(ε, ε ∨ q) -Q-Fuzzy Subgroups and Normal Subgroups

D. Hazarika1, K. D. Choudhury2
1Department of Mathematics, DHSK College. Dibrugarh, Assam, India.
2Department of Mathematics, Assam University, Srich, Assam, India.

Abstract: In this paper, the notions of (ε, ε ∨ q) -Q-fuzzy subgroup and normal subgroup are introduced and some of their properties are investigated.

Keywords: Q-fuzzy subgroup and normal subgroup, Fuzzy point, (ε, ε ∨ q) -Q-fuzzy subgroup.

I. Introduction

The idea of fuzzy subgroups was initiated by Rosenfield [1]. Goguen [9] generalised the notion of fuzzy subset of X to that of an L-fuzzy subset namely a function from X to a lattice L. Muthuraj et al. [10] introduced the notion of Q-fuzzy set. Solaiaraj and Nagaranjan [3] brought in the concept of Q-fuzzy groups. Priya, Ramachandran and Nagalakshmi [7] extended this idea to Q-fuzzy normal subgroups. The concept of “belongs to’ and ‘ quasi coincident with’ between fuzzy point and fuzzy set was introduced by Bakhat and Das [6] with the study of (ε, ε ∨ q) -fuzzy subgroups and (ε, ε ∨ q) -fuzzy subrings. Herein, (ε, ε ∨ q) -Q-fuzzy subgroup and normal subgroup are defined and some results obtained.

II. Preliminaries

Definition 2.1 : A mapping μ : G × Q → [0,1] where G is a group and Q a non empty set, is called a Q-fuzzy set in G. For any Q-fuzzy set μ in G and t ∈ [0,1], the set U(μ, t) = {μ(x, q) ≥ t, q ∈ Q} is called the upper cut of μ.

Definition 2.2 : A Q-fuzzy set μ in a group G is called a Q-fuzzy subgroup if ∀x, y ∈ G and q’ ∈ Q, μ(xy⁻¹, q’) ≥ min{μ(x, q’), μ(y, q’)}

Example 2.3 : Let (Z, +) be the additive group and Q denotes the set of integers. Define

μ : Z × Q → [0,1] with μ(x, q’) =

= 0.7 if x is even

= 0.3 if x is odd

for all q’ in Q.

Then μ is a Q-fuzzy subgroup of Z.

Solution : (i) Let x, y be even, The x – y is even. Therefore μ(x – y, q) = 0.7 and

min{μ(x, y), μ(y, q)} = min{0.7, 0.7} = 0.7 so that μ(x – y, q) = min{μ(x, q), μ(y, q)}

(ii) Let x, y be odd. Then x – y is even. Therefore μ(x – y, q) = 0.7 and

min{μ(x, q), μ(y, q)} = min{0.3, 0.3} = 0.3 so that μ(x – y, q) > min{μ(x, q), μ(y, q)}

(iii) Let x be even (odd) and y odd (even). Therefore, x – y must be odd. Now μ(x – y, q) = 0.3 and min{μ(x, q), μ(y, q)} = min{0.7, 0.3} = 0.3 so that

μ(x – y, q’) = min{μ(x, q’), μ(y, q’)} . Thus, ∀x, y ∈ (Z, +) and q’ ∈ Q,

μ(x – y, q’) ≥ min{μ(x, q’), μ(y, q’)} . Hence μ is a Q-fuzzy subgroup of Z.
Example 2.4: Let us take the multiplicative group $G$ where $G = \{1, -1, i, -i\}$. We define $\mu : G \times Q \to [0,1]$ where $Q$ denotes the set of real numbers, by setting $\mu(1, q') = 0.8, \mu(-1, q') = 0.5, \mu(i, q') = 0.3 = \mu(-i, q')$. Then $\mu$ is a $Q$-fuzzy subgroup of $G$.

Solution: Clearly,

$\mu(1(-1)^{-1}, q') = \mu(-1, q') = 0.5 \min \{\mu(1, q'), \mu(-1, q')\} = 0.5. \mu(1(-1)^{-1}, q') = \min \{\mu(1, q'), \mu(-1, q')\}$

$\mu((-1)i^{-1}, q') = \mu(i, q') = 0.8 \min \{\mu(1, q'), \mu(i, q')\} = 0.8. \mu((-1)i^{-1}, q') = \min \{\mu(1, q'), \mu(i, q')\}$

$\mu((i)i^{-1}, q') = \mu(1, q') = 0.3 \min \{\mu(1, q'), \mu(i, q')\} = 0.3. \mu((i)i^{-1}, q') = \min \{\mu(1, q'), \mu(i, q')\}$

Thus, $\forall x, y \in G$ and $q' \in Q, \mu(xy^{-1}, q') \geq \min \{\mu(x, q'), \mu(y, q')\}$. Hence, $\mu$ is a $Q$-fuzzy subgroup of $G$.

Definition 2.5: A $Q$-fuzzy set $\mu$ of a group $G$ is called a $Q$-fuzzy normal subgroup of $G$ if $\forall x, y \in G$ and $q' \in Q$.

$\mu(xy^{-1}, q') \geq \min \{\mu(x, q'), \mu(y, q')\}$

Or equivalently $\mu(xy, q') = \mu(y, q')$.

Definition 2.6: Let $\mu$ be a $Q$-fuzzy set in a group $G$. Let us define,

$\mu : G \times Q \to G \times Q$

$\mu(a, q') = (xa, q')$

A $Q$-fuzzy left coset $\mu_x$ is defined as $\mu_x = f_x(\mu)$. Likewise, the $Q$-fuzzy right coset is defined as $\mu_x = f_x(\mu)$.

It can be readily seen that $\mu(y, q') = \mu(x^{-1}y, q')$ and $\mu_y(y, q') = \mu(xy^{-1}, q') \forall (y, q') \in G \times Q$.

Definition 2.7: Let $G$ be a group and $Q$ a nonempty set. A $Q$-fuzzy point $(x, q')$ is a function defined as

$(x, q') : G \times Q \to [0,1]$ where $\theta(x, q'), (y, q') = \begin{cases} t & \text{if } (x, q') = (y, q') \\ 0 & \text{if } (x, q') \neq (y, q') \end{cases}$
A Q-fuzzy point \((x, q')\), is said to belong to Q-fuzzy set \(\mu\) i.e. \((x, q')_t \in \mu\) if \(\mu(x, q') \geq t\) and a Q-fuzzy point \((x, q')_t\) is said to quasi coincident with a Q-fuzzy set \(\mu\) written as \((x, q')_t \in \mu\) if \(\mu(x, q') + t > 1\).

If \((x, q')_t \in \mu, or(x, q'), q \mu\), we write \((x, q')_t \in \vee q \mu\).

III. \((\varepsilon, \varepsilon \in \vee q)\)-Q-fuzzy subgroup

**Definition 3.1**: Let \(G\) be a group. A Q-fuzzy subset \(\mu : G \times Q \rightarrow [0, 1]\) is called \((\varepsilon, \varepsilon \in \vee q)\)-Q-fuzzy subgroup of \(G\) if \((x, q')_t \in \mu, (y, q')_t \in \mu \Rightarrow \{(xy^{-1}, q')_{m(t,s)} \in \vee q \mu\}

where \(m(t,s) = \min[t,s]\).

**Theorem 3.2**: Intersection of two \((\varepsilon, \varepsilon \in \vee q)\) subgroups of -Q fuzzy a group \(G\), is again a \((\varepsilon, \varepsilon \in \vee q)\)-Q-fuzzy subgroup of \(G\).

**Proof**: Let \(\mu\) and \(\nu\) be two \((\varepsilon, \varepsilon \in \vee q)\)-Q-fuzzy subgroups of a group \(G\).

Let, \((x, q')_t \in \mu \cap \nu, (y, q')_s \in \mu \cap \nu\) where \(t, s \in [0, 1]\)

So, \((x, q')_t \in \mu \cap \nu, (y, q')_s \in \mu \cap \nu\)

\(\Rightarrow (x, q')_t \in \nu, (y, q')_s \in \nu\)

\(\Rightarrow (xy^{-1}, q')_{m(t,s)} \in \vee q \mu \cap \nu\)

Hence the proof.

**Remark 3.3**: i) The result can be extended to a family of \((\varepsilon, \varepsilon \in \vee q)\)-Q-fuzzy subgroups.

ii) However, the union of two \((\varepsilon, \varepsilon \in \vee q)\)-Q-fuzzy subgroups of a group \(G\) is not necessarily a \((\varepsilon, \varepsilon \in \vee q)\)-Q-fuzzy subgroups of \(G\).

**Theorem 3.4**: A Q-fuzzy subset \(\mu\) in a group \(G\) is a Q-fuzzy subgroup of \(G\) if and only if \(\mu\) is a \((\varepsilon, \varepsilon)\)-Q-fuzzy subgroup of \(G\).

**Proof**: Let \(\mu\) be a Q-fuzzy subgroup of \(G\). Let \(x, y \in G\) such that \((x, q')_t \in \mu, (y, q')_s \in \mu\) where \(t, s \in [0, 1]\). Then \(\mu(x, q') \geq t, \mu(y, q') \geq s\).

Now \(\mu(xy^{-1}, q') \geq \min\{\mu(x, q'), \mu(y, q')\} \geq \min\{t, s\} = m(t,s) \Rightarrow \{(xy^{-1}, q')_{m(t,s)} \in \mu \Rightarrow \mu\) is a \((\varepsilon, \varepsilon)\)-Q-fuzzy subgroup of \(G\).

Conversely, let \(\mu\) be a \((\varepsilon, \varepsilon)\)-Q-fuzzy subgroup of \(G\). Let \(x, y \in G\). Let \(\mu(x, q') = t, \mu(y, q') = s\) where \(t, s \in [0, 1]\). Then \(\mu(x, q') \geq t, \mu(y, q') \geq s\) \(\Rightarrow (x, q')_t \in \mu, (y, q')_s \in \mu\), where \(\mu\) is a \((\varepsilon, \varepsilon)\)-Q-fuzzy subgroup of \(G\).

So, \((xy^{-1}, q')_{m(t,s)} \in \mu \Rightarrow \{(xy^{-1}, q')_{m(t,s)} \geq m\{t, s\} = \min\{\mu(x, q'), \mu(y, q')\} \Rightarrow \mu\) is a Q-fuzzy subgroup of \(G\).

**Remark 3.5**: If \(\mu\) is \((\varepsilon, \varepsilon)\)-Q-fuzzy subgroup of \(G\) then it is also a \((\varepsilon, \varepsilon \in \vee q)\)-Q-fuzzy subgroup of \(G\).
Theorem 3.6: If $\mu$ is a $(q,q)$-Q-fuzzy subgroup of $G$ then $\mu$ is also $(\in,\in)\cdot Q$-fuzzy subgroup of $G$.

Proof: Let $\mu$ be a $(q,q)$-Q-fuzzy subgroup of $G$. Let $x, y \in G$ such that $(x,q') \in \mu(y,q')$ where $t,s \in [0,1]$. Then, $x, y \in G$ such that $(x,q') \in \mu(y,q')$ where $t,s \in [0,1]$. Then, $\mu(x,q') \geq t, \mu(y,q') \geq s \Rightarrow (x,q') + \delta > t, \mu(y,q') + \delta > s$, for any $\delta > 0$.

$\Rightarrow \mu(x,q') + 1 - t + \delta > 1, \mu(y,q') + 1 - s + \delta > 1 \Rightarrow (x,q')_{(t-s)} \mu(y,q')_{(t-s)} \mu$. But $\mu$ is a $(q,q)$-Q-fuzzy subgroup of $G$. So,

$\mu(x,q')_{(t,s)} \subseteq \mu$ \Rightarrow $\mu$ is a $(\in,\in)\cdot Q$-fuzzy subgroup of $G$.

Theorem 3.7: A Q-fuzzy subgroup $\mu$ in $G$ is a $(\in,\in\vee q)$ -Q-fuzzy subgroup of $G$ if and only if $\mu(x^{-1},q') \geq \min \{\mu(x,q'), \mu(y,q'), 0.5\} \forall x, y \in G$.

Proof: Let $\mu$ be a $(\in,\in\vee q)$ -Q-fuzzy subgroup of $G$.

Case 1: Let $\min \{\mu(x,q'), \mu(y,q')\} < 0.5$.

Then, $\min \{\mu(x,q'), \mu(y,q'), 0.5\} = \min \{\mu(x,q'), \mu(y,q')\}$. If possible, let $\mu(x^{-1},q') < \min \{\mu(x,q'), \mu(y,q')\}$. Let us choose a real number $t$ such that

$\mu(x^{-1},q') < t < \min \{\mu(x,q'), \mu(y,q')\}$

$\Rightarrow \mu(x,q') > t, \mu(y,q') > t \Rightarrow (x,q')_{t} \subseteq \mu(y,q')_{t} \subseteq \mu$. But $\mu(x^{-1},q') < t \Rightarrow (x^{-1},q')_{t} \subseteq \mu$ and $\mu(x^{-1},q') + t < 2t < 2 \min \{\mu(x,q'), \mu(y,q')\} < 1$, a contradiction to the fact that $\mu$ is a $(\in,\in\vee q)$ -Q-fuzzy subgroup of $G$. Thus we must have $\mu(x^{-1},q') \geq \min \{\mu(x,q'), \mu(y,q')\} = \min \{\mu(x,q'), \mu(y,q'), 0.5\} \forall x, y \in G$.

Case 1: Let $\min \{\mu(x,q'), \mu(y,q')\} \geq 0.5 \forall x, y \in G$. Then $\min \{\mu(x,q'), \mu(y,q'), 0.5\} = 0.5$ If possible, let $\mu(x,q') < \min \{\mu(x,q'), \mu(y,q'), 0.5\} = 0.5$. Therefore $\mu(x,q') \geq 0.5$ and $\mu(y,q') \geq 0.5 \Rightarrow (x,q')_{0.5} \subseteq \mu$, and $(y,q')_{0.5} \subseteq \mu$. But $\mu(x^{-1},q') < 0.5 \Rightarrow (x^{-1},q')_{0.5} \subseteq \mu$ and so $\mu(x^{-1},q') + 0.5 < 0.5 + 0.5 = 1$, a contradiction to the fact that $\mu$ is a $(\in,\in\vee q)$ -Q-fuzzy subgroup of $G$.

Hence we have, $\mu(x^{-1},q') \geq 0.5 = \min \{\mu(x,q'), \mu(y,q'), 0.5\}$.

Conversely, let $\mu(x^{-1},q') \geq \min \{\mu(x,q'), \mu(y,q'), 0.5\}$.

Let $\forall x, y \in G$ such that $(x,q')_{t} \subseteq \mu$ and $(y,q')_{s} \subseteq \mu$ where $t,s \in [0,1]$. Then $\mu(x,q') \geq t$ and $\mu(y,q') \geq s \Rightarrow \min \{\mu(x,q'), \mu(y,q')\} \geq t, s$. But $\mu(x^{-1},q') \geq \min \{\mu(x,q'), \mu(y,q'), 0.5\} \geq m(t,s, 0.5)$

If $m(t,s) \leq 0.5$ then $m(t,s, 0.5) = m(t,s)$. So, $\mu(x^{-1},q') \geq m(t,s) \Rightarrow (x^{-1},q')_{m(t,s)} \subseteq \mu$. 

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If \( m(t,s) > 0.5 \) then \( m(t,s,0.5) = 0.5 \). So, \( \mu(xy^{-1},q') \geq 0.5 \Rightarrow \mu(xy^{-1},q') + m(t,s) \geq 0.5 + m(t,s) > 1 \).

\[ \Rightarrow (xy^{-1},q')_{m(t,s)} q^\mu So(xy^{-1},q')_{m(t,s)} \in \vee q^\mu. \] Therefore, \( \mu \) is a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G \).

**Theorem 3.8**: If the \( Q \)-fuzzy subgroup \( \mu \) of \( G \) is a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G \) and \( \mu(xy^{-1},q') < 0.5 \) \( \forall x \in G \), then \( \mu \) is also a \((\varepsilon, \varepsilon)\) - fuzzy subgroup of \( G \).

**Proof**: Since, \( \mu \) is a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G \), \( \forall x, y \in G \),

\[ \mu(xy^{-1},q') \geq \min \{ \mu(x,q'), \mu(y,q') \} \Rightarrow 0.5 > \mu(xy^{-1},q') \geq \min \{ \mu(x,q'), \mu(y,q') \} \]

\[ \Rightarrow \mu(x,q') < 0.5 \text{ and } \mu(y,q') < 0.5. \] Now let \((x,q'),(y,q') \in \mu \) where \( t,s \in [0,1] \).

Then \( \mu(x,q') \geq t, \mu(y,q') \geq s \ i.e. t < 0.5, s < 0.5 \Rightarrow m(t,s) < 0.5 \).

\[ *: \mu \text{ is a } (\varepsilon, \varepsilon^\vee q) \text{- fuzzy subgroup of } G. \]

\[ \mu(xy^{-1},q') \geq m(t,s) \text{ or } \mu(xy^{-1},q') + m(t,s) > 1. \]

Since, \( m(t,s) < 0.5 \) we must have in both situations, \( \mu(xy^{-1},q') \geq m(t,s) \). Therefore \( \mu \) is also a \((\varepsilon, \varepsilon)\) - fuzzy subgroup of \( G \).

**Theorem 3.9**: Let \( \mu \) be \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G \) and \( g \in G \). Then, \( g \mu_g^{-1} \) is also a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G \).

**Proof**: Let \((x,q'),(y,q') \in g \mu_g^{-1} \) where \( t,s \in [0,1] \). Then,

\[ (g \mu_g^{-1})(x,q') \geq t, (g \mu_g^{-1})(y,q') \geq s \Rightarrow \mu(g^{-1}xg,q') \geq t, \mu(g^{-1}yg,q') \geq s \Rightarrow (g^{-1}xg,q'),(g^{-1}yg,q') \in \mu \]

Since, \( \mu \) is a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G \),

\[ \mu((g^{-1}xg)(g^{-1}yg^{-1}),q') \geq m(t,s) \vee \mu((g^{-1}xg)(g^{-1}yg^{-1}),q') + m(t,s) > 1. \] \( \text{(A)} \)

Now,

\[ \mu((g^{-1}xg)(g^{-1}yg^{-1}),q') = \mu((g^{-1}xg)(yg^{-1}g),q') = \mu((g^{-1}xg)(g^{-1}y^{-1}g),q') = \mu(g^{-1}(xg)(y^{-1}g),q') \]

\[ = \mu(g^{-1}(xy^{-1}g),q') = (g \mu_g^{-1})(xy^{-1},q') \]

Therefore from \( \text{(A)} \),

\[ (g \mu_g^{-1})(xy^{-1},q') \geq m(t,s) \vee (g \mu_g^{-1})(xy^{-1},q') + m(t,s) > 1 \]

Therefore, \( g \mu_g^{-1} \) is also a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G \).

**IV. Homomorphism of \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup**

**Theorem 4.1**: Let \( f \) be a homomorphism. If \( \mu' \) be a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( f : G \times Q \rightarrow G' \times Q \) then \( f^{-1}(\mu') \) is a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G \).

**Proof**: We recall that \( f^{-1}(\mu') \) as defined as \( (f^{-1}(\mu'))(x,q') = \mu'(f(x),q') \ \forall (x,q') \in G \times Q \) where \( \mu' \) is a \((\varepsilon, \varepsilon^\vee q)\) - fuzzy subgroup of \( G' \). Let \( x, y \in G \).
Then, \((x, q'), (y, q') \in f^{-1}(\mu') \forall t, s \in [0, 1]\) implies
\[(f^{-1}(\mu')(x, q')) \geq t, (f^{-1}(\mu')(y, q')) \geq s\]
\[\Rightarrow \mu'(f(x), q') \geq t, \mu'(f(y), q') \geq s\]
\[\Rightarrow (f(x), q') \in \mu', (f(y), q') \in \mu'\]
\[\Rightarrow \left(f(x)(f(y))^{-1}, q'\right)_{m(t,s)} \in \mu' \text{ or } \left(f(x)(f(y))^{-1}, q'\right)_{m(t,s)} q' \mu'\]

(Since \(\mu'\) of a \((\in \in \vee q)\) -Q-fuzzy of subgroup of G)
\[\Rightarrow \mu'(f(x)(f(y))^{-1}, q') \geq m(t, s) \text{ or } \mu'(f(x)(f(y))^{-1}, q') + m(t, s) > 1\]
\[\Rightarrow \mu'(f(x)f(y^{-1}), q') \geq m(t, s) \text{ or } \mu'(f(x)f(y^{-1}), q') + m(t, s) > 1\]
\[\Rightarrow \mu'(f(xy^{-1}, q') \geq m(t, s) \text{ or } \mu'(f(xy^{-1}, q') + m(t, s) > 1\]
\[\Rightarrow (f^{-1}(\mu'))(xy^{-1}, q') \geq m(t, s) \text{ or } (f^{-1}(\mu'))(xy^{-1}, q') + m(t, s) > 1\]
\[\Rightarrow (xy^{-1}, q')_{m(t,s)} \in f^{-1}(\mu') \text{ or } (xy^{-1}, q')_{m(t,s)} qf^{-1}(\mu')\]
\[\Rightarrow f^{-1}(\mu') \text{ is a } (\in \in \vee q) \text{ -Q-fuzzy of subgroup of G}\]

**Theorem 4.2**: Let \(f : G \times Q \rightarrow G' \times Q\) be an epimorphism, where G and G’ are two groups, and Q, a non-empty set. If \(f^{-1}(\mu')\) is a \((\in \in \vee q)\) -Q-fuzzy of subgroup of G where \(\mu'\) is a Q-fuzzy subgroup of G’, then \(\mu'\) is also a \((\in \in \vee q)\) -Q-fuzzy of subgroup of G’.

**Proof**: Let \(u, v \in G' \times t(u, q'), (v, q') \in \mu'\) where \(t, s \in [0.1]\). Now a \(f\) being onto \(\exists x, y \in G, s.t. f(x) = u, f(y) = v\). Since \(\mu'\) is a Q-fuzzy subset of G,
\[\mu'(u, q') \geq t, \mu'(v, q') \geq s\]
\[\Rightarrow (f^{-1}(\mu'))(x, q') \geq t, (f^{-1}(\mu'))(y, q') \geq s\]
\[\Rightarrow (x, q'), \in f^{-1}(\mu') \text{ of } (x, q'), \in f^{-1}(\mu')\]
\[\text{where } f^{-1}(\mu') \text{ is a } (\in \in \vee q) \text{ -Q-fuzzy of subgroup of G.}\]
\[\vdash (xy^{-1}, q')_{m(t,s)} \in f^{-1}(\mu') \text{ or } (xy^{-1}, q')_{m(t,s)} qf^{-1}(\mu')\]
\[\Rightarrow (f^{-1}(\mu'))(xy^{-1}, q') \geq m(t, s) \text{ or } (f^{-1}(\mu'))(xy^{-1}, q') + m(t, s) > 1\]
\[\Rightarrow \mu'(f(xy^{-1}), q') \geq m(t, s) \text{ or } \mu'(f(xy^{-1}), q') + m(t, s) > 1\]
\[\Rightarrow \mu'(f(x)f(y^{-1}), q') \geq m(t, s) \text{ or } \mu'(f(x)f(y^{-1}), q') + m(t, s) > 1\]
\[\Rightarrow \mu'(f(x)(f(y))^{-1}, q') \geq m(t, s) \text{ or } \mu'(f(x)(f(y))^{-1}, q') + m(t, s) > 1\]
\[\Rightarrow \mu'(uv^{-1}, q') \geq m(t, s) \text{ or } \mu'(uv^{-1}, q') + m(t, s) > 1\]
\[\Rightarrow \mu'(uv^{-1}, q')_{m(t,s)} \in \mu' \text{ or } \mu'(uv^{-1}, q')_{m(t,s)} q' \mu'\]
\[\Rightarrow \mu' \text{ is a } (\in \in \vee q) \text{ -Q-fuzzy of subgroup of G'.} \]
5.0 \((\in, \in \lor q)\) -Q-Fuzzy Subgroups and Normal Subgroups

**Definition 5.1**: A Q-fuzzy set \(\mu : G \times Q \rightarrow [0,1]\) where \(G\) is a group and \(Q\) a non-empty set is called \((\in, \in \lor q)\)-Q-fuzzy normal subgroup of \(G\) if \((x, q') \in \mu, (y, q') \in \mu \Rightarrow \left(xy^{-1}, q'\right)_{m(t,s)} \in \lor q \mu\) where \(m(t,s) = \min \{t, s\}\).

**Theorem 5.2**: The intersection of two \((\in, \in \lor q)\)-Q-fuzzy normal subgroups of \(G\) is a \((\in, \in \lor q)\)-Q-fuzzy normal subgroup of \(G\).

Proof: Similar to that of Theorem 3.2.

**Remark 5.3**: i) The result can be extended to a family of \((\in, \in \lor q)\)-Q-fuzzy normal subgroups of \(G\).

ii) However, the union of two \((\in, \in \lor q)\)-Q-fuzzy normal subgroups of \(G\) is not necessarily a \((\in, \in \lor q)\)-Q-fuzzy normal subgroup of \(G\).

**Theorem 5.4**: Any Q-fuzzy subset \(\mu\) in a group \(G\) is a Q-fuzzy normal subgroup of \(G\) if and only if \(\mu\) is a \((\in, \in)\)-Q-fuzzy normal subgroup of \(G\).

Proof: Let \(\mu\) be a \((\in, \in)\)-Q-fuzzy normal subgroup of \(G\). Let \(x, y \in G, q' \in Q, s.t. (x, q') \in \mu, (y, q') \in \mu\), where \(t, s \in [0,1]\). Then, \(\mu(x, q') \geq t\) and \(\mu(y, q') \geq s\).

Now, \(\mu(xy^{-1}, q') \geq \min \{\mu(x, q'), \mu(y, q')\} \geq \min \{t, s\} = m(t, s)\)

\(\Rightarrow (xy^{-1}, q')_{m(t,s)} \in \mu \Rightarrow \mu\) is a \((\in, \in)\)-Q-fuzzy normal subgroup of \(G\).

Conversely, let \(\mu\) be a \((\in, \in)\)-Q-fuzzy normal subgroup of \(G\). Let \(x, y \in G, q' \in Q, s.t. \mu(x, q') = t\) and \(\mu(y, q') = s\) where \(t, s \in [0,1]\).

Then \(\mu(x, q') \geq t\) and \(\mu(y, q') \geq s\).

\(\Rightarrow (x, q') \in \mu\) and \((x, q') \in \mu\). Since \(\mu\) is a \((\in, \in)\)-Q-fuzzy normal subgroup of \(G\), we have \((xy^{-1}, q')_{m(t,s)} \in \mu\)

\(\Rightarrow \mu(xy^{-1}, q') \geq \min \{\mu(x, q'), \mu(y, q')\}\)

\(\Rightarrow \mu\) is a Q-fuzzy normal subgroup of \(G\).

**Theorem 5.5**: If \(\mu\) is a \((q,q)\)-Q-fuzzy normal subgroup of \(G\) then \(\mu\) is a \((\in, \in)\)-Q-fuzzy normal subgroup of \(G\).

Proof: Let \(\mu\) be a \((q,q)\)-Q-fuzzy normal subgroup of \(G\).

Let \(x, y \in G\) and \(q \in Q, s.t. (x, q') \in \mu, (y, q') \in \mu\) where \(t, s \in [0,1]\). Then, \(\mu(x, q') \geq t\) and \(\mu(y, q') \geq s\).

\(\Rightarrow \mu(x, q') \delta > t, \mu(y, q') + \delta > s\) for any \(\delta > 0\)

\(\Rightarrow \mu(x, q') + 1 - t + \delta > 1, \mu(y, q') + 1 - s + \delta > 1\)

\(\Rightarrow (x, q')_{(1-t+\delta)} q \mu, \mu(y, q')_{(1-s+\delta)} q \mu\)

Since \(\mu\) is a \((q,q)\)-Q-fuzzy normal subgroup of \(G\), \((xy^{-1}, q')_{m(1-t+\delta,1-s+\delta)} q \mu\) where \(m(1-t+\delta,1-s+\delta) = \min \{1-t+\delta,1-s+\delta\}\)

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\( \mu(xyx^{-1},q') + m(1-t + \delta, 1-s + \delta) > 1 \)
\( \Rightarrow \mu(xyx^{-1},q') + 1 + \delta - M(t,s) > 1 \) where \( M(t,s) = \max \{t,s\} \)
\( \Rightarrow \mu(xyx^{-1},q') > M(t,s) - \delta \)
\( \Rightarrow \mu(xyx^{-1},q') \geq M(t,s) \) as \( \delta \) is arbitrary.
\( \Rightarrow \mu(xyx^{-1},q') \geq m(t,s) \) as \( M(t,s) \geq m(t,s) \)
\( \Rightarrow \mu(xyx^{-1},q')_{m(t,s)} \in \mu \)
\( \Rightarrow \mu \) is a \( (\varepsilon, \varepsilon) \) -\( Q \)-fuzzy normal subgroup of \( G \).

**Theorem 5.6**: A \( Q \)-fuzzy normal subgroup \( \mu \) of \( G \) is a \( (\varepsilon, \varepsilon \vee q) \) -\( Q \)-fuzzy normal subgroup of \( G \) if and only if
\( \mu(xyx^{-1},q') \geq \min \{\mu(x,q'),\mu(y,q'),0.5\} \) \( \forall x, y \in G \).

**Proof**: Let \( \mu \) be a \( (\varepsilon, \varepsilon \vee q) \) -\( Q \)-fuzzy normal (QFN) subgroup of \( G \).

**Case I**: Let \( \min \{\mu(x,q'),\mu(y,q')\} < 0.5 \) \( \forall x, y \in G \).
Then, \( \min \{\mu(x,q'),\mu(y,q'),0.5\} = \min \{\mu(x,q'),\mu(y,q')\} \)
If possible, let \( \mu(xyx^{-1},q') < \min \{\mu(x,q'),\mu(y,q')\} \).
Then, \( \exists \) a real number \( t \) such that,
\( \mu(xyx^{-1},q') < t < \min \{\mu(x,q'),\mu(y,q')\} \)
\( \Rightarrow \mu(x,q') > t, \mu(y,q') > t \)
\( \Rightarrow (x,q'), (y,q') \in \mu \)
Now \( \mu(xyx^{-1},q') < t \)
\( \Rightarrow (xyx^{-1},q') \in \mu \)
and \( \mu(xyx^{-1},q') + t < 2t < 2 \min \{\mu(x,q'),\mu(y,q')\} < 2 \times 0.5 = 1 \), a contradiction to the fact that \( \mu \) is a \( (\varepsilon, \varepsilon \vee q) \) -\( Q \)-fuzzy normal subgroup of \( G \). Hence, we must have,
\( \mu(xyx^{-1},q') \geq \min \{\mu(x,q'),\mu(y,q')\} = \min \{\mu(x,q'),\mu(y,q'),0.5\} \)

**Case II**: Let \( \min \{\mu(x,q'),\mu(y,q')\} \geq 0.5 \) \( \forall x, y \in G \).
Then, \( \min \{\mu(x,q'),\mu(y,q'),0.5\} = 0.5 \)
If possible, let \( \mu(xyx^{-1},q') < \min \{\mu(x,q'),\mu(y,q'),0.5\} = 0.5 \)
\( \Rightarrow \mu(x,q') \geq 0.5, \mu(y,q') \geq 0.5 \)
\( \Rightarrow (x,q'), (y,q') \in \mu \)
Now \( \mu(xyx^{-1},q') < 0.5 \)
\( \Rightarrow (xyx^{-1},q') \not\in \mu \)
and \( \mu(xyx^{-1},q') + 0.5 < 0.5 + 0.5 = 1 \), a contradiction to the fact that \( \mu \) is a \( (\varepsilon, \varepsilon \vee q) \) -\( Q \)-fuzzy normal subgroup of \( G \). Hence, we must have,
\( \mu(xyx^{-1},q') \geq 0.5 = \min \{\mu(x,q'),\mu(y,q'),0.5\} \)
Conversely, let $\mu(\text{xy}^{-1}, q') \geq \min \{\mu(x, q'), \mu(y, q'), 0.5\}$
Let $\mu(x, q)_t \in \mu(y, q)_s \in \mu$ where $t, s \in [0,1]$.
Then, $\mu(x, q)_t \geq \mu(y, q)_s \geq s$
$\Rightarrow \min \{\mu(x, q'), \mu(y, q')\} \geq m(t, s)$
But $\mu(\text{xy}^{-1}, q') \geq \min \{\mu(x, q'), \mu(y, q'), 0.5\} \geq m(t, s, 0.5)$
If $m(t, s, 0.5) \leq 0.5$ then $m(t, s, 0.5) = s \Rightarrow \mu(x, q)_t \geq \mu(y, q)_s \geq m(t, s)$
If $m(t, s, 0.5) > 0.5$ then $m(t, s, 0.5) = 0.5 s \Rightarrow \mu(x, q)_t \geq \mu(y, q)_s \geq 0.5 + m(t, s) > 1$
$\Rightarrow \mu(x, q)_t \geq \mu(y, q)_s \geq m(t, s)$
Thus, $(\text{xy}^{-1}, q')_{m(t, s)} \in q \mu \Rightarrow \mu$ is a $(\varepsilon, \varepsilon \lor \lor q')$ -Q-fuzzy normal (QFN) subgroup of $G$.

**Theorem 5.7**: of $Q$-fuzzy subgroup $\mu$ of $G$ is a $(\varepsilon, \varepsilon \lor \lor q')$ -Q-fuzzy normal subgroup of $G$ and
$\mu(\text{xy}^{-1}, q') < 0.5 \quad \forall x, y \in G$ then $\mu$ is also a $(\varepsilon, \varepsilon \lor \lor q')$ -Q-fuzzy normal subgroup of $G$.

**Proof**: Since $\mu$ as a $(\varepsilon, \varepsilon \lor \lor q')$ -Q-fuzzy normal subgroup of $G, \forall x, y \in G$
$\mu(\text{xy}^{-1}, q') \geq \min \{\mu(x, q'), \mu(y, q')\}$
$\Rightarrow 0.5 > \min \{\mu(x, q'), \mu(y, q')\}$
$\Rightarrow \mu(x, q') < 0.5$ and $\mu(y, q') < 0.5$
Let $(x, q)_t \in \mu(y, q)_s \in \mu$ where $t, s \in [0,1]$.
Then $\mu(x, q)_t \geq \mu(y, q)_s \geq s$
$\Rightarrow t < 0.5, s < 0.5$
$\Rightarrow m(t, s) < 0.5$
Since $\mu$ is a $(\varepsilon, \varepsilon \lor \lor q')$ -Q-fuzzy normal subgroup of $G, \forall x, y \in G$
$(x, q')_t \in \mu(y, q)_s \in \mu \Rightarrow \mu(\text{xy}^{-1}, q') \geq m(t, s)$ or $\mu(\text{xy}^{-1}, q') + m(t, s) > 1$
Since, $m(t, s) < 0.5$ we must have $\mu(\text{xy}^{-1}, q') \geq m(t, s)$ so that $\mu$ is a $(\varepsilon, \varepsilon \lor \lor q')$ -Q-fuzzy normal subgroup of $G$.

**Theorem 5.8**: Let $\mu$ be $(\varepsilon, \varepsilon \lor \lor q')$-fuzzynormal subgroup of $G$ and $g \in G$.Then, $g \mu g^{-1}$ is also a $(\varepsilon, \varepsilon \lor \lor q')$-fuzzy normal subgroup of $G$.

**Proof**: Let $(x, q)_t \in g \mu g^{-1}(y, q)_s \in g \mu g^{-1}$ where $t, s \in [0,1]$. Then,
$(g \mu g^{-1})_t \geq t, (g \mu g^{-1})_s \geq s \Rightarrow \mu(g^{-1} x, g^{-1} y)_t \geq t, \mu(g^{-1} y, g^{-1} x)_s \geq s \Rightarrow (g^{-1} x, g^{-1} y)_t \in \mu(g^{-1} y, g^{-1} x)_s \in \mu$
Since, $\mu$ is a $(\varepsilon, \varepsilon \lor \lor q')$ -Q-fuzzy normal subgroup of $G$. 

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\( \mu((g^{-1}xg)(g^{-1}yg)(g^{-1}xg)^{-1},q') \geq m(t,s) \lor \mu((g^{-1}xg)(g^{-1}yg)(g^{-1}xg)^{-1},q') + m(t,s) > 1 \ldots (B) \)

Now,
\[
\begin{align*}
\mu((g^{-1}xg)(g^{-1}yg)(g^{-1}xg)^{-1},q') &= \mu((g^{-1}xg)((g^{-1}yg)((xg)^{-1}g),q') = \mu((g^{-1}xg)(g^{-1}yg)(g^{-1}x^{-1}g),q') \\
= \mu((g^{-1}xg)(g^{-1}yg)(g^{-1}x^{-1}g),q') &= \mu((g^{-1}(xy)x^{-1})g,q') = (g_{\mu_{\cdot}})(xyx^{-1},q')
\end{align*}
\]

Therefore from (B),
\[
(g_{\mu_{\cdot}})(xyx^{-1},q') \geq m(t,s) \lor (g_{\mu_{\cdot}})(xyx^{-1},q') + m(t,s) > 1
\]

Therefore, \(g_{\mu_{\cdot}}\) is also a \((\in,\in \lor q)\) -Q-fuzzy normal subgroup of G.

V. Homomorphism of \((\in,\in \lor q)\) -Q-fuzzy normal subgroup

**Theorem 6.1:** \(f: G \times Q \rightarrow G' \times Q\) be a homomorphism, where \(G,G'\) are two groups and \(Q\) a non empty set. If \(G'\) is a \((\in,\in \lor q)\) -Q-fuzzy normal sub group of \(G'\) then \(f^{-1}(\mu')\) is also a \((\in,\in \lor q)\) -Q-fuzzy normal sub group of G.

**Proof:** Let us recall that \(f^{-1}(\mu')\) is defined as
\[
(f^{-1}(\mu'))((x,q')) = \mu'(f(x),q') \quad \forall (x,q') \in G \times Q\text{ where } \mu'\text{ is a } (\in,\in \lor q) \text{-Q-fuzzy normal subgroup of } G'.
\]

Let \(x,y \in G\) and \(q' \in Q\). Then, \((x,q'),(y,q') \in f^{-1}(\mu')\quad \forall t,s \in [0,1]\]

implies \((f^{-1}(\mu'))((x,q')) \geq t,(f^{-1}(\mu'))((y,q')) \geq s\]

\[\Rightarrow \mu'(f(x),q') \geq t,\mu'(f(y),q') \geq s\]

\[\Rightarrow (f(x),q'), \mu'(f(y),q') \in \mu'\text{ where } \mu'\text{ is a } (\in,\in \lor q) \text{-Q-fuzzy normal subgroup of } G'.\]

So, \((f(x)f(y)f(x^{-1}),q')_{m(t,s)} \in \mu'\text{ or } (f(x)f(y)f(x^{-1}),q')_{m(t,s)} q\mu'\]

\[\Rightarrow \mu'(f(x)f(y)f(x^{-1}),q') \geq m(t,s)\text{ or } \mu'(f(x)f(y)f(x^{-1}),q') + m(t,s) > 1\]

\[\Rightarrow \mu'(f(x)f(y)f(x^{-1}),q') \geq m(t,s)\text{ or } \mu'(f(x)f(y)f(x^{-1}),q') + m(t,s) > 1\]

\[\Rightarrow \mu'(f(xy)f(x^{-1}),q') \geq m(t,s)\text{ or } \mu'(f(xy)f(x^{-1}),q') + m(t,s) > 1\]

\[\Rightarrow \mu'(f(xy)f(x^{-1}),q') \geq m(t,s)\text{ or } \mu'(f(xy)f(x^{-1}),q') + m(t,s) > 1\]

\[\Rightarrow \mu'(f(xy),q') \geq m(t,s)\text{ or } \mu'(f(xy),q') + m(t,s) > 1\]

\[\Rightarrow \mu'(f(xy)f(x^{-1}),q') \geq m(t,s)\text{ or } \mu'(f(xy)f(x^{-1}),q') + m(t,s) > 1\]

\[\Rightarrow (f^{-1}(\mu'))((xyx^{-1},q') \geq m(t,s)\text{ or } \mu'(f^{-1}(\mu'))((xyx^{-1},q') + m(t,s) > 1\]

\[\Rightarrow (xyx^{-1},q')_{m(t,s)} \in f^{-1}(\mu') \text{ or } (xyx^{-1},q')_{m(t,s)} qf^{-1}(\mu')\]

\[\Rightarrow (xyx^{-1},q')_{m(t,s)} \in qf^{-1}(\mu') \Rightarrow f^{-1}(\mu') \text{ is } (\in,\in \lor q) \text{-Q-fuzzy normal subgroup of } G'.\]

**Theorem 6.2:** \(f: G \times Q \rightarrow G' \times Q\) be a homomorphism, where \(G\) and \(G'\) are two groups and \(Q\) a non empty set. If \(f^{-1}(\mu')\) is a \((\in,\in \lor q)\) -Q-fuzzy normal sub group of \(G'\) then \(\mu'\) is also a \((\in,\in \lor q)\) -Q-fuzzy normal sub group of \(G'\).

**Proof:** Let \(u,v \in G'\) s.t. \((u,q'),(u,q') \in \mu'\text{ where } t,s \in [0,1].\)
Now $f$ being onto, $\exists u, v \in G'$ s.t. $f(x) = u$ and $f(y) = v$. Since $\mu'$ is a $(\varepsilon, \mu \vee \eta)$-fuzzy normal subgroup of $G'$, $\mu'(u, q') \geq t, \mu'(v, q') \geq s$

$\Rightarrow \mu'(f(x), q') \geq t, \mu'(f(y), q') \geq s$

$\Rightarrow (f^{-1}(\mu'))(x, q') \geq t, (f^{-1}(\mu'))(y, q') \geq s$

$\Rightarrow (x, q') \in f^{-1}(\mu'), (y, q') \in f^{-1}(\mu')$ where $f^{-1}(\mu')$ is a $(\varepsilon, \mu \vee \eta)$-fuzzy normal subgroup of $G$.

So, $(xyx^{-1}, q') \in f^{-1}(\mu')$ or $(xyx^{-1}, q')_{m(t, s)} \in f^{-1}(\mu')$

$\Rightarrow (f^{-1}(\mu'))(xyx^{-1}, q') \geq m(t, s)$ or $\mu'(f^{-1}(\mu'))(xyx^{-1}, q') + m(t, s) > 1$

$\Rightarrow \mu'(f(xy), q') \geq m(t, s) \quad \text{or} \quad \mu'(f(xy), q') + m(t, s) > 1$

$\Rightarrow \mu'(f(x) f(y), q') \geq m(t, s) \quad \text{or} \quad \mu'(f(x) f(y), q') + m(t, s) > 1$

$\Rightarrow \mu'(f(x) f(y), x q^{-1}, q') \geq m(t, s) \quad \text{or} \quad \mu'(f(x) f(y), x q^{-1}, q') + m(t, s) > 1$

$\Rightarrow \mu'(uvu^{-1}, q') \geq m(t, s) \quad \text{or} \quad \mu'(uvu^{-1}, q') + m(t, s) > 1$

$\Rightarrow (uvu^{-1}, q') \in \varepsilon \quad \Rightarrow \mu'$ is a $(\varepsilon, \mu \vee \eta)$-fuzzy normal subgroup of $G'$.

References


