A Note on “$\alpha - \phi$ Geraghty contraction type mappings”

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Abstract: In this paper, a fixed point result for $\alpha - \phi$ Geraghty contraction type mappings has been proved. Karapinar [2] assumes $\phi$ to be continuous. In this paper, the continuity condition of $\phi$ has been replaced by a weaker condition and fixed point result has been proved. Thus the result proved generalizes many known results in the literature [2-7].

Keywords: Fixed point, $\alpha$ - Geraghty contraction type map, $\alpha$ - $\psi$ Geraghty contraction type, $\alpha - \phi$ Geraghty contraction type, metric space

I. Introduction

The Banach contraction principle [1], which is a useful tool in the study of many branches of mathematics, is one of the earlier and fundamental results in fixed point theory. A number of authors have improved and extended this result either by defining a new contractive mapping or by investigating the existing contractive mappings in various abstract spaces, see, for e.g.,[2-10]. Geraghty [3] obtained a generalization of Banach contraction principle by considering an auxiliary function $\beta : \mathbb{R} \to [0,1)$ satisfying the condition that $\beta(t_n) \to 1$ implies $t_n \to 0$.

He proved the following theorem:

**Theorem 1.1:** Let $(X,d)\) be a metric space and let $T : X \rightarrow X$ be a map. Suppose there exists $\beta \in \mathcal{A}$ such that for all $x, y \in X$:

$$d(Tx, Ty) \leq \beta(d(x,y))d(x,y).$$

Then $T$ has a unique fixed point $x_0 \in X$ with $\{T^n x_0 \}$ converges to $x_0$ for each $x_0 \in X$.

Cho et al. [5] used the concept of $\alpha$ - admissible and triangular $\alpha$ - admissible maps to generalize the result of Geraghty [3].

**Definition 1.1:** Let $T : X \rightarrow X$ be a map and $\beta : \mathbb{R} \times \mathbb{R} \rightarrow [0,1)$ satisfying the condition that $\beta(t_n) \to 1$ implies $t_n \to 0$.

**Definition 1.2:** An $\alpha$ - admissible map is said to be triangular $\alpha$ - admissible if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ and $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

**Definition 1.3:** A map $T : X \rightarrow X$ is called a generalized $\alpha$ - Geraghty contraction type if there exists $\beta \in \mathcal{A}$ such that for all $x, y \in X$:

$$\alpha(x, y)d(Tx, Ty) \leq \beta(M(x,y))M(x,y)$$

Where $M(x, y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$

Cho et al. [5] proved the following theorem:

**Theorem 1.2:** Let $(X,d)$ be a complete metric space. $\alpha : X \times X \rightarrow \mathbb{R}$ be a map and let $T : X \rightarrow X$ be a map. Suppose the following conditions are satisfied:

1) $T$ is generalized $\alpha$ - Geraghty contraction type map
2) $T$ is triangular $\alpha$ - admissible
3) There exists $x_0 \in X$ such that $\alpha(Tx_0, Ty_0) \geq 1$
4) $T$ is continuous

Then $T$ has a fixed point $x_0 \in X$ with $\{T^n x_0 \}$ converges to $x_0$.

Popescu [6] extended this result using concept of $\alpha$ - orbitally admissible and triangular $\alpha$ - orbitally admissible maps:

**Definition 1.4:** Let $T : X \rightarrow X$ be a map. $\alpha : X \times X \rightarrow \mathbb{R}$ be a map. $T$ is said to be $\alpha$ - orbitally admissible if
\( \alpha(x, Tx) \geq 1 \) implies \( \alpha(Tx, T^2 x) \geq 1 \)

**Definition 1.5:** Let \( T : X \rightarrow X \) be a map. \( \alpha : X \times X \rightarrow \mathbb{R} \) be a map. \( T \) is said to be triangular \( \alpha \)-admissible if \( T \) is \( \alpha \)-orbital admissible and \( \alpha(x, y) \geq 1 \) and \( \alpha(y, Ty) \geq 1 \) implies \( \alpha(x, Ty) \geq 1 \).

Popescu [6] proved the following theorem:

**Theorem 1.3:** Let \((X,d)\) be a complete metric space. \( \alpha : X \times X \rightarrow \mathbb{R} \) be a function. Let \( T : X \rightarrow X \) be a map. Suppose the following conditions are satisfied:
1) \( T \) is generalized \( \alpha \)-Geraghty contraction type map
2) \( T \) is triangular \( \alpha \)-orbital admissible map
3) There exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq 1 \), \( T \) is continuous

Then \( T \) has a fixed point \( x, x \in X \) and \( \{T^n x_1\} \) converges to \( x \).

Karapinar [4], introduced the notion of \( \alpha \)-Geraghty contraction type map to extend the result:

Let \( \Psi \) denote the class of the functions \( \psi : [0, \infty) \rightarrow [0, \infty) \) which satisfy the following conditions:

(a) \( \psi \) is non-decreasing.
(b) \( \psi \) is subadditive, that is, \( \psi(s+t) \leq \psi(s) + \psi(t) \) for all \( s, t \).
(c) \( \psi \) is continuous.
(d) \( \psi(t) = 0 \iff t = 0 \).

**Definition 1.6:** Let \((X,d)\) be a metric space, and let \( \alpha : X \times X \rightarrow \mathbb{R} \) be a function. A mapping \( T : X \rightarrow X \) is said to be a generalized \( \alpha \)-\( \psi \)-Geraghty contraction if there exists \( \beta \in \mathfrak{A} \) such that

\[ \alpha(x, y) \psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y))) \psi(M(x, y)) \]

for any \( x, y \in X \) where \( M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\} \) and \( \psi \in \Psi \).

Karapinar, E. [4] proved the following theorem:

**Theorem 1.4:** Let \((X,d)\) be a complete metric space, \( \alpha : X \times X \rightarrow \mathbb{R} \) be a function and let \( T : X \rightarrow X \) be a map. Suppose that the following conditions are satisfied:
1) \( T \) is generalized \( \alpha \)-\( \psi \)-Geraghty contraction type map
2) \( T \) is triangular \( \alpha \)-admissible
3) There exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq 1 \)
4) \( T \) is continuous.

Then, \( T \) has a fixed point \( x^* \in X \), and \( \{T^n x_1\} \) converges to \( x^* \).

Later Karapinar [2] observed that condition of subadditivity of \( \psi \) can be removed:

Let \( \Phi \) denote the class of functions \( \phi : [0, \infty) \rightarrow [0, \infty) \) which satisfy the following conditions:

1) \( \phi \) is nondecreasing
2) \( \phi \) is continuous
3) \( \phi(t) = 0 \iff t = 0 \)

**Definition 1.7:** Let \((X,d)\) be a metric space. \( \alpha : X \times X \rightarrow \mathbb{R} \) be a map. A mapping \( T : X \rightarrow X \) is said to be generalized \( \alpha \)-\( \phi \)-Geraghty contraction type map if there exists \( \beta \in \mathfrak{A} \) such that

\[ \alpha(x, y) \phi(d(Tx, Ty)) \leq \beta(\phi(M(x, y))) (\phi(M(x, y))) \]

for all \( x, y \in X \) where \( M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\} \) and \( \phi \in \Phi \).

Karapinar [2] proved the following theorem:

**Theorem 1.5:** Let \((X,d)\) be a complete metric space, \( \alpha : X \times X \rightarrow \mathbb{R} \) be a function, and let \( T : X \rightarrow X \) be a map. Suppose that the following conditions are satisfied:
1) \( T \) is generalized \( \alpha \)-\( \phi \)-Geraghty contraction type map
2) \( T \) is triangular \( \alpha \)-admissible
3) There exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq 1 \)
4) \( T \) is continuous.

Then \( T \) has a fixed point \( x^* \in X \) and \( \{T^n x_1\} \) converges to \( x^* \).

In this paper, we have shown that above result is true even if the continuity condition of \( \phi \) is replaced by the following weaker condition:
Claim \( r \neq 0 \) from Eq. (3), we get,

\[
\phi (d(x_{n+1}, x_{n+2})) < \phi (d(x_{n+1}, x_{n+1})) \Rightarrow d(x_{n+1}, x_{n+2}) \neq d(x_{n+1}, x_{n+1}) \text{ for all } n
\]

Thus the sequence \( \{d(x_{n}, x_{n+1})\} \) is non-negative and monotonically decreasing.

This implies that \( \lim_{n \to \infty} d(x_{n}, x_{n+1}) = r \geq 0 \)

Claim \( r = 0 \)

If \( r > 0 \), from Eq. (3),
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\[
\frac{\phi(d(x_{n+1}, x_n))}{\phi(M(x_n, x_{n+1}))} \leq \beta(\phi(M(x_n, x_{n+1})) < 1
\]

\[\Rightarrow \lim \beta(\phi(M(x_n, x_{n+1}))) = 1\]

\[\Rightarrow \lim \phi(M(x_n, x_{n+1})) = 0\]

\[\Rightarrow r = \lim d(x_n, x_{n+1}) = 0\] (4)

Now let \((x_n)\) be not Cauchy. Thus, there exists \(\varepsilon > 0\) such that
Given \(k\) there exists \(m(k) > n(k) > k\) such that
\[d(x_{n(k)}, x_{m(k)}) \geq \varepsilon \text{ but } d(x_{n(k)}, x_{m(k)-1}) < \varepsilon\]
\[\Rightarrow d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) < \varepsilon + d(x_{m(k)-1}, x_{m(k)})\]

This implies \(\lim_{k \to \infty} d(x_{n(k)}, x_{m(k)}) = \infty\)
\[\Rightarrow \lim_{k \to \infty} \phi(d(x_{n(k)}, x_{m(k)})) > 0\]
Also \(\lim_{k \to \infty} d(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon\)

Now \(\phi(d(x_{m(k)}, x_{n(k)})) = \phi(d(Tx_{m(k)-1}, Tx_{n(k)-1})) \leq \alpha(x_{m(k)-1}, x_{n(k)-1}) \phi(d(Tx_{m(k)-1}, Tx_{n(k)-1}))\)
\[\leq \beta(\phi(M(x_{m(k)-1}, x_{n(k)-1}))) \phi(M(x_{m(k)-1}, x_{n(k)-1}))\]
\[\Rightarrow \frac{\phi(d(x_{m(k)}, x_{n(k)}))}{\phi(M(x_{m(k)-1}, x_{n(k)-1}))} \leq \beta(\phi(M(x_{m(k)-1}, x_{n(k)-1})))\] (5)

Now \(d(x_{m(k)}, x_{n(k)}) \to \infty\) and \(M(x_{m(k)-1}, x_{n(k)-1}) \to \infty\)
Thus by assumption:
\[\lim \phi(d(x_{m(k)}, x_{n(k)})) = \lim \phi(M(x_{m(k)-1}, x_{n(k)-1}))\] and it will be +ve.
Thus by (5), \(\lim \beta(\phi(M(x_{m(k)-1}, x_{n(k)-1}))) = 1\)
\[\Rightarrow \phi(M(x_{m(k)-1}, x_{n(k)-1})) \to 0\]
\[\Rightarrow M(x_{m(k)-1}, x_{n(k)-1}) \to 0\]
\[\Rightarrow d(x_{m(k)-1}, x_{n(k)-1}) \to 0\] which is a contradiction.
Thus the sequence \((x_n)\) is Cauchy.
Hence the result.

Example: define a map, \(\phi: \mathbb{R} \to \mathbb{R}\) as follows:
\[\phi(x) = 1 \text{ if } x > 0 \text{ & } \phi(x) = 0 \text{ if } x \leq 0\]
Clearly, \(\phi\) is discontinuous but it satisfies the condition given in Eq. (1)
Thus our result applies to a wider class of mappings.

References

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