A study on Ricci soliton in $\mathcal{S}$-manifolds.

K.R. Vidyavathi and C.S. Bagewadi

Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, India.

Abstract: In this paper, we study semi symmetric and pseudo symmetric conditions in $\mathcal{S}$-manifolds, those are $R \cdot R = 0$, $R \cdot C = 0$, $C \cdot R = 0$, $C \cdot C = 0$, $R \cdot R = L_{Q}(g,R)$, $R \cdot C = L_{Q}(g,C)$, $C \cdot R = L_{Q}(g,R)$, and $C \cdot C = L_{Q}(g,C)$, where $C$ is the Concircular curvature tensor and $L_{1}, L_{2}, L_{3}, L_{4}$ are the smooth functions on $M$, further we discuss about Ricci soliton.

Keywords: $\mathcal{S}$-manifold, $\eta$-Einstein manifold, Einstein manifold, Ricci soliton.

1. Introduction

The notion of $f$-structure on a $(2n+s)$-dimensional manifold $M$, i.e., a tensor field of type $(1,1)$ on $M$ of rank $2n$ satisfying $f^{3} + f = 0$, was firstly introduced in 1963 by K. Yano [28] as a generalization of both (almost) contact (for $s = 1$) and (almost) complex structures (for $s = 0$). During the subsequent years, this notion has been furtherly developed by several authors [3], [4], [11], [12], [15], [16], [17]. Among them, H. Nakagawa in [16] and [17] introduced the notion of framed $f$-manifold, later developed and studied by S.I. Goldberg and K. Yano ([11], [12]) and others with the denomination of globally framed $f$-manifolds.

A Riemannian manifold $M$ is called locally symmetric if its curvature tensor $R$ is parallel, i.e., $\nabla R = 0$, where $\nabla$ denotes the Levi-Civita connection. As a generalization of locally symmetric manifolds the notion of semisymmetric manifolds was defined by

$$ (R(X,Y) \cdot R)(U,V)W = 0, \quad X,Y,U,V,W \in TM $$

and studied by many authors [18], [19], [26], [20]. Z.I. Szabo [25] gave a full intrinsic classification of these spaces. R. Deszcz [8, 9] weakened the notion of semisymmetry and introduced the notion of pseudosymmetric manifolds by

$$ (R(X,Y) \cdot R)(U,V)W = L_{R}[(X \wedge Y) \cdot R](U,V)W, \quad (1.1) $$

where $L_{R}$ is smooth function on $M$ and $X \wedge Y$ is an endomorphism defined by

$$ (X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y. \quad (1.2) $$

Definition 1 A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold $(M, g)$. A Ricci soliton is a triple $(g, V, \lambda)$ with $g$ is a Riemannian metric, $V$ is a vector field and $\lambda$ is a real scalar such that

$$ (L_{V})g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \quad (1.3) $$

where $S$ is a Ricci tensor of $M$ and $L_{V}$ denotes the Lie derivative operator along the vector field $V$.

The Ricci soliton is said to be shrinking, steady and expanding according as $\lambda$ is negative, zero and positive respectively. The authors R.Sharma [22, 23, 24] and M.M.Tripathi [27] initiated the study of Ricci solitons in contact manifold. The authors Călín and Crasmareanu [6], Bagewadi and Ingalahalli [14, 1], S.Debnath and A.Battacharya [7] have studied the existence and also obtained results on Ricci solitons in $f$-kenmotsu manifolds, $\alpha$-Sasakian manifolds, Lorentzian $\alpha$-Sasakian manifolds, Trans-Sasakian manifolds using L.P.Eisenhart problem [10]. But C.S.Bagewadi, Ingalahalli and Ashok, C.S.Bagewadi and K.R.Vidyavathi have studied Ricci solitons in Kenmotsu manifolds, almost $C(\alpha)$ manifolds using semi-symmetric and pseudosymmetric conditions [2]. In the present paper, we study Ricci solitons in $\mathcal{S}$-manifolds satisfying semi symmetric and pseudo symmetric conditions those are $R \cdot R = 0$, $R \cdot C = 0$, $C \cdot R = 0$, $C \cdot C = 0$, $R \cdot R = L_{Q}(g,R)$, $R \cdot C = L_{Q}(g,C)$, $C \cdot R = L_{Q}(g,R)$, and $C \cdot C = L_{Q}(g,C)$, where $C$ is the
Concircular curvature tensor and $L_1, L_2, L_3, L_4$ are the smooth functions on $M$.

II. Preliminaries

Let $M$ be a $(2n+s)$-dimensional manifold with an $f$-structure of rank $2n$. If there exists global vector fields $\xi^\alpha, \alpha = (1,2,3,\ldots,s)$ on $M$ such that:

$$f^2 = -I + \sum \xi^\alpha \otimes \eta^\alpha, \quad \eta^\alpha(\xi^\beta) = \delta^\alpha_\beta, \quad (2.1)$$

$$f \xi^\alpha = 0, \quad \eta^\alpha \circ f = 0, \quad (2.2)$$

$$g(X, \xi^\alpha) = \eta^\alpha(X), \quad g(X, fY) = -g(fX, Y), \quad (2.3)$$

where $\eta^\alpha$ are the dual 1-forms of $\xi^\alpha$, we say that the $f$-structure has complemented frames. For such a manifold there exists a Riemannian metric $g$ such that

$$g(X,Y) = g(fX,fY) + \sum \eta^\alpha(X)\eta^\alpha(Y) \quad (2.4)$$

for any vector fields $X$ and $Y$ on $M$.

An $f$-structure $f$ is normal, if it has complemented frames and

$$[f, f] + 2\sum \xi^\alpha \otimes d\eta^\alpha = 0,$$

where $[f, f]$ is Nijenhuis torsion of $f$.

Let $F$ be the fundamental 2-form defined by $F(X,Y) = g(X, fY), X,Y \in T(M)$. A normal $f$-structure for which the fundamental form $F$ is closed, $\eta^\alpha \wedge \ldots \wedge \eta^\alpha \wedge (d\eta)^s \neq 0$ for any $\alpha$, and $d\eta = \ldots = d\eta^s = F$ is called to be an $S$-structure. A smooth manifold endowed with an $S$-structure will be called an $S$-manifold. These manifolds introduced by Blair [3]. We have to remark that if we take $s = 1$, $S$-manifolds are natural generalizations of Sasakian manifolds. In the case $s \geq 2$ some interesting examples are given [3], [13].

If $M$ is an $S$-manifold, then the following relations hold true [3];

$$\nabla_X \xi^\alpha = -fX, \quad X \in T(M), \alpha = 1,2,\ldots,s \quad (2.5)$$

$$(\nabla \xi^\alpha f)Y = \sum \{g(fX,fY)\eta^\alpha + \eta^\alpha(Y)f^2X\}, \quad X,Y \in T(M), \quad (2.6)$$

where $\nabla$ is the Riemannian connection of $g$. Let $\Omega$ be the distribution determined by the projection tensor $f^2$ and let $N$ be the complementary distribution which is determined by $f^2 + I$ and spanned by $\xi^1,\ldots,\xi^s$. It is clear that if $X \in \Omega$ then $\eta^\alpha(X) = 0$ for any $\alpha$, and if $X \in N$, then $fX \equiv 0$. A plane section $\pi$ on $M$ is called an invariant $f$-section if it is determined by a vector $X \in \Omega(x), x \in M$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of $\pi$ is called the $f$-sectional curvature. If $M$ is an $S$-manifold of constant $f$-sectional curvature $k$, then its curvature tensor has the form

$$R(X,Y,Z,W) = \sum_{\alpha,\beta} \{g(fX,fW)\eta^\alpha(Y)\eta^\beta(Z) - g(fX,fZ)\eta^\alpha(Y)\eta^\beta(W) + g(fY,fZ)\eta^\alpha(X)\eta^\beta(W)$$

$$- g(fY,fW)\eta^\alpha(X)\eta^\beta(Z)\} + \frac{1}{4} (k + 3s)\{g(fX,fW)g(fY,fZ) - g(fX,fZ)g(fY,fW)\}$$

$$+ \frac{1}{4} (k - s)\{F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\}, \quad (2.7)$$

where $X,Y,Z,W \in T(M)$. Such a manifold $N(K)$ will be called an $S$-space form. The Euclidean space $E^{2n+s}$ and the hyperbolic space $H^{2n+s}$ are examples of $S$-space forms.
A study on Ricci soliton in S-manifolds.

**Definition 2** S-manifold $(M, f, \eta_\alpha, g, \xi_\alpha)$ is said to be $\eta$-Einstein if the Ricci tensor $S$ of $M$ is of the form

$$S = ag + b\sum\eta_\alpha \otimes \eta_\alpha,$$

where $a, b$ are constants on $M$.  

Now contracting equation (2.7) we get

$$S(Y, Z) = \sum_{\alpha}(\eta_\alpha(Y)X - \eta(Y)X)\eta_\alpha(Z), (2.8)$$

$$S(Y, \xi_\alpha) = \frac{1}{4}[s^2(13 - 6n - k - 3s) + 2s(7n - 5) + k(2 - s) + 2nk(1 - s)]. (2.9)$$

From (2.7) we have

$$R(X, Y)\xi_\alpha = s\sum\eta_\alpha(Y)X - \eta_\alpha(X)Y), (2.10)$$

$$R(\xi_\alpha, Y)Z = s\sum\eta_\alpha(Y)Z - \eta_\alpha(Z)Y), (2.11)$$

$$\eta_\alpha(R(X, Y))Z = s\sum\eta_\alpha(Y)Z - \eta_\alpha(Z)Y). (2.12)$$

**III. Ricci Soliton In Semi-Symmetric S-Manifolds**

An $S$-manifold is said to be semi-symmetric if $R \cdot R = 0$.

$$(R(\xi_\alpha, Y) \cdot R)(U, V)W = 0, (3.1)$$


Using (2.11) in (3.2), we get

$$s\sum\eta_\alpha(Y)R(U, V)W + \eta_\alpha(U)R(Y, V)W$$

$$- g(Y, V)R(U, \xi_\alpha)W + \eta_\alpha(V)R(U, V)W - g(Y, W)R(U, V)\xi_\alpha + \eta_\alpha(W)R(U, V)Y = 0 (3.3)$$

By taking an inner product with $\xi_\alpha$ then we get

$$\sum_{\alpha}[sR(U, V, W, Y) - \eta_\alpha(R(U, V)W)\eta_\alpha(Y) - g(Y, U)\eta_\alpha(R(\xi_\alpha, V)W) + \eta_\alpha(U)\eta_\alpha(R(Y, V)W)$$

$$- g(Y, V)\eta_\alpha(R(U, \xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(R(U, Y)W) - g(Y, W)\eta_\alpha(R(U, V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(R(U, V)Y)\} = 0. (3.4)$$

By using (2.10), (2.12) in (3.4) we have

$$sR(U, V, W, Y) + s^2g(Y, V)g(U, W) - s^2g(Y, U)g(V, W) = 0. (3.5)$$

Taking $U = Y = e_1$ in (3.5) and summing over $i = 1, 2, \ldots, 2n + s$ we get

$$S(V, W) = s(2n + s - 1)g(V, W) (3.6)$$

Thus we state the following;

**Theorem 1** Semi symmetric S-manifold is an Einstein manifold.

If $V$ is co-linear with $\xi$, then Ricci soliton along $\xi$ is given by

$$(L_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y)$$

**Definition 3** Let $(f, \xi_1, \xi_2, \ldots, \xi_s, \eta_1, \eta_2, \ldots, \eta_s, g)$ is the contact S-frame manifold, if $V$ is in the linear span (combination) of $\xi_1, \xi_2, \ldots, \xi_s$, then $V = c_1 \xi_1 + c_2 \xi_2 + \ldots + c_s \xi_s$ and the Ricci soliton is a...
triple \((g, \xi_{\alpha}, \lambda)\) with \(g\) is a Riemannian metric, \(\xi_{\alpha}, (\alpha = 1, 2, \ldots, s)\) is a vector field and \(\lambda\) is a real scalar such that

\[
\sum_{i=1}^{k} c_i L_{\xi_i} g(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0 \tag{3.7}
\]

From (3.7) we have

\[
c_i g(\nabla_X \xi_{\alpha}, Y) + c_i g(\nabla_Y \xi_{\alpha}, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{3.8}
\]

Using (2.5) in (3.8) we get

\[
c_i g(-fX, Y) + c_i g(-fY, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0 \tag{3.9}
\]

From (3.6) and (3.9) we have

\[
(s(2n + s - 1) + \lambda) g(X, Y) = 0 \tag{3.10}
\]

Taking \(X = Y = e_i\) in (3.10) and summing over \(i = 1, 2, \ldots, 2n + s\), we get the value of \(\lambda\)

\[
\lambda = -s(2n + s - 1)(< 0)
\]

Thus we state the following:

**Theorem 2.** Ricci soliton in semi-symmetric \(S\)-manifold is shrinking.

**Corollary 1.** Ricci soliton in semi symmetric \(S\)-manifold is steady if \(s = 0\) (Kaehler manifold) and is shrinking if \(s = 1\) (Sasakian manifold).

**IV.** **Ricci soliton in \(S\)-manifolds satisfying** \(R \cdot C = 0\).

The concircular curvature tensor \(C\) is given by

\[
C(X, Y)Z = R(X, Y)Z - \frac{r}{2n(2n+1)} \{g(Y, Z)X - g(X, Z)Y\} \tag{4.1}
\]

Using (2.10), (2.11) and (2.12) in (4.1) we get

\[
C(X, Y)\xi_{\alpha} = \left[ s - \frac{r}{2n(2n+1)} \right] \sum_{\alpha} \{X\eta_{\alpha}(Y) - \eta_{\alpha}(X)Y\}, \tag{4.2}
\]

\[
C(\xi_{\alpha}, Y)Z = \left[ s - \frac{r}{2n(2n+1)} \right] \sum_{\alpha} \{g(Y, Z)\xi_{\alpha} - Y\eta_{\alpha}(Z)\}, \tag{4.3}
\]

\[
\eta_{\alpha}(C(X, Y)Z) = \left[ s - \frac{r}{2n(2n+1)} \right] \sum_{\alpha} \{g(Y, Z)\eta_{\alpha}(X) - g(X, Z)\eta_{\alpha}(Y)\}. \tag{4.4}
\]

Let us assume that the condition \(R((\xi_{\alpha}, Y) \cdot C)(U, V)W = 0\) hold on \(M\), then

\[
R(\xi_{\alpha}, Y)C(U, V)W - C(R(\xi_{\alpha}, Y)U, V)W - C(U, R(\xi_{\alpha}, Y)V)W - C(U, V)R(\xi_{\alpha}, Y)W = 0. \tag{4.5}
\]

Using (2.11) in (4.5), we get

\[
s\sum_{\alpha} \{g(Y, C(U, V)W)\xi_{\alpha} - \eta_{\alpha}(C(U, V)W)Y - g(Y, U)C(\xi_{\alpha}, V)W + \eta_{\alpha}(U)C(Y, V)W
\]

\[
- g(Y, V)C(U, \xi_{\alpha})W + \eta_{\alpha}(V)C(U, Y)W - g(Y, W)C(U, V)\xi_{\alpha} + \eta_{\alpha}(W)C(U, V)Y\} = 0 \tag{4.6}
\]

By taking an inner product with \(\xi_{\alpha}\) then we get

\[
\sum_{\alpha} \{sC(U, V, W, Y) - \eta_{\alpha}(C(U, V)W)\eta_{\alpha}(Y) - g(Y, U)\eta_{\alpha}(C(\xi_{\alpha}, V)W) + \eta_{\alpha}(U)\eta_{\alpha}(C(Y, V)W)
\]

\[
- g(Y, V)\eta_{\alpha}(C(U, \xi_{\alpha})W) + \eta_{\alpha}(V)\eta_{\alpha}(C(U, Y)W) - g(Y, W)\eta_{\alpha}(C(U, V)\xi_{\alpha}) + \eta_{\alpha}(W)\eta_{\alpha}(C(U, V)Y)\} = 0. \tag{4.7}
\]

By using (4.2), (4.4) in (4.7) we have
A study on Ricci soliton in $S$-manifolds.

\[ C(U, V, W, Y) = \left[ s - \frac{r}{2n(2n+1)} \right] \{ g(Y, U) g(V, W) - g(Y, V) g(U, W) \}. \] (4.8)

Taking $U = Y = e_1$ in (4.8) and summing over $i = 1, 2, \ldots, 2n + s$ and using (4.1) we get

\[ S(V, W) = s(2n + s - 1) g(V, W) \] (4.9)

Thus, we state the following:

**Theorem 3** $S$-manifold satisfying the condition $R \cdot C = 0$ is an Einstein manifold.

From (4.9) and (3.9) we have

\[(s(2n + s - 1) + \lambda) g(X, Y) = 0 \] (4.10)

Taking $X = Y = e_i$ in (4.10) and summing over $i = 1, 2, \ldots, 2n + s$, we get the value of $\lambda$

\[ \lambda = -s(2n + s - 1)(< 0) \]

Thus, we state the following:

**Theorem 4** Ricci soliton in $S$-manifold satisfying the condition $R \cdot C = 0$ is shrinking.

**Corollary 2** Ricci soliton in $S$-manifold satisfying $R \cdot C = 0$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).

**V. Ricci soliton in $S$-manifolds satisfying $C \cdot R = 0$.**

Let us assume that the condition $C((\xi_\alpha, Y) \cdot R)(U, V)W = 0$ hold on $M$, then


Using (4.3) in (5.1), we get

\[
\left[ s - \frac{r}{2n(2n+1)} \right] \sum_\alpha \{ g(Y, R(U, V)W)\xi_\alpha - \eta_\alpha (R(U, V)W)Y - g(Y, U)R(\xi_\alpha, V)W + \eta_\alpha (U)R(Y, V)W
\]

\[ - g(Y, V)R(U, \xi_\alpha)W + \eta_\alpha (V)R(Y, U)W - g(Y, W)R(U, V)\xi_\alpha + \eta_\alpha (W)R(U, V)Y \} = 0 \] (5.2)

By taking an inner product with $\xi_\alpha$ then we get

\[
\sum_\alpha \{ sR(U, V, W, Y) - \eta_\alpha (R(U, V)W)\eta_\alpha (Y) - g(Y, U)\eta_\alpha (R(\xi_\alpha, V)W) + \eta_\alpha (U)\eta_\alpha (R(Y, V)W)
\]

\[ - g(Y, V)\eta_\alpha (R(U, \xi_\alpha)W) + \eta_\alpha (V)\eta_\alpha (R(Y, U)W) - g(Y, W)\eta_\alpha (R(U, V)\xi_\alpha) + \eta_\alpha (W)\eta_\alpha (R(U, V)Y) \} = 0. \] (5.3)

By using (4.2), (4.4) in (5.3) we have

\[ R(U, V, W, Y) = s \{ g(Y, U) g(V, W) - g(Y, V) g(U, W) \}. \] (5.4)

Taking $U = Y = e_i$ in (5.4) and summing over $i = 1, 2, \ldots, 2n + s$ we get

\[ S(V, W) = s(2n + s - 1) g(V, W) \] (5.5)

Thus, we state the following:

**Theorem 5** $S$-manifold satisfying the condition $C \cdot R = 0$ is an Einstein manifold.

From (5.5) and (3.9) we have

\[(s(2n + s - 1) + \lambda) g(X, Y) = 0 \] (5.6)

Taking $X = Y = e_i$ in (5.6) and summing over $i = 1, 2, \ldots, 2n + s$, we get the value of $\lambda$

\[ \lambda = -s(2n + s - 1)(< 0) \]

Thus, we state the following:

**Theorem 6** Ricci soliton in $S$-manifold satisfying the condition $C \cdot R = 0$ is shrinking.
Corollary 3 Ricci soliton in $S$-manifold satisfying $C \cdot R = 0$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).

Ricci soliton in $S$-manifolds satisfying $C \cdot C = 0$.

Let us assume that the condition $C((\xi_\alpha, Y) \cdot C(U, V)W = 0$ hold on $M$, then


(6.1)

Using (4.3) in (6.1), we get

$$[s - \frac{r}{2n(2n + 1)}] \sum_\alpha (g(Y, C(U, V)W)\xi_\alpha - \eta_\alpha (C(U, V)W)Y - g(Y, U)C(\xi_\alpha, V)W + \eta_\alpha(U)C(Y, V)W$$

$$- g(Y, V)C(U, \xi_\alpha)W + \eta_\alpha(V)C(U, Y)W - g(Y, W)C(U, V)\xi_\alpha + \eta_\alpha(W)C(U, V)Y) = 0.$$ 

(6.2)

By taking an inner product with $\xi_\alpha$, then we get

$$\sum_\alpha (sC(U, V, W, Y) - \eta_\alpha(C(U, V)W)\eta_\alpha(Y) - g(Y, U)\eta_\alpha(C(\xi_\alpha, V)W) + \eta_\alpha(U)\eta_\alpha(C(Y, V)W)$$

$$- g(Y, V)\eta_\alpha(C(U, \xi_\alpha)W) + \eta_\alpha(V)\eta_\alpha(C(U, Y)W) - g(Y, W)\eta_\alpha(C(U, V)\xi_\alpha) + \eta_\alpha(W)\eta_\alpha(C(U, V)Y)) = 0.$$ 

(6.3)

By using (4.2), (4.4) in (6.3) we have

$$C(U, V, W, Y) = \left[ s - \frac{r}{2n(2n + 1)} \right] \{g(Y, U)g(V, W) - g(Y, V)g(U, W)\}. \quad (6.4)$$

Taking $U = Y = e_i$ in (4.8) and summing over $i = 1, 2, \ldots, 2n + s$ and using (4.1) we get

$$S(V, W) = s(2n + s - 1)g(V, W). \quad (6.5)$$

Thus we state the following:

Theorem 7 $S$-manifold satisfying the condition $C \cdot C = 0$ is an Einstein manifold.

From (6.5) and (3.9) we have

$$\sum_\alpha (s(2n + s - 1) + \lambda) g(X, Y) = 0 \quad (6.6)$$

Taking $X = Y = e_i$ in (6.6) and summing over $i = 1, 2, \ldots, 2n + s$, we get the value of $\lambda$

$$\lambda = -s(2n + s - 1)(< 0).$$

Thus we state the following:

Theorem 8 Ricci soliton in $S$-manifold satisfying the condition $C \cdot C = 0$ is shrinking.

Corollary 4 Ricci soliton in $S$-manifold satisfying $C \cdot C = 0$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).

VI. Ricci soliton in Pseudo-symmetric $S$-manifolds

An $S$-manifold is said to be Pseudo-symmetric if $R \cdot R = L_1Q(g, R)$.

$$(R(\xi_\alpha, Y) \cdot R)(U, V)W = L_1[(\xi_\alpha \wedge Y) \cdot R](U, V)W, \quad (7.1)$$


$$= L_1[(\xi_\alpha \wedge Y)R(U, V)W - R((\xi_\alpha \wedge Y)U, V)W - R(U, (\xi_\alpha \wedge Y)V)W - R(U, V)(\xi_\alpha \wedge Y)W]. \quad (7.2)$$

Using (2.11) L.H.S of (7.2) is

DOI: 10.9790/5728-1301011222 www.iosrjournals.org 17 | Page
\[ s \sum_{\alpha} \{ g(Y, R(U, V)\hat{W})\xi_{\alpha} - \eta_{\alpha}(R(U, V)\hat{W})Y - g(Y, U)R(\xi_{\alpha}, V)W + \eta_{\alpha}(U)R(Y, V)W \\
- g(Y, V)R(U, \xi_{\alpha})W + \eta_{\alpha}(V)R(U, Y)W - g(Y, W)R(U, V)\xi_{\alpha} + \eta_{\alpha}(W)R(U, Y)\}. \] (7.3)

By taking an inner product with \( \xi_{\alpha} \) then we get

\[ s \sum_{\alpha} \{ sR(U, V, W, Y) - \eta_{\alpha}(R(U, V)\eta_{\alpha}(Y) - g(Y, U)\eta_{\alpha}(R(\xi_{\alpha}, V)W) + \eta_{\alpha}(U)\eta_{\alpha}(R(Y, V)W) \\
- g(Y, V)\eta_{\alpha}(R(U, \xi_{\alpha})W) + \eta_{\alpha}(V)\eta_{\alpha}(R(U, Y)W) - g(Y, W)\eta_{\alpha}(R(U, V)\xi_{\alpha} + \eta_{\alpha}(W)\eta_{\alpha}(R(U, V)Y)\}. \] (7.4)

By using (2.10), (2.12) in (??) we have

\[ s\{sR(U, V, W, Y) + s^2 g(Y, V)g(U, W) - s^2 g(Y, U)g(V, W)\}. \] (7.5)

Again using (2.11) R.H.S of (??), we get

\[ L_1 \{ \sum_{\alpha} \{ g(Y, R(U, V)\hat{W})\xi_{\alpha} - \eta_{\alpha}(R(U, V)\hat{W})Y - g(Y, U)R(\xi_{\alpha}, V)W + \eta_{\alpha}(U)R(Y, V)W \\
- g(Y, V)R(U, \xi_{\alpha})W + \eta_{\alpha}(V)R(U, Y)W - g(Y, W)R(U, V)\xi_{\alpha} + \eta_{\alpha}(W)R(U, Y)\} \}. \] (7.6)

By taking an inner product with \( \xi_{\alpha} \) then we get

\[ L_1 \{ \sum_{\alpha} \{ sR(U, V, W, Y) - \eta_{\alpha}(R(U, V)\eta_{\alpha}(Y) - g(Y, U)\eta_{\alpha}(R(\xi_{\alpha}, V)W) + \eta_{\alpha}(U)\eta_{\alpha}(R(Y, V)W) \\
- g(Y, V)\eta_{\alpha}(R(U, \xi_{\alpha})W) + \eta_{\alpha}(V)\eta_{\alpha}(R(U, Y)W) - g(Y, W)\eta_{\alpha}(R(U, V)\xi_{\alpha} + \eta_{\alpha}(W)\eta_{\alpha}(R(U, V)Y)\}. \] (7.7)

By using (2.10), (2.12) in (7.7) we have

\[ L_1 \{ s\{ sR(U, V, W, Y) + s^2 g(Y, V)g(U, W) - s^2 g(Y, U)g(V, W)\}. \] (7.8)

From (7.5) and (7.8) we get

\[ [L_1 - s] \{ sR(U, V, W, Y) + s^2 g(Y, V)g(U, W) - s^2 g(Y, U)g(V, W)\} = 0. \] (7.9)

Therefore either \( L_1 = s \) or

\[ R(U, V, W, Y) = s\{ g(Y, U)g(V, W) - g(Y, V)g(U, W)\}. \] (7.10)

Taking \( U = Y = e_i \) in (7.10) and summing over \( i = 1, 2, \ldots, 2n + s \) we get

\[ S(V, W) = s(2n + s - 1) g(V, W) \] (7.11)

Thus we state the following:

**Theorem 9** Pseudo symmetric \( S \)-manifold is an Einstein manifold provided \( L_1 \neq s \)

From (7.11) and (3.9) we have

\[ (s(2n + s - 1) + \lambda) g(X, Y) = 0 \] (7.12)

Taking \( X = Y = e_i \) in (7.12) and summing over \( i = 1, 2, \ldots, 2n + s \), we get the value of \( \lambda \)

\[ \lambda = -s(2n + s - 1)(< 0) \]

Thus we state the following:

**Theorem 10** Ricci soliton in pseudo symmetric \( S \)-manifold is shrinking.

**Corollary 5** Ricci soliton in pseudo symmetric \( S \)-manifold is steady if \( s = 0 \) (Kaeahler manifold) and is shrinking if \( s = 1 \) (Sasakian manifold).

**VII.** Ricci soliton in \( S \)-manifolds satisfying \( R \cdot C = L_0 Q(g, C) \).

Let us assume that the condition \( R((\xi_{\alpha}, Y)\cdot C(U, V)W = L_0[(\xi_{\alpha} \wedge Y)\cdot C(U, V)W \) hold on \( M \), then

\[ = L_0[(\xi_{\alpha} \wedge Y)C(U, V)W - C((\xi_{\alpha} \wedge Y)U, V)W - C(U, (\xi_{\alpha} \wedge Y)V)W - C(U, V)(\xi_{\alpha} \wedge Y)W] \]

DOI: 10.9790/5728-1301011222  www.iosrjournals.org 18 | Page
A study on Ricci soliton in $S$-manifolds.

Using (2.11) L.H.S of (??) is

$$s \sum_{\alpha} \left[ g(Y, C(U, V)W, \xi_{\alpha}) - \eta_{\alpha}(C(U, V)W)Y - g(Y, U)C(\xi_{\alpha}, V)W + \eta_{\alpha}(U)C(Y, V)W 
- g(Y, V)C(U, \xi_{\alpha})W + \eta_{\alpha}(V)C(U, Y)W - g(Y, W)C(U, V)\xi_{\alpha} + \eta_{\alpha}(W)C(U, V)Y \right].$$

(8.2)

By taking an inner product with $\xi_{\alpha}$ then we get

$$s \sum_{\alpha} \left[ sC(U, V, W, Y) - \eta_{\alpha}(C(U, V)W)\eta_{\alpha}(Y) - g(Y, U)\eta_{\alpha}(C(\xi_{\alpha}, V)W) + \eta_{\alpha}(U)\eta_{\alpha}(C(Y, V)W) 
- g(Y, V)\eta_{\alpha}(C(U, \xi_{\alpha})W) + \eta_{\alpha}(V)\eta_{\alpha}(C(U, Y)W) - g(Y, W)\eta_{\alpha}(C(U, V)\xi_{\alpha}) + \eta_{\alpha}(W)\eta_{\alpha}(C(U, V)Y) \right].$$

(8.3)

By using (4.2), (4.4) in (??) we have

$$s^{2} \left\{ C(U, V, W, Y) - \left[ s - \frac{r}{2n(2n+1)} \right] [g(Y, U)g(V, W) - g(Y, V)g(U, W)] \right\}. (8.4)$$

Again using (2.11) R.H.S of (8.1) is

$$L_{2} \sum_{\alpha} \left[ g(Y, C(U, V)W, \xi_{\alpha}) - \eta_{\alpha}(C(U, V)W)Y - g(Y, U)C(\xi_{\alpha}, V)W + \eta_{\alpha}(U)C(Y, V)W 
- g(Y, V)C(U, \xi_{\alpha})W + \eta_{\alpha}(V)C(U, Y)W - g(Y, W)C(U, V)\xi_{\alpha} + \eta_{\alpha}(W)C(U, V)Y \right].$$

(8.5)

By taking an inner product with $\xi_{\alpha}$ then we get

$$L_{2} \sum_{\alpha} \left[ sC(U, V, W, Y) - \eta_{\alpha}(C(U, V)W)\eta_{\alpha}(Y) - g(Y, U)\eta_{\alpha}(C(\xi_{\alpha}, V)W) + \eta_{\alpha}(U)\eta_{\alpha}(C(Y, V)W) 
- g(Y, V)\eta_{\alpha}(C(U, \xi_{\alpha})W) + \eta_{\alpha}(V)\eta_{\alpha}(C(U, Y)W) - g(Y, W)\eta_{\alpha}(C(U, V)\xi_{\alpha}) + \eta_{\alpha}(W)\eta_{\alpha}(C(U, V)Y) \right].$$

(8.6)

By using (4.2), (4.4) in (8.6) we have

$$sL_{2} \left\{ C(U, V, W, Y) - \left[ s - \frac{r}{2n(2n+1)} \right] [g(Y, U)g(V, W) - g(Y, V)g(U, W)] \right\}. (8.7)$$

From (8.4) and (8.7) we get

$$[sL_{2} - s^{2}] \left\{ C(U, V, W, Y) - \left[ s - \frac{r}{2n(2n+1)} \right] [g(Y, U)g(V, W) - g(Y, V)g(U, W)] \right\} = 0$$

(8.8)

Therefore either $L_{2} = s$ or

$$C(U, V, W, Y) = \left[ s - \frac{r}{2n(2n+1)} \right] [g(Y, U)g(V, W) - g(Y, V)g(U, W)]$$

(8.9)

Taking $U = Y = e_{i}$ in (8.9) and summing over $i = 1, 2, ..., 2n + s$ we get

$$S(V, W) = s(2n + s - 1)g(V, W)$$

(8.10)

Thus we state the following:

Theorem 11 $S$-manifold satisfying the condition $R \cdot C = L_{2}Q(g, C)$ is an Einstein manifold provided $L_{2} \neq s$.

From (8.10) and (3.9) we have

$$s(2n + s - 1 + \lambda)g(X, Y) = 0 \ (8.11)$$

Taking $X = Y = e_{i}$ in (8.11) and summing over $i = 1, 2, ..., 2n + s$, we get the value of $\lambda$

$$\lambda = -s(2n + s - 1)(< 0)$$

Thus we state the following;

DOI: 10.9790/5728-1301011222 www.iosrjournals.org 19 | Page
Theorem 13 Ricci soliton in $S$-manifold satisfying the condition $C \cdot R = L_3 Q(g, R)$ is shrinking.

Corollary 7 Ricci soliton in $S$-manifold satisfying $C \cdot R = L_3 Q(g, R)$ is steady if $s = 0$ (Kaehler manifold) and is shrinking if $s = 1$ (Sasakian manifold).

IX. Ricci soliton in $S$-manifolds satisfying $C \cdot C = L_4 Q(g, C)$.

Let us assume that the condition $C((\xi_\alpha, Y) \cdot C(U,V)W = L_4[(\xi_\alpha \wedge Y) \cdot C](U,V)W$ hold on $M$, then
(10.1)
Using (6.2), (6.3), (8.5) and (8.6) in (10.1) we get
\[
\left\{ sL_4 - \left[ s - \frac{r}{2n(2n+1)} \right] \right\} \left\{ C(U,V,W,Y) - \left[ s - \frac{r}{2n(2n+1)} \right] g(Y,U)g(V,W) - g(Y,V)g(U,W) \right\} = 0
\]
(10.2)
Therefore, either \( L_4 = s - \frac{r}{2n(2n+1)} \) or \( C(U, V, W, Y) = \left[ s - \frac{r}{2n(2n+1)} \right] \{ g(Y, U) g(V, W) - g(Y, V) g(U, W) \}. \) (10.3)

Taking \( U = Y = e_i \) in (10.3) and summing over \( i = 1, 2, \ldots, 2n + s \), using (4.1) we get
\[
S(V, W) = s(2n + s - 1) g(V, W)
\] (10.4)
Thus we state the following:

**Theorem 15** \( S \)-manifold satisfying the condition \( C \cdot C = L_4 Q(g, C) \) is an Einstein manifold provided
\[
L_4 \neq s - \frac{r}{2n(2n+1)}.
\]

From (10.4) and (3.9) we have
\[
s(2n + s - 1 - \lambda) g(X, Y) = 0 \quad (10.5)
\]
Taking \( X = Y = e_i \) in (10.5) and summing over \( i = 1, 2, \ldots, 2n + s \), we get the value of \( \lambda \)
\[
\lambda = -s(2n + s - 1)(< 0)
\]

Thus we state the following:

**Theorem 16** Ricci soliton in \( S \)-manifold satisfying the condition \( C \cdot C = L_4 Q(g, C) \) is shrinking.

**Corollary 8** Ricci soliton in \( S \)-manifold satisfying \( C \cdot C = L_4 Q(g, C) \) is steady if \( s = 0 \) (Kaehler manifold) and is shrinking if \( s = 1 \) (Sasakian manifold).

X. Conclusion

It is shown that Ricci soliton in \( S \)-manifold satisfying semi-symmetric and pseudo-symmetric conditions are shrinking. Hence if \( S = 1 \), then Sasakian manifolds are shrinking which in accordance with [1], [5], [14], and if \( S = 0 \), then Kaehler manifolds are steady [21]

References


A study on Ricci soliton in $S$-manifolds.


[23]. R. Sharma, Second order parallel tensor on contact manifolds, Algebra, Groups and Geometries, 7 (1990), 787-790.


[28]. K. Yano, On a structure defined by a tensor field $f$ of type $(1,1)$ satisfying $f^3 + f = 0$. Tensor N.S. 14 (1963), 99–109.