Totally geodesic submanifolds of \((k, \mu)\)- contact manifold

M.S. Siddhesha and C.S. Bagewadi
Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, INDIA.

Abstract: In this paper we study invariant submanifolds of \((k, \mu)\)-contact manifold. Here we investigate the conditions for invariant submanifolds of \((k, \mu)\)-contact manifold satisfying \(Q(\sigma, R) = 0\), \(Q(S, \sigma) = 0\) and \(Q(\sigma, C) = 0\) to be totally geodesic, where \(S, R, C\) are the Ricci tensor, curvature tensor and concircular curvature tensor respectively and \(\sigma\) is the second fundamental form.

Keywords: Invariant submanifold, \((k, \mu)\)-contact manifold, totally geodesic.

I. INTRODUCTION

The study of invariant submanifold of \((k, \mu)\)-contact manifold was initiated by Mukut Mani Tripathi et al., [17]. They proved that, an odd dimensional invariant submanifold of a \((k, \mu)\)-contact manifold is a submanifold for which the structure tensor field \(\phi\) maps tangent vectors into tangent vectors. This submanifold inherits a contact metric structure from the ambient space and it is, in fact, a \((k, \mu)\)-contact manifold.

In general, an invariant submanifold of a Sasakian manifold is not totally geodesic. Forexample the circle bundle \((S, Q_n)\) over an \(n\)-dimensional complex projective space \(CP^{n+1}\) is an invariant submanifold of a \((2n + 3)\)-dimensional Sasakian space form with \(c > -3\), which is not totally geodesic [19]. Kon studied invariant submanifold of Sasakian manifold and obtained the well-known result that an invariant submanifold of a Sasakian manifold is totally geodesic, provided that the second fundamental form of the immersion is covariantly constant [9]. Generalizing this Kon’s result, the authors of [17] proved that if the second fundamental form of an invariant submanifold in a \((k, \mu)\)-contact manifold is covariantly constant, then it is totally geodesic.

The authors Montano et al [11] have studied invariant submanifold of \((k, \mu)\)-contact manifold and obtained the main result that every invariant submanifold of a non-Sasakian \((k, \mu)\)-contact manifold is totally geodesic. Conversely, every totally geodesic submanifold of a non-Sasakian \((k, \mu)\)-contact manifold, with \(\mu \neq 0\), and characteristic vector field \(\phi\) is tangent to the submanifold is invariant. Recently, the authors of [2] and [14] find the necessary and sufficient conditions for an invariant submanifold of a \((k, \mu)\)-contact manifold to be totally geodesic, when the second fundamental form is recurrent, 2-recurrent, generalized 2-recurrent, and when the submanifold is semiparallel, pseudoparallel, Ricci-generalized pseudoparallel, 2-Ricci-generalized pseudoparallel. Also in [7], the authors studied invariant submanifolds of Kenmotsu manifold satisfying \(Q(\sigma, R) = 0\) and \(Q(S, \sigma) = 0\). It is seen that invariant submanifolds of various types of contact manifolds have been studied by several authors like [1, 7, 9, 12, 15, 20].

Motivated by these works, in the present paper we consider invariant submanifold of \((k, \mu)\)-contact manifold satisfying \(Q(\sigma, R) = 0\), \(Q(S, \sigma) = 0\), and \(Q(\sigma, C) = 0\), where \(S, R\) and \(C\) are the Ricci tensor, curvature tensor and concircular curvature tensor respectively and \(\sigma\) is the second fundamental form.

The paper is organized as follows.

In section 2, we give necessary details about submanifolds and the concircular curvature tensor. In section 3, we recall the notion of \((k, \mu)\)-contact manifold and the related results. In section 4, we define invariant submanifold of \((k, \mu)\)-contact manifold and review some basic results. Sections 5, 6, 7 deals with the study of invariant submanifolds of \((k, \mu)\)-contact manifold satisfying \(Q(\sigma, R) = 0\), \(Q(S, \sigma) = 0\), and \(Q(\sigma, C) = 0\), where \(S, R, C\) are the Ricci tensor, curvature tensor and concircular curvature tensor respectively.

II. PRELIMINARIES

Let \(M\) be an \(n\)-dimensional submanifold immersed in a \(m\)-dimensional Riemannian manifold \(\bar{M}\), we denote by the same symbol \(g\) the induced metric on \(M\). Let \(TM\) be the set of all vector fields tangent to \(M\) and \(T^\perp M\) is the set of all vector fields normal to \(M\). Then Gauss and Weingarten formulæ are given by [6]

\[
\bar{\nabla}_XY = \nabla_XY + \sigma(X, Y),
\]
(2.1)

\[
\bar{\nabla}_XN = -A_NX + \nabla_X^N,
\]
(2.2)

for all vector fields \(X, Y\) tangent to \(M\) and normal vector field \(N\) on \(M\), where \(\nabla\) is the Riemannian connection on \(M\) determined by the induced metric \(g\) and \(\nabla^N\) is the normal connection on \(T^\perp M\) of \(M\). The second fundamental form \(\sigma\) and \(A_N\) are related by

\[
\bar{g}(\sigma(X, Y), N) = g(A_NX, Y).
\]

DOI: 10.9790/5728-1206068489 www.iosrjournals.org 84 | Page
Totally geodesic submanifolds of \((k, \mu)\) – contact manifold

If \(\sigma = 0\) then the manifold is said to be totally geodesic. Now for a \((0, k)\)-tensor \(T\), \(k \geq 1\) and a \((0, 2)\)-tensor \(B\), \(Q(B, T)\) is defined by [18]

\[
Q(B, T)(X_1, X_2, \ldots, X_k; X, Y) = -T((X \wedge_B Y)X_1, X_2, \ldots, X_k) - T((X \wedge_B Y)X_2, \ldots, X_k),
\]

where \((X \wedge_B Y)Z = B(Y, Z)X - B(X, Z)Y\).

For an \(n\)-dimensional, \((n \geq 3)\), Riemannian manifold \((M, g)\), the concircular curvature tensor \(C\) of \(M\) is defined by [19]

\[
C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)} [g(Y, Z)X - g(X, Z)Y],
\]

for all vector fields \(X, Y\) and \(Z\) on \(M\), where \(r\) is the scalar curvature of \(M\).

III. \((k, \mu)\) – CONTACT MANIFOLD

A manifold \(M^n\) (\(n\)-odd) is said to be a contact manifold if it is equipped with a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^{(n-1)/2}\) everywhere on \(M^n\). For a contact form \(\eta\), it is well known that there exists a vector field \(\xi\), called the characteristic vector field of \(\eta\), such that \(\eta(\xi) = 1\) and \(d\eta(X, \xi) = 0\) for any vector field \(X\) on \(M^n\). A Riemannian metric \(g\) is said to be associated metric if there exists a tensor field \(\phi\) of type \((1, 1)\) such that

\[
\begin{align*}
\eta & = g(\phi X, Y), \\
\phi^2 & = -I + \eta \otimes \xi, \\
\eta(\xi) & = 1, \\
\eta(X) & = g(X, \xi), \\
g(\phi(X, Y), \phi(Y)) & = g(X, Y) - \eta(\xi)g(Y, Y), \\
g(\phi(X), Y) & = -g(\phi(X), Y),
\end{align*}
\]

for all vector fields \(X, Y\) on \(M^n\). The manifold equipped with a contact metric structure is called a contact metric manifold [4].

Given a contact metric manifold \(M^n(\phi, \xi, \eta, g)\), we define a \((1, 1)\) tensor field \(h\) by \(h = \frac{1}{2} \xi \otimes \phi\), where \(\xi\) denotes the Lie differentiation. Then \(h\) is symmetric and satisfies \(h\phi = -\phi h\). Hence, if \(\lambda\) is an eigen value of \(h\) with eigenvector \(X\), \(-\lambda\) is also an eigen value with eigenvector \(\phi X\). Also, we have \(\nabla h = h = 0\) and \(h\xi = 0\). Moreover, if \(\mathcal{V}\) denotes the Riemannian connection of \(g\), then the following relation holds:

\[
\nabla_X^Y \xi = -\phi X - \phi hX.
\]

A contact metric manifold is Sasakian if and only if the relation \(R(X, Y)\xi = \eta(Y)X - \eta(X)Y\) holds for all \(X, Y\), where \(R\) denotes the curvature tensor of the manifold. It is well known that there exists contact metric manifolds for which the curvature tensor \(R\) and the direction of the characteristic vector field \(\xi\) satisfy \(R(X, Y)\xi = 0\) for every vector fields \(X\) and \(Y\).

As a generalization of both \(R(X, Y)\xi = 0\) and the Sasakian case: Blair, Koufogiorgos and Papantoniou introduced the notion of \((k, \mu)\)-nullity distribution and is defined by

\[
N(k, \mu) = \{W \in \mathcal{T}_pM \mid R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\}
\]

for all \(X, Y \in TM\), where \((k, \mu) \in R^2\). A contact metric manifold \(M^n\) with \(\xi \in N(k, \mu)\) is called a \((k, \mu)\)-contact metric manifold. Then, we have

\[
R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].
\]

In a \((k, \mu)\)-contact metric manifold the following relations hold:

\[
\begin{align*}
h^2 & = (k - 1)\phi^2, k \leq 1, \\
(\nabla_X^Y)\phi & = g(X + hY, Y) - \eta(Y)(X + hX), \\
S(X, \xi) & = (n - 1)\eta(X), \\
r & = (n - 1)(n - 3 + k - \frac{(n - 1)}{2})\mu,
\end{align*}
\]

where \(S\) is the Ricci tensor of type \((0, 2)\), \(Q\) is the Ricci operator, i.e., \(g(QX, Y)\) and \(r\) is the scalar curvature of the \((k, \mu)\)-contact manifold have been studied by several authors such as [5, 8, 13, 16] and many others.

From (2.5), we have

\[
C(X, Y)\xi = \left(k - \frac{r}{n(n-1)}\right)[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY].
\]

IV. IN Variant SUB MAN IFOLD OF \((k, \mu)\)-CONTACT MANIFOLD

A submanifold \(M\) of is said to be invariant if the structure vector field \(\xi\) is tangent to \(M\), at every point of \(M\) and \(\phi X\) is tangent to \(M\) for any vector field \(X\) tangent to \(M\) at every point on \(M\), that is, \(\phi(TM) \subset TM\) at every point on \(M\).
Totally geodesic submanifolds of \((k, \mu)\)–contact manifold

**Proposition-1:**[17] Let \(M\) be an invariant submanifold of a \((k, \mu)\)-contact manifold \(\tilde{M}\). Then the following equalities hold on \(M\).

\[
\bar{\nabla}_\xi X = -\phi X - \phi hX, \quad \bar{\nabla}_X \xi = 0, \quad \bar{\nabla}_X Y = \bar{\nabla}_Y X = 0, \quad \bar{\nabla}_X \sigma = 0, \quad \bar{\nabla}_X \varphi = 0, \quad \bar{\nabla}_X \psi = 0
\]

\[
\bar{\nabla}_X \varphi = \phi \bar{\nabla}_X (Y, Z) + \psi \bar{\nabla}_X (X, Z) - \mu \bar{\nabla}_X (Y, Z)
\]

\[
\bar{\nabla}_X \psi = \psi \bar{\nabla}_X (Y, Z) - \mu \bar{\nabla}_X (Y, Z)
\]

**Theorem-2:**[17] An invariant submanifold \(M\) of a \((k, \mu)\)-contact manifold \(\tilde{M}\) is a \((k, \mu)\)-contact manifold.

V. IN Variant SUBMANIFOLDS OF \((k, \mu)\)-CONTACT MINVariant SUBMANIFOLDS OF \((k, \mu)\)-CONTACT MANIFOLDS SATISFYING \(Q(\sigma, R) = 0\)

This section is devoted with the study of invariant submanifolds of \((k, \mu)\)-contact manifolds satisfying \(Q(\sigma, R) = 0\). Therefore

\[
0 = Q(\sigma, R)(X, Y, Z; U, V)
\]

Using (5.2) in (5.1) we have

\[
0 = (U \wedge V) \cdot (X, Y, Z) = -R(U \wedge V, X, Y)Z - R(X, U \wedge V)Z - R(X, Y)U = 0
\]

This implies

\[
k(\sigma(U, X)X - \mu \sigma(U, X)Y)X - \mu \sigma(U, X)YX - k\eta(X)hX = 0
\]

VI. IN Variant SUBMANIFOLDS OF \((k, \mu)\)-CONTACT MANIFOLDS SATISFYING \(Q(S, \sigma) = 0\)

In this section we study invariant submanifolds of \((k, \mu)\)-contact manifold satisfying \(Q(S, \sigma) = 0\). Therefore

\[
0 = Q(S, \sigma)(X, Y, U, V)
\]

This implies

\[
(n - 1)k\sigma(X, V) = 0
\]

DOI: 10.9790/5728-1206068489 www.iosrjournals.org 86 | Page
It follows that $\sigma(X, Y) = 0$, provided $k \neq 0$. Hence $M$ is totally geodesic. Conversely, let $M$ be totally geodesic, then from (6.2) we get $Q(S, \sigma) = 0$.

Thus we can state the following:

**Theorem 4:** An invariant submanifold of a $(k, \mu)$-contact manifold with $k \neq 0$ satisfies $Q(S, \sigma) = 0$ if and only it is totally geodesic.

**Corollary 2:** An invariant submanifold of a Sasakian manifold satisfies $Q(S, \sigma) = 0$ if and only it is totally geodesic.

### VII. INVARIANT SUBMANIFOLDS OF $(k, \mu)$-

#### CONTACT MANIFOLDS SATISFYING $Q(\sigma, C) = 0$

In this section we study invariant submanifolds of $(k, \mu)$-contact manifold satisfying $Q(\sigma, C) = 0$. Therefore

$$0 = Q(\sigma, C)(X, Y, Z; U, V)$$

Using (5.2) in (7.1) we have

$$-\sigma(V, X)(C(U, Y)Z + C(U, C)(Y, Z)Z - C(X, U)Z)Z - \sigma(Y, Z)C(X, Y)Z - \sigma(V, Z)C(X, Y)U + \sigma(U, Z)C(X, Y)V = 0.$$ 

Putting $Z = V = \xi$ in (7.2) and in view of (4.2), we obtain

$$\sigma(U, X)C(\xi, Y)\xi + \sigma(U, Y)C(\xi, X)\xi = 0.$$ 

Using (3.10) in (7.3) we have

$$(k - \frac{r}{n(n-1)})[\eta(\sigma(U, X) - \mu \sigma(U, X)hY + (k - \frac{r}{n(n-1)})[X - \sigma(U, X)]\sigma(U, Y)$$

Using inner product with $W$ yields

$$(k - \frac{r}{n(n-1)})[\eta(\sigma(U, X) - \mu \sigma(U, X)hY + (k - \frac{r}{n(n-1)})[X - \sigma(U, X)]\sigma(U, Y)$$

Taking inner product with $W$ yields

$$(k - \frac{r}{n(n-1)})[\eta(\sigma(U, X) - \mu \sigma(U, X)hY + (k - \frac{r}{n(n-1)})[X - \sigma(U, X)]\sigma(U, Y)$$

Contracting $Y$ and $W$, we get

$$(k - \frac{r}{n(n-1)})[\sigma(U, X) + \mu \sigma(U, X)hX = 0.$$ 

This implies

$$(2-n)k - \frac{r}{n(n-1)}[\sigma(U, X) = 0,$$ 

and hence $Q(U, X) = 0$, provided $r \neq n(n-1)/(2-n), k \pm \mu l$. Thus the manifold is totally geodesic. Conversely, if $Q(U, X) = 0$, then from (7.2), it follows that $Q(\sigma, C) = 0$. Therefore in view of the above results we get

**Theorem 5:** An invariant submanifold of a $(k, \mu)$-contact manifold with $r \neq n(n-1)/(2-n), k \pm \mu l$ satisfies $Q(\sigma, C) = 0$ if and only it is totally geodesic.

**Corollary 3** An invariant submanifold of a Sasakian manifold with $r \neq n(n-1)$ satisfies $Q(\sigma, C) = 0$ if and only it is totally geodesic.

### VIII. EXAMPLE

We consider five dimensional manifold $M = \{(x_1, x_2, y_1, y_2, z) \in R^5: z \neq 0\}$, where $(x_1, x_2, y_1, y_2, z)$ are standard coordinates in $R^5$. We choose the vector fields

$$e_1 = 2\frac{\partial}{\partial x_1}, \quad e_2 = 2\frac{\partial}{\partial x_2}, \quad e_3 = 2\left(\frac{\partial}{\partial y_1} + x_1\frac{\partial}{\partial z}\right), \quad e_4 = 2\left(\frac{\partial}{\partial y_2} + x_1\frac{\partial}{\partial z}\right), \quad e_5 = 2\frac{\partial}{\partial z},$$

which are linearly independent at each point of $M$. Let $g$ be the Riemannian metric defined by

$$g = \frac{1}{4}(dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + dy_1 \otimes dy_1 + dy_2 \otimes dy_2) + \eta \otimes \eta,$$

where $\eta$ is the 1-form defined by $\eta(X) = g(X, e_5)$ for any vector field $X$ on $M$. Hence $(e_1, e_2, e_3, e_4, e_5)$ is an orthonormal basis of $M$. We define the $(1,1)$ tensor field $\phi$ as

$$\phi(e_1) = e_2, \quad \phi(e_2) = e_3, \quad \phi(e_3) = -e_1, \quad \phi(e_4) = -e_2, \quad \phi(e_5) = 0.$$ 

The linear property of $g$ and $\phi$ yields that

$$\eta(e_5) = 1, \quad \phi X = -X + \eta(X)e_5, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(\eta(Y), Y),$$

for any vector fields $X, Y$ on $M$. Thus $\Sigma = \xi, M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.

DOI: 10.9790/5728-1206068489 www.iosrjournals.org 87
Moreover, we get 
\[ [e_i, e_j] = 2e_5, \quad [e_2, e_4] = 2e_5, \]
and remaining \([e_i, e_j] = 0\) for all \(1 \leq i, j \leq 5\).

The Riemannian connection \(\nabla\) of the metric tensor \(g\) is given by Koszula formula which is given by,
\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Z, [X, Y]) \]

Using Koszul’s formula we get the following:
\[ \nabla_{e_1}e_3 = e_5, \quad \nabla_{e_1}e_5 = -e_3, \quad \nabla_{e_2}e_4 = e_5, \quad \nabla_{e_2}e_5 = -e_4, \]
\[ \nabla_{e_3}e_1 = -e_5, \quad \nabla_{e_3}e_5 = e_1, \quad \nabla_{e_4}e_2 = -e_5, \quad \nabla_{e_4}e_5 = e_2, \]
and the remaining \(\nabla_{e_i}e_j = 0\), for all \(1 \leq i, j \leq 5\).

From the above results it is easy to verify that \(\bar{M}\) is a \((k, \mu)\)-contact manifold with \(k = 1\) and \(\mu = 0\).

Let \(M\) be a subset of \(\bar{M}\) and consider the isometric immersion \(f: M \rightarrow \bar{M}\) defined by \(f(x^1, y^1, z) = f(x^1, 0, y^1, 0, z)\).

It can be easily prove that\(M = \{(x^1, y^1, z) \in R^5: (x^1, y^1, z) \neq 0\}\), where \((x^1, y^1, z)\) are standard coordinates in \(R^5\) is a 3-dimensional submanifold of the 5-dimensional \((k, \mu)\)-contact manifold \(\bar{M}\).

We choose the vector fields
\[ e_1 = 2\frac{\partial}{\partial x^1}, \quad e_3 = 2\left(\frac{\partial}{\partial y^1} + x^1\frac{\partial}{\partial z}\right), \quad e_5 = 2\frac{\partial}{\partial z}, \]

which are linearly independent at each point of \(M\). Let \(g\) be the Riemannian metric defined by
\[ g = \frac{1}{4}(dx^1 \otimes dx^1 + dy^1 \otimes dy^1) + \eta \otimes \eta, \]
where \(\eta\) is the 1-form defined by \(\eta(X) = g(X, e_5)\) for any vector field \(X\) on \(M\). Hence \((e_1, e_3, e_5)\) is an orthonormal basis of \(M\). We define the \((1, 1)\) tensor field \(\phi\) as
\[ \phi(e_1) = e_3, \quad \phi(e_3) = -e_1, \quad \phi(e_5) = 0. \]

The linear property of \(g\) and \(\phi\) yields that
\[ \eta(e_5) = 1, \quad \phi^2 X = -X + \eta(X)e_5, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \]
for any vector fields \(X, Y\) on \(M\). Thus \(\phi e_5 = \xi\). \(\phi\) is also an almost contact metric manifold. Taking \(e_5 = \xi\), and using Koszul’s formulae for the metric \(g\), it can be easily calculated that
\[ \nabla_{e_1}e_3 = e_5, \quad \nabla_{e_1}e_5 = -e_3, \quad \nabla_{e_3}e_1 = -e_3, \]
\[ \nabla_{e_3}e_5 = e_1, \quad \nabla_{e_5}e_3 = e_1, \]
and the remaining \(\nabla_{e_i}e_j = 0\), for all \(1 \leq i, j \leq 5\) and \(i, j \neq 2, 4, 5\).

Let us consider,
\[ TM = D \oplus D^\perp \oplus <\xi>, \]
where \(D = <e_1> \supset D^\perp = <e_3>\). Then we see that \(\sigma(e_1, e_5) = 0\), \(e_1 \in D\) and \(\phi(e_3) = -e_1 \in D\), \(fore e_3 \in D^\perp\). Hence the submanifold is invariant. Now from the values of \(\nabla_{e_i}e_j\), we see that \(\sigma(e_i, e_j) = 0\), for all \(i, j = 1, 3, 5\). This means that the submanifold is totally geodesic. Thus the theorems 3-5 are verified.

REFERENCES


