is \(x_j\). Since \(v_n\) and \(x_j\) both are in \(V_i\), we get a path \(P_0\) from \(v_n\) to \(x_j\) of even length. We consider a walk \(W : v = v_0 \xrightarrow{b_1} v_1 \xrightarrow{b_2} \cdots \xrightarrow{b_{4i+2}} v_{4i+2} \xrightarrow{b_{4i+2}} v'_n = v'\) between \(v\) and \(v'\) in \(H\). Then, \(l(W) = l(P_1) + 2l(P_0) + l(C_{4i+2}) = l(P_2) + 2l(P_0) + (4i + 2) = l(P_2) + 4l + 2).\)

Thus, if \(l(P_1) = 4l + 2\) or \(4l + 4\), then \(l(W) = 4q + 4l + 2\). So, in any case, \(l(P_0)\) and \(l(W)\) are of the same form.

Next, we assume that the graphs \(G\) and \(H\) do not contain a cycle of the form \(C_{4i+2}\). We prove that the number of components in \(G\otimes_2 H\) is depending upon \(\Delta(G)\) as well as \(\Delta(H)\).

Let \(\Delta(U_i) = \max\{d(u) : u \in U_i\}\) and \(\Delta(V_i) = \max\{d(v) : v \in V_i\}\), \(i = 1, 2\) and for \(a \in V(G)\), \(N(a) = \{b \in V(G) : d_\theta(a, b) = 1\}\).

Proposition 2.6 Let \(G\) and \(H\) be connected bipartite graphs with \(\Delta(G) \leq 2\) and \(\Delta(H) \geq 3\).

(a) If \(\Delta(V_i) \leq 2\) and \(\Delta(V_j) \geq 3\), then \(G\otimes_2 H\) has six components.

(b) If \(\Delta(V_i) \geq 3\) and \(\Delta(V_j) \geq 3\), then \(G\otimes_2 H\) has four components.

Proof. We know that \(U_i\times V_i\), \(U_j\times V_j\), \(U_k\times V_k\) and \(U_k\times V_k\) give disconnected subgraphs in \(G\otimes_2 H\).

(a) Fixed \(U_i\times V_i\). We shall show that \(U_i\times V_i\) gives connected subgraph of \(G\otimes_2 H\). Let \(a \in V_i\) with \(d(a) \geq 3\) and \(N(a) = \{w_0, w_1, w_2, \ldots\} \subset V_i\).

Fixed \((u,v)\) in \(U_i\times V_i\) with \(v_0 = w_0\). Let \((u',v')\) be any vertex in \(U_i\times V_i\). Suppose, \(P_1\) and \(P_2\) are paths between \(u = u_0\) and \(u'\) and \(v = v_0\) to \(v'\) in \(G\) and \(H\), as we have considered in Proposition 2.3. If \(l(P_1) = 4k\) or \(4k + 2\) and \(l(P_2) = 4t + 2\), then the result is clear.
Next, suppose \( |P_1| = 4k \) and \( |P_2| = 4t + 2 \); \( k \leq t \) with \( k \neq 0 \neq t \). First, we show that for \( v_0 \rightarrow v_1 \rightarrow v_2 \) there is a path between \((u_0, v_0)\) and \((u_0, v_2)\) in \(G \otimes_2 H\). 

Case (1) Suppose \( z \neq v_1 \) in \( V_2 \).
If \( v_2 \neq w_1 \) in \( V_1 \) as given in figure 1, then \((u_0, v_0) \rightarrow (u_2, w_1) \rightarrow (u_0, v_2) \) is a path between \((u_0, v_0)\) and \((u_0, v_2)\) in \(G \otimes_2 H\).

If \( v_2 = w_1 \) in \( V_1 \), then \((u_0, v_0) \rightarrow (u_2, w_2) \rightarrow (u_0, v_2) \) is the required path.

Case (2) Suppose \( z = v_1 \) in \( V_2 \).
If \( w_1 \neq v_1 \neq w_2 \) in \( V_1 \), then \((u_0, v_0) \rightarrow (u_2, w_2) \rightarrow (u_0, v_2) \) is the path and if \( v_2 = w_2 \), then consider the path \((u_0, v_0) \rightarrow (u_2, w_1) \rightarrow (u_0, v_2) \) in \(G \otimes_2 H\).

Thus in each case there is a path between \((u_0, v_0)\) and \((u_0, v_2)\) in \(G \otimes_2 H\). Also as in Proposition 2.3 there is a path from \((u_0, v_0)\) to \((u_4, v_{n+2})\) in \(G \otimes_2 H\). Hence there is a path from \((u_0, v_0)\) to \((u', v')\) in \(G \otimes_2 H\). By similar arguments if \( |P_1| = 4k + 2 \) and \( |P_2| = 4t \), then also there is a path between \((u_0, v_0)\) and \((u', v')\) in \(G \otimes_2 H\). So, \( U_1 \times V_1 \) gives a connected component in \(G \otimes_2 H\).

Thus if \( d(z) \geq 3 \) with \( z \in V_2 \), then the other partite set \( V_1 \) contribute connected components \( U_1 \times V_1 \) and \( U_2 \times V_1 \) in \(G \otimes_2 H\).

Here as \( \Delta(U_1) \leq 2 \) and \( \Delta(V_1) \leq 2 \), \( U_1 \times V_2 \) as well as \( U_2 \times V_2 \) each give two components in \(G \otimes_2 H\). So, the graph \(G \otimes_2 H\) has six components.

(b) In this case \( \Delta(U_1) \geq 3 \) and \( \Delta(V_1) \geq 3 \). So \( U_1 \times V_2 \), \( U_1 \times V_1 \) and \( U_2 \times V_1 \) will give connected components in \(G \otimes_2 H\). So, the graph \(G \otimes_2 H\) has four components.

**Corollary 2.7** Let \( G \) and \( H \) be connected bipartite graphs with \( \Delta(G) \geq 3 \) and \( \Delta(H) \geq 3 \).

(a) If \( \Delta(U_1) \leq 2 \) and \( \Delta(U_2) \geq 3 \) as well as \( \Delta(V_1) \leq 2 \) and \( \Delta(V_2) \geq 3 \), then \( G \otimes_2 H \) has five components.

(b) If \( \Delta(U_1) \geq 3 \) and \( \Delta(V_1) \geq 3 \); \( i = 1,2 \), then \( G \otimes_2 H \) has four components.

**Proof.** (a) In this case \( \Delta(U_1) \geq 3 \) and \( \Delta(V_1) \geq 3 \). Since \( \Delta(U_2) \geq 3 \), the other partite set \( U_1 \) contribute connected components \( U_1 \times V_1 \) and \( U_1 \times V_2 \) in \(G \otimes_2 H\). Similarly as \( \Delta(V_2) \geq 3 \), \( U_2 \times V_1 \) and \( U_2 \times V_2 \) give connected components in \(G \otimes_2 H\).

Further as \( \Delta(U_1) \leq 2 \) as well as \( \Delta(V_1) \leq 2 \), corresponding to other partite set \( U_2 \) and \( V_2 \), we get two components for \( U_2 \times V_2 \) in \(G \otimes_2 H\). Thus the graph \(G \otimes_2 H\) has five components.

(b) By similar arguments as given in Proposition 1.6, for \( \Delta(U_i) \geq 3 \); \( i = 1,2 \), we get connected components \( U_i \times V_1 \), \( U_i \times V_2 \), \( U_1 \times V_1 \) and \( U_2 \times V_2 \) in \(G \otimes_2 H\). Thus, the graph \(G \otimes_2 H\) has four components.

In general from Remarks 2.4, Proposition 2.6 and Corollary 2.7, we can summarize the number of components in \(G \otimes_2 H\) as follows:
Next, we discuss connectedness of $G \otimes_2 H$ for non-bipartite graphs. First we shall prove the following Proposition:

**Proposition 2.8** Let $G$ be a non-bipartite connected graph with $N^2(u) \neq \emptyset, \forall u \in V(G)$. Assume that $G$ contains $C_{2l+1}$, $l > 1$. Then between every pair of vertices, there exists a walk of length $4k$ as well as $4k + 2$; $(k \in \mathbb{N} \cup \{0\})$ form in $G$.

**Proof.** Since $G$ is non-bipartite, it contains an odd cycle. Suppose $G$ contains $C_{2l+1}$ with $V(C_{2l+1}) = \{x_1, \ldots, x_{2l+1}\}$, $l > 1$. Let $u$ and $u'$ be in $V(G)$ with path $P: u = u_0 \rightarrow u_1 \rightarrow \ldots \rightarrow u_{2l+1} = u'$, where $l(P) = d_G(u, u') = 2l + 1$.

Suppose $u$ and $u'$ are on $C_{2l+1}$. Then clearly there is a path between $u$ and $u'$ of even length. Next, assume that $u, u' \in V(C_{2l+1})$. Assume that $u_i$ is the nearest vertex from the cycle $C_{2l+1}$ and $x_j$ is the corresponding nearest vertex of $V(C_{2l+1})$. Suppose $P_i$ is the path between $u_i$ and $x_j$ in $G$. Then there is a walk $W'$ between $u$ and $u'$ in $G$ as follows:

$W': u = u_0 \rightarrow \text{path of } G \rightarrow u_i \rightarrow x_j \rightarrow \text{cycle } C_{2l+1} \rightarrow x_j \rightarrow \text{path of } G \rightarrow u_i \rightarrow \text{path of } G \rightarrow u'$.

Then $l(W') = l(P) + l(C_{2l+1}) + 2l(P_i) = (2l + 1) + (2l + 1) + 2l(P_i)$, which is of even length.

If necessary travelling on the cycle more than once, we get the length of the walk in both the form $4k$ as well as $4k + 2$ in $G$. If $l(P)$ is even, then by same arguments as above we get a walk of length $4k$ and $4k + 2$ in $G$.

Thus in all cases there is a walk between $u$ and $u'$ of length $4k$ as well as $4k + 2$ form in $G$.

Note that since $l > 1$, in every walk $W': u = w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_p = u'$ between $u$ and $u'$ in above cases, we get $d_G(w_i, w_{i+1}) = 2$.

Now onwards, whenever we consider a non-bipartite graph, we assume that it contain a cycle $C_{2l+1}$, $(l > 1)$.

**Proposition 2.9** Let $G$ and $H$ be two connected graphs. Suppose $G$ is a non-bipartite graph. Then,

(a) the graph $G \otimes_2 H$ has two components, if $H$ is a bipartite graph.

(b) the graph $G \otimes_2 H$ is connected, if $H$ is a non-bipartite graph.

**Proof.** (a) Suppose $H$ is a bipartite graph. It is clear that $U \times V_1$ and $U \times V_2$ give two disconnected subgraphs in $G \otimes_2 H$.

Let $(u, v)$ and $(u', v')$ be in $U \times V_1$.

Let path $P_1$ between $u$ and $u'$ in $G$ and path $P_2: v = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_n = v'$; $(n$ is an even integer) between $v$ and $v'$ in $H$ be as follows:

Since $G$ is non-bipartite graph, by Proposition 1.11, there are walks between $u$ and $u'$ of length of the form $4k$ as well as $4k + 2$. Since $n$ is an even integer, as we have discussed in Proposition 2.3, we get a path between $(u, v)$ and $(u', v')$ in $G \otimes_2 H$. Thus $U \times V_1$ gives a connected component, which proves (a).

(b) Let $(u, v)$ and $(u', v')$ be in $U \times V$. Since $G$ and $H$ both are non-bipartite graphs, there exist walks between $v$ and $v'$ of length $4k$ as well as $4k + 2$ form. So, as above we get the result.

**Corollary 2.10** Let $G$ and $H$ be two connected graphs. Then $G \otimes_2 H$ is connected if and only if $G$ and $H$ both are non-bipartite graphs.

Note that the result of usual Tensor product is as follows:

**Proposition 2.11** Let $G$ and $H$ be connected graphs. Then $G \otimes H$ is connected graph if and only if either $G$ or $H$ is non-bipartite.

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III. Distance between two vertices in $G \otimes_2 H$

In this section, we discuss the distance between two vertices in $G \otimes_2 H$ for $G$ and $H$ both are connected and $N^2(w) \neq \emptyset, \forall w \in V(G) \cup V(H)$.

First we define $d_G^*(u, u')$ and $d^*_G(u, u')$ for $u$ and $u'$ in $V(G)$, where $G$ is a connected graph.

**Definition 2.1** Let $G = (U, E)$ be a connected graph and $u, v \in U$. Then,

1. $d_G^*(u, u')$ is defined as the length of a shortest walk $W: u = w_0 \to w_1 \to \ldots \to w_p = u'$ between $u$ and $u'$ of the form $4k (k \in \mathbb{N})$ in which $d_G(w_i, w_{i+1}) = 2$ for $i = 0, 2, 4, \ldots, 4k - 2$.
2. $d^*_G(u, u')$ is defined as the length of a shortest walk $W: u = w_0 \to w_1 \to \ldots \to w_p = u'$ between $u$ and $u'$ of the form $4k + 2 (k \geq 0)$ in which $d_G(w_i, w_{i+1}) = 2$ for $i = 0, 2, 4, \ldots, 4k$.

Note that

\[
\begin{align*}
\forall u, u' \in U, & \quad d_G(u, u') = d_G(u, u')' < d^*_G(u, u')', \\
& \quad \text{if } d_G(u, u') = 4k \\
& \quad \text{if } d_G(u, u') = 4k + 2
\end{align*}
\]

If there is no such shortest walk, then we write $d^*_G(u, u') = \infty (d^*_G(u, u') = \infty)$.

**Remark 3.2**

[i] If $G$ is a non-bipartite graph, then $d_G^*(u, u') < \infty$ and $d^*_G(u, u') < \infty$ for every $u, u' \in V(G)$, by [Proposition 1.8].

[ii] If $G$ is a bipartite graph and even, also if $d_G(u, u')$ is an even number $4k$, then $d^*_G(u, u')$ may not be finite.

For example, if $G = K_2$, then $d_G(u, u') = 4$ but $d^*_G(u, u') = \infty$. However, if $G = C_4$, then $d_G(u, u') = 4$ but $d^*_G(u, u') = 6 < \infty$.

Now we fix the following notations:

Let $G = (U, E)$ and $H = (V, E)$ be connected graphs. Then $V(G \otimes_2 H) = U \times V$. Fix $u, u' \in U$, suppose $d_G = d_G(u, u') = m$ with path $P_G: u = u_0 \rightarrow u_1 \rightarrow \ldots \rightarrow u_m = u'$ and $v, v' \in V$. $d_H = d_H(v, v') = n$ with path $P_H: v = v_0 \rightarrow v_1 \rightarrow \ldots \rightarrow v_n = v'$. Denote $d = d_{G \otimes_2 H}(u, (v, (u', v')))$. We assume that $(u, v)$ and $(u', v')$ are in the same component of $G \otimes_2 H$, i.e., $d < \infty$.

**Proposition 3.3** If $d_G$ and $d_H$ are of the same form $4k$ or $4k + 2$, then $d = \text{Max} \left\{ \frac{1}{2} d_G, \frac{1}{2} d_H \right\}$.

**Proof.** Let $d_G = 4k$ and $d_H = 4t$; $k \leq t$. Then using paths $P_G$ and $P_H$ from $u$ to $u'$ and $v$ to $v'$ in $G$ and $H$ respectively, there is a path $P = (u, v) = (u_0, v_0) \rightarrow (u_1, v_1) \rightarrow \ldots \rightarrow (u_{4k}, v_{4k}) \rightarrow (u_{4k+2}, v_{4k+2}) \rightarrow (u_{2k+2}, v_{2k+2}) \rightarrow \ldots \rightarrow (u_{2t}, v_{2t}) = (u', v')$ between $(u, v)$ and $(u', v')$ of length $2t$ in $G \otimes_2 H$. So, $d \leq 2t = \frac{1}{2} d_H = \text{Max} \left\{ \frac{1}{2} d_G, \frac{1}{2} d_H \right\}$. Similarly if $d_G = 4k + 2$ and $d_H = 4t + 2$, then path $P = (u, v) = (u_0, v_0) \rightarrow (u_1, v_1) \rightarrow \ldots \rightarrow (u_{4k+2}, v_{4k+2}) \rightarrow (u_{2k+2}, v_{2k+2}) \rightarrow \ldots \rightarrow (u_{2t+1}, v_{2t+1}) = (u', v')$ between $(u, v)$ and $(u', v')$ of length $2t + 1$ in $G \otimes_2 H$. So, $d \leq 2t + 1 = \frac{1}{2} d_H = \text{Max} \left\{ \frac{1}{2} d_G, \frac{1}{2} d_H \right\}$.

Conversely suppose that $d < \infty$ with the path $(u, v) = (u_0, v_0) \rightarrow (u_1, v_1) \rightarrow \ldots \rightarrow (u_{4k}, v_{4k}) = (u', v')$ in $G \otimes_2 H$. Then $d_G(u, u_{4k}) = 4d_H(v, v_{4k})$. So, there is a walk $W_G: u = u_0 \rightarrow u_1 \rightarrow \ldots \rightarrow u_4k = u'$ of length $2d$ between $u$ and $u'$ in $G$ with $u_i \neq u_{4k+i}$. Similarly we get a walk $W_H$ between $(v, v')$ in $H$. Hence $d_G \leq 2d$ and $d_H \leq 2d$. So, $\text{Max} \left\{ \frac{1}{2} d_G, \frac{1}{2} d_H \right\} \leq d$. Thus we get $d = \text{Max} \left\{ \frac{1}{2} d_G, \frac{1}{2} d_H \right\}$.

Next, we consider the case in which $d_G$ and $d_H$ are not in same form, but both are even.

**Proposition 3.4** If $d_G = 4k$ and $d_H = 4t + 2$, then $d = \text{Min} \left\{ \frac{1}{2} d_G, \frac{1}{2} d_H, \frac{1}{2} d_G \right\}$.
Proof. First we prove that \( d \leq \min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} \).

If \( d_h = \infty = d_u^{**} \), then it is clear.

Suppose \( d_h < \infty \). Suppose \( d_u = 4k \) and \( d_h^{**} = 4t'; k \leq t' \). Then there is a shortest walk \( W_2: v = w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_d = v' \) such that \( d_y(w_i, w_{i+1}) = 2 \) for \( i = 0, 2, 4, \ldots, 4t' - 2 \). So, using path \( P_t \) and walk \( W_2 \), we get a path between \((u, v)\) and \((u', v')\) in \( G \otimes H \), as in Proposition 3.3. So,
\[
d \leq 2t' = \frac{1}{2} d_u^{**} = \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\} .\]
Similarly, if \( d_u^{**} = 4k' + 2 \), then \( d \leq \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\} .\)

Hence we get \( d \leq \min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} .\)

For the reverse inequality, as \( d < \infty \), as we have seen in Proposition 3.3, there are walks \( W_0 \) and \( W_2 \) between \( u - u' \) and \( v - v' \) respectively with \( l(W_0) = 2d = l(W_2) \). Also, as \( d_u = 4k \) and \( d_h = 4t + 2 \), we get \( d_u = d_u^* < d_h^* \) and \( d_h = d_h^{**} < d_h^{**} \).

Suppose \( d \) is even. Let \( d = 2p \). Then \( l(W_0) = 4p = l(W_2) \). So, \( d_u \leq 4p \) as well as \( d_h^{**} \leq 4p \). Thus
\[
\max \left\{ d_u, d_h^{**} \right\} = \max \left\{ d_u^*, d_h^* \right\} \leq 4p = 2d .\]
If \( d = 2p + 1 \), then \( l(W_0) = 4p + 2 = l(W_2) \). So, \( d_u^{**} \leq 4p + 2 \) and \( d_h^{**} \leq 4p + 2 \) and therefore \( \max \left\{ d_u^{**}, d_h^{**} \right\} \leq 4p + 2 = 2d \). Hence
\[
\min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} \leq d .\]

**Corollary 3.5** Let \( d_h \) be an odd integer.

(i) If \( d_u \) is odd, then \( d = \min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} .\)

(ii) If \( d_h = 4k \), then \( d = \min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} .\)

(iii) If \( d_u = 4k + 2 \), then \( d = \min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} .\)

**Proof.** (i) Suppose \( d_u \) and \( d_h \) both are odd integers.

First we prove that \( d \leq \min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} .\)

Suppose \( \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\} = \infty = \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} .\) Then it is clear.

Suppose \( \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\} < \infty .\) Therefore \( d_u^* = 4k' \) and \( d_h^* = 4t' ; k' \leq t' .\) Then using walks \( W_0 \) and \( W_2 \), we get a path between \((u, v)\) and \((u', v')\), as in Proposition 3.4 by replacing \( P_t \) by \( W_1 \). So,
\[
d \leq \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} .\]
Similarly, if \( d_u^{**} = 4k' + 2 \) and \( d_h^{**} = 4t' + 2 \), then \( d \leq \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} .\) Hence we get
\[
d \leq \min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} .\]

Conversely, as \( d < \infty \), as we have seen in Proposition 3.4, we get
\[
\min \left\{ \max \left\{ \frac{1}{2} d_u^*, \frac{1}{2} d_h^* \right\}, \max \left\{ \frac{1}{2} d_u^{**}, \frac{1}{2} d_h^{**} \right\} \right\} \leq d .\]

(ii) If \( d_u = 4k \), then \( d_u = d_u^* \) and hence the result follows.

(iii) If \( d_u = 4k + 2 \), then \( d_u = d_u^{**} \) and hence the result follows.
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References

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