On ray properties of Hurwitz polynomials

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Abstract: In this paper, we investigate some geometric properties of the Hurwitz set which corresponds to the set of stable monic polynomials in a parameter space. We firstly consider the segment stability. After we study properties of rays in the Hurwitz sets, which corresponds with inclusion or non-inclusion of certain rays in the Hurwitz sets.

Keywords: Hurwitz polynomials, monic polynomials, ray properties, segment stability

I. Introduction

The celebrated theorem Kharitonov [1] on the stability of prisms of polynomials gave an impetus to the research in this old and ever-important field and in the last decades many new results concerning stability of diamonds, edges, segments, polygons, polytopes etc. have been obtained (see [2-15]). A remarkable new approach has been towards understanding the geometry (and topology) of (all or part of) stable polynomials.

First of all, we identify a non-monic polynomial \( p(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \) with the point (or vector) \( (a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1} \). A stable (or Hurwitz) polynomial is a polynomial with roots lying in the open left half of the complex plane. (A necessary but not sufficient condition for stability is that all of \( a_0, a_1, \ldots, a_n \) have the same sign. There are well-known necessary and sufficient conditions for stability such as the Routh-Hurwitz and Hermite-Bieler criteria and the separation property [16-17])

We will denote the set of such vectors by \( \mathcal{H}^+ \subseteq \mathbb{R}^{n+1} \) and the subset of \( \mathcal{H}^+ \) with positive leading coefficients \( a_0 > 0 \) with \( \mathcal{H}_0^+ \). The important special case of monic polynomials \( a_0 = 1 \), which for the consideration of stability are equivalent to the general case, are thus identified with vectors of the form \( (1, a_1, \ldots, a_n) \). On the other hand, they are often identified with the vector \( (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) and this causes a minor nuisance of notation. To prevent ambiguity, we will denote the set of stable monic polynomials by \( \mathcal{H}_0^+ \) if they are taken as elements \( (1, a_1, \ldots, a_n) \in \mathbb{R}^{n+1} \), and by \( \mathcal{H}_0^+ \) if they are taken as elements \( (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \). Unless explicitly stated otherwise, we will represent the \( n \)th order monic polynomials

\[
p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n
\]

with \( (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \).

Thus, the open sets \( \mathcal{H}_0^+ \subseteq \mathbb{R}^{n+1} \) and \( \mathcal{H}_0^+ \subseteq \mathbb{R}^n \) are defined as follows:

- \( (a_0, a_1, \ldots, a_n) \in \mathcal{H}_0^+ \iff a_0 > 0 \) and the polynomial \( p(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \) is stable.
- \( (a_1, a_2, \ldots, a_n) \in \mathcal{H}_0^+ \iff \) the polynomial \( p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \) is stable.

It is obvious that for \( k > 0 \) and \( p = (a_1, a_2, \ldots, a_n) \in \mathcal{H}_0^+ \)

\[
kp \in \mathcal{H}_0^+ \iff \text{the polynomial } p_k(s) = s^n + ka_1 s^{n-1} + \ldots + ka_n s^{n-k+1} \text{ is stable.
}

The first geometric property of interest is the convexity and it is well-known that \( \mathcal{H}_0^+ \) (and thus \( \mathcal{H}_0^+ \)) is non-convex. The next question of interest is the following: Given two elements from \( \mathcal{H}_0^+ \) (or \( \mathcal{H}_0^+ \)), under which conditions it can be stated that the segment in \( \mathbb{R}^{n+1} \) (or \( \mathbb{R}^n \)) with these end points belong to \( \mathcal{H}_0^+ \) (or \( \mathcal{H}_0^+ \))? Several authors gave results and discussions in this direction (see [4,6]), but the most important result is due to Rantzer [3] and implies the others. In Section 2, we give a simple new case (Remark 1) and some important consequence (Corollary 1 and Corollary 2) not obtainable by Rantzer’s theorem.

Section 3 contains the main results where we investigate some other geometric properties of rays, but before stating them we want to introduce some additional terminology. Given a vector \( p \in \mathbb{R}^n \) (which corresponds to a monic polynomial of degree \( n \)), we call the set \( \{kp : k > 0\} \subseteq \mathbb{R}^n \) the radial ray through \( p \). Likewise, we will call the set \( \{kp : k \geq 1\} \subseteq \mathbb{R}^n \) the radial ray starting at \( p \) and the set \( \{kp : 0 < k < 1\} \subseteq \mathbb{R}^n \) the radial ray till \( p \). Now we state the properties proven in Section 3. Given any vector \( p \in \mathcal{H}_0^+ \) \( (n \geq 3) \), there exists \( k_0 \in (0,1) \) such that the part \( \{kp : 0 < k \leq k_0\} \) of the radial ray till \( p \) lies outside \( \mathcal{H}_0^+ \) and the part \( \{kp : k_0 < k \leq 1\} \) lies inside \( \mathcal{H}_0^+ \) (Theorem 1).

On the other hand, for every \( n \geq 2 \) there is a vector \( p \in \mathcal{H}_0^+ \) (actually infinitely many) such that the radial ray starting at \( p \) lies completely inside \( \mathcal{H}_0^+ \) (Theorem 2). For \( n = 2,3 \) and 4 all radial rays starting at any \( p \in \mathcal{H}_0^+ \) lie completely in \( \mathcal{H}_0^+ \).
II. Segment-Stability And Properties Concerning Rays

The following result comes from [6]: Given two stable polynomials \( p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \) and \( q(s) = b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n \), then the segment \([p, q]\) is stable if \( a_i = b_i \) either for even entries or odd entries (consult also [8,9,13]).

**Proposition 1** Let \( p(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \) and \( q(s) = b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n \) be stable polynomials. If even (or odd) part of \( q(s) \) is a positive scalar multiple of the even (or odd) part of \( p(s) \) then the segment \([p, q]\) of their convex combinations is also stable.

It is enough to see this for the case of even parts, the case of odd parts being similar. One can re-arrange \( p(s) \) and \( q(s) \) as \( p(s) = h(s^2) + sg_1(s^2), q(s) = kh(s^2) + sg_2(s^2) \) where \( k > 0 \) is a fixed scalar. Denote \( q_1(s) = q(s) \), then the convex combination of \( p(s) \) and \( q(s) \) is stable by [6]. Hence for every \( \lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1 + \lambda_2 > 0 \) the polynomial \( \lambda_1 p(s) + \lambda_2 q(s) \) is stable, since

\[
\lambda_1 p(s) + \lambda_2 q(s) = (\lambda_1 + \lambda_2) \frac{\lambda_1}{\lambda_1 + \lambda_2} p(s) + \frac{\lambda_2}{\lambda_1 + \lambda_2} q(s)
\]

Therefore, assigning \( \lambda_1 = (1 - \lambda) \) and \( \lambda_2 = k\lambda \) the polynomial \( \lambda_1 p(s) + \lambda_2 q_1(s) = (1 - \lambda)p(s) + \lambda q(s) \) is stable for all \( \lambda \in [0,1] \).

**Corollary 1** Let \( p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \) and \( q(s) = s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n \) be stable polynomials. Identify \( p(s) \) with \( (a_1, a_2, \ldots, a_n) \) and \( q(s) \) with \( (b_1, b_2, \ldots, b_n) \) and assume that \( b_1, b_2, \ldots, b_n \) is \( k \) times \( a_1, a_2, \ldots, a_n \) for a positive scalar \( k \). Then the segment \([p, q]\) in \( \mathbb{R}^n \) is stable. In other words, segments on radial rays with stable end points are stable.

**Proof.** Either the even or odd parts of \( p \) and \( q \) are proportional according to \( n \) being odd or even. The result follows from Proposition 1. \( \square \)

**Corollary 2** If the radial ray emanating from the origin enters the \( \mathbb{H}^n \) and then leaves it, it cannot re-enter it. In other words, for \( p \in \mathbb{H}^n \) if \( k_0 \in \mathbb{H}^n \) for \( k_0 < 1 \) then \( kp \in \mathbb{H}^n \) for any \( k < k_0 \) and similarly if \( k_1 p \in \mathbb{H}^n \) for \( k_1 > 1 \), then \( kp \notin \mathbb{H}^n \) for any \( k > k_1 \).

We now prove the theorems stated in the introduction.

**Theorem 1** For any vector \( p \in \mathbb{H}^n \): (\( n \geq 3 \), there exists \( k_0 \in (0,1) \) such that

- \( kp \notin \mathbb{H}^n \) for all \( k \) with \( 0 < k \leq k_0 \)
- \( kp \in \mathbb{H}^n \) for all \( k \) with \( k_0 < k < 1 \)

**Proof.** By the separation property of stable polynomials, a necessary and sufficient condition for \( p(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \) to be stable is that the curve \( p(j\omega), \) where \( 0 \leq \omega < \infty \), cuts the real and imaginary axes alternatively \( n \) times precisely.

If \( n = 4m \) then for

\[
k_1 = -\frac{\omega^n}{a_n - a_{n-2} \omega^2 + \cdots - a_2 \omega^{n-2}}
\]

we have \( 0 < k_1 < 1 \) and \( p_{k_1}(j\omega) = 0 \), where \( p_{k_1}(s) = s^n + ka_1 s^{n-1} + ka_2 s^{n-2} + \cdots + ka_n \) and \( \omega \) corresponds with the point of intersection with the real axis. If \( n = 4m + 1 \) then for

\[
k_2 = -\frac{\omega^n}{a_{n-1} \omega + a_{n-3} \omega^3 + \cdots - a_2 \omega^{n-2}}
\]

we have \( 0 < k_2 < 1 \) and \( p_{k_2}(j\omega) = 0 \), where \( \omega \) corresponds with the point of intersection with the imaginary axis. Similar procedure can be applied to the cases \( n = 4m + 2 \) and \( n = 4m + 3 \). Thus for any \( n \geq 3 \) and any \( p \in \mathbb{H}^n \) there exists \( k \in (0,1) \) such that \( k(a_1, a_2, \ldots, a_n) \notin \mathbb{H}^n \). From Corollary 2 the desired result follows. \( \square \)

Theorem 1 shows that if we move radially towards the origin starting from an arbitrary polynomial \( p \in \mathbb{H}^n \), then we certainly leave \( \mathbb{H}^n \).

The following properties are about what can happen when we move in reverse direction.

**Theorem 2** For \( n \geq 2 \) there exists infinitely many \( p \in \mathbb{H}^n \) such that \( kp \in \mathbb{H}^n \) for all \( k \geq 1 \). To prove this theorem we first prove the following proposition.

**Proposition 2** Let \( q(s) = a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n \) be a stable polynomial. Then there exists \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \) the polynomial \( p_\varepsilon(s) = \varepsilon s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \) is stable.

**Proof.** Let \( n \) be an even number. Then we can write \( q(s) = q_1(s^2) + q_2(s^2) \), where \( q_1(u) \) and \( q_2(u) \) are polynomials of order \( m = \frac{n}{2} \). Let \( u_1, u_2, \ldots, u_m \) and \( v_1, v_2, \ldots, v_m \) denote the roots of \( q_1(u) \) and \( q_2(u) \) respectively. Then by the Hermite-Biehler theorem.

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The polynomial $p_k(s)$ can be written as $p_k(s) = \left[s^n + a_1s^{n-1} + \ldots + a_n\right]$ where $n \geq 5$ and $a_n$ is a stable polynomial. From Proposition 2 it follows that for all $k \geq 1$ the polynomial

$$p_k(s) = \frac{1}{t} s^n - a_1 s^{n-1} + \ldots + a_n$$

is stable. If we choose $p = (t_0 a_1, t_0 a_2, \ldots, t_0 a_n)$, then $p \in \mathcal{H}_1^n$ and for all $k \geq 1$ we have $kp \in \mathcal{H}_1^n$. □

**Proposition 3** For $n = 2, 3$ and 4 the property stated in Theorem 2 is true for all $p \in \mathcal{H}_1^n$.

The proof is omitted.

**Remark 1** It might seem that the Proposition 2 could plausibly be expected to be "naturally" true but the situation is more intricate than it seems, because there comes a surprise when we add two small terms: Let $s^n + 2s^{n-1} + \ldots$ be stable polynomial, then for no $\varepsilon > 0$ the polynomial $\varepsilon s^{n+2} + \varepsilon s^{n+1} + s^n + 2s^{n-1} + \ldots$ is stable.

**Theorem 3** Let $n \geq 5$. Then for all $k > 0$, $k \neq 1$, there exists $p = (a_1, a_2, \ldots, a_n) \in \mathcal{H}_1^n$ such that $kp = (ka_1, ka_2, \ldots, ka_n) \in \mathcal{H}_1^n$. That is to say the polynomial $p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$ is stable but $p_k(s) = s^n + ka_1 s^{n-1} + \ldots + ka_{n-1} s + ka_n$ is not stable.

**Proof.** The proof is based on the Hermite-Biehler theorem. Suppose that $n$ is an odd integer and $m = \frac{n-1}{2}$.

Choose arbitrary numbers $v_1, v_2, \ldots, v_m$ satisfying $v_1 < v_2 < \ldots < v_m < 0$ and define the polynomial $g(u) = (u - v_1)(u - v_2) \cdots (u - v_m) = u^m - b_m u^{m-1} - \ldots - b_1 u + b_0$. Let $k > 0, k \neq 1$ is given. Consider the polynomials $g_k(u) = u^m + kb_m u^{m-1} + \ldots + kb_1 u + kb_0$. Firstly suppose that the roots of $g_k(u)$ satisfies the condition $v_1 < v_2 < \ldots < v_m < 0$. It is not difficult to see that $g(u)$ and $g_k(u)$ have no common root. Then we can find $u_1, u_2, \ldots, u_m$ satisfying $v_1 < u_1 < v_2 < u_2 < \ldots < v_m < u_m < 0$ and not satisfying at least one of the following inequities $v_1 < u_1 < v_2 < u_2 < \ldots < v_m < u_m < 0$ (here we use $m \geq 2$). The Hermite-Biehler theorem ensures that $p(s) = h(s^2) + sg(s^2)$ is stable, where $h(u) = (u - u_1)(u - u_2) \cdots (u - u_m)$. If we write down $p(s)$ as $p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n$ then $p_k(s) = s^n + ka_1 s^{n-1} + \ldots + ka_{n-1} s + ka_n = kh(s^2) + sg_k(s^2)$ and the Hermite-Biehler theorem also guarantees the unstability of $p_k(s)$.

If the roots of $g_k(u)$ does not satisfy $v_1 < v_2 < \ldots < v_m < 0$ then $p_k(s)$ is also unstable. By a similar scheme one may prove the theorem for even $n$. □

**Remark 2** As it is seen from the proof of Theorem 3, the point $p$ depends on $v_1, v_2, \ldots, v_m$. By changing these numbers we can obtain infinitely many $p$ satisfying Theorem 3.

**Corollary 3** There exists a point $p \in \mathcal{H}_1^n, (n \geq 5)$ with the following property: There exists a number $k_0 > 1$ such that

- $kp \in \mathcal{H}_1^n$ for all $1 \leq k < k_0$,
- $kp \in \mathcal{H}_1^n$ for all $k \geq k_0$.

**Proof.** Choose $k = 2$. Then by Theorem 3 there exists $p \in \mathcal{H}_1^n$ such that $2p \not\in \mathcal{H}_1^n$. Then the claim follows from Corollary 2. □

**Remark 3** There exists a radial ray in the positive quadrant of $\mathbb{R}^n$ which lies completely outside $\mathcal{H}_1^n (n \geq 4)$. The polynomial $p_k(s) = s^n + ks^{n-1} + ks^{n-2} + \ldots + ks + k$ is unstable for all $k > 0$. But for $n = 3$ there is no such ray.

**III. Conclusion**

In this paper it is established that in a parameter space of polynomials segments on radial rays with stable end points are stable. We show that there is a stable vector such that the radial ray starting at this point lies completely inside the stability region. We also show that for any positive scalar differing one, there exists a stable vector such that the multiplication of this vector by this scalar is not stable.

**References**


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