On Independent Equitable Cototal Dominating set of graph

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Abstract: A subset $D$ of $V(G)$ is an independent set if no two vertices in $D$ are adjacent. A dominating set $D$ which is also an independent dominating set. An independent dominating set $D$ of vertex set $V(G)$ is called independent equitable cototal dominating set, if it satisfies the following condition:

i) For every vertex $u \in D$ there exist a vertex $v \in V - D$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$.

ii) $<V - D>$ contains no isolated vertex.

The minimum cardinality of independent equitable cototal dominating set is called independent equitable cototal domination number of a graph and it is denoted by $\gamma_{ie}^{\#}(G)$. In this paper, we initiate the study of new degree equitable domination parameter.

Keywords: Domination number, Equitable domination number, Cototal domination number, independent equitable cototal domination number.

I. Introduction

All graphs considered here are simple, finite, connected and nontrivial. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the vertex set and $E(G)$ be the edge set of $G$. A subset $D \subseteq V$ is said to be a dominating set of $G$ if every vertex $v \in V - D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a minimal dominating set is called the domination number of $G$ [2]. A subset $D$ of $V(G)$ is an independent set if no two vertices in $D$ are adjacent. A dominating set $D$ which is also an independent dominating set. The independent domination number $i(G)$ is the minimum cardinality of an independent domination set [2,3]. A subset $D$ of $V$ is called an equitable dominating set if for every $v \in V - D$ there exist a vertex $u \in D$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$, where $\deg(u)$ and $\deg(v)$ denotes the degree of a vertex $u$ and $v$ respectively. The minimum cardinality of such a dominating set is denoted by $\gamma^{\#}$ and is called the equitable domination number [7].

A dominating set $D$ is said to be a cototal dominating set if the induced subgraph $<V - D>$ has no isolated vertex. The cototal domination number $\gamma_{ct}(G)$ of $G$ is the minimum cardinality of a cototal dominating set of $G$ [6].

Analogously, we introduce new concept on independent equitable cototal dominating set as follows.

Definition 1.

An independent dominating set $D$ of vertex set $V(G)$ is called independent equitable cototal dominating set, if it satisfies the following condition:

1. For every vertex $u \in D$ there exist a vertex $v \in V - D$ such that $uv \in E(G)$ and $|\deg(u) - \deg(v)| \leq 1$.

2. $<V - D>$ contains no isolated vertex.

The minimum cardinality of independent equitable cototal dominating set is called independent equitable cototal domination number of a graph and it is denoted by $\gamma_{ie}^{\#}(G)$. 

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Example:

\[ G: \]

The dominating set \( D = \{ v_2 \} \) which is also an independent dominating set.

Independent equitable cototal dominating set \( D = \{ v_3 \} \).

Hence \( \gamma_{ic}^e(G) = |D| = 1 \).

Remark: Let \( G \) be any graph with independent dominating set \( D \) for some \( u, v, w \in V(G) \) and \( u, w \in D \).

If \( N(v) \cap N(v) = \{ u, w \} \) then \( G \) does not contain independent equitable cototal dominating set.

For example:

\[ G = C_4 \]

In this graph independent equitable cototal dominating set does not exist.

In general,

i) For \( P_n, n \neq 4 \) independent equitable cototal dominating set does not exist.

ii) For \( C_n, n \neq 3 \) independent equitable cototal dominating set does not exist.

iii) \( G = H \cup K_2 \) where \( H \) is any connected graph with \( \delta(G) \geq 3 \) we cannot define independent equitable cototal dominating set.

Example:

\[ G = K_4 \cup K_1 \]

Firstly, we obtain the independent equitable cototal domination number \( \gamma_{ic}^e(G) \) of some standard class of graphs. Which are listed in the following proposition.
Proposition 1:

i) For any complete graph $G = K_n$, $n \geq 3$, $\gamma_{ic}^e(K_n) = 1$

ii) For any complete bipartite graph $G = K_{m,n}$

\[
\gamma_{ic}^e(K_{m,n}) = \begin{cases} 
2 & \text{if } |m - n| \leq 1 \\
\text{does not exist} & \text{otherwise}
\end{cases}
\]

iii) For any complete bipartite graph $G = W_n$

\[
\gamma_{ic}^e(W_n) = \begin{cases} 
1 & \text{if } n = 4 \\
\text{does not exist} & \text{otherwise}
\end{cases}
\]

Proof:

i) Let $G$ be a complete graph of order at least 4. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of $K_n$. Let $D = \{v\}$ be independent cototal dominating set of $G$. Since $K_n$ is a $(n - 1)$-regular graph. Therefore for every vertex $u \in V - D$, $|\deg(u) - \deg(v)| = 0$. Hence $D$ acts as an independent equitable cototal dominating set.

Therefore $\gamma_{ic}^e(G) = 1$.

ii) Let $G = K_{m,n}$ be a complete bipartite graph with partite sets of cardinality $m$ & $n$ respectively. We consider the following cases.

Case i) If $|m - n| \leq 1$

Let $V'(G) = m$ and $V''(G) = n$. $V'(G) \cup V''(G) = m + n$. By definition of complete bipartite graph no two vertices of the same partite sets are adjacent. Since $|m - n| \leq 1$, therefore one vertex from each partite set is sufficient to dominate vertex set of $G$. Therefore any independent cototal dominating set acts as an independent equitable cototal dominating set of $G$. Hence $\gamma_{ic}^e(G) = 2$.

Case ii) if $|m - n| \leq 2$, then for every vertex $u \in D$ there exist a vertex $v \in V - D$ such that $|\deg(u) - \deg(v)| \geq 2$. Therefore the independent equitable cototal dominating set does not exist.

iii) Let $G$ be wheel graph $W_n$. By definition of wheel graph $W_n = C_{n-1} + K_1$. We consider the following cases.

Case i) For $n = 4$, $W_n$ is isomorphic to $K_4$. Therefore by (i) $\gamma_{ic}^e(W_n) = 1$.

Case ii) For $n \geq 5$, we can observe that $|\deg(u) - \deg(v)| \geq 2$ where $u$ is the cototal vertex of $W_n$. Further $\deg(u) = n - 1$. Hence $G$ does not contain independent equitable cototal dominating set.

II. Bounds For Independent Equitable Cototal Dominating Set

Theorem 1: For any graph $G$ without isolated vertices, $1 \leq \gamma_{ic}^e(G) \leq \frac{n}{2}$ equality of lower bound holds if and only if $\Delta(G) = n - 1$ and $\delta(G) \geq n - 2$. Further equality of upper bound holds if $G = P_4$.

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Proof: Let $G$ be any graph without isolated vertices, then by Proposition 1, it is easy to see that $\gamma^{e}_{ic}(G) \geq 1$.

For equality, suppose $\Delta(G) = n - 1$ and $\delta(G) \geq n - 2$ then $G$ contains a vertex $u$ which as degree $n - 1$ and a vertex of minimum degree $v$ which as degree at least $n - 2$.

Clearly, $\{u\} = D$ is an independent equitable cototal dominating set.

Such that $|\deg(u) - \deg(v)| \leq 1$

Conversely, Suppose $\gamma^{e}_{ic}(G) = 1$ and $\Delta(G) = n - 1$ and $\delta(G) \leq n - 3$. Then for every vertex $u \in D$ there is no vertex $v \in V - D$ such that $|\deg(u) - \deg(v)| \leq 1$. This is a contradiction. Therefore $\delta(G) \geq n - 2$.

Now, the upper bound follows from the fact that, for any graph $G$ contains at most $\frac{n}{2}$ independent vertices.

Hence $\gamma^{e}_{ic}(G) \leq \frac{n}{2}$

Equality case is easy to follow.

Theorem 2: For any graph $G$ without isolated vertices, $\gamma^{e}_{ic}(G) \leq \beta(G)$ equality holds if $G = K_n, n \geq 3$ where $\beta(G)$ is the vertex independent number.

Proof: Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of a graph $G$. Let $S$ be a collection of all independent vertices of $G$. Such that $|S| = \beta(G)$. If for every vertex $v \in S$ there exist $u \in V - D$ such that $|\deg(u) - \deg(v)| \leq 1$ and $<V - D>$ contains no isolated vertices, then $S$ act as a minimal independent equitable cototal dominating set of $G$.

Hence, $\gamma^{e}_{ic}(G) \leq |S| \leq \beta(G)$

$\gamma^{e}_{ic}(G) \leq \beta(G)$.

Theorem 3: For any graph $G$ without isolated vertices, $\gamma^{e}_{ic}(G) \leq n - \alpha(G)$, where $\alpha(G)$ is the vertex covering number of $G$.

Proof: Let $G$ be a graph without isolated vertices. We know from famous Gallia’s theorem $\alpha(G) + \beta(G) = n$.

Hence by theorem (2) and using this result we get the required inequality.

Theorem 4: For any graph $G$ without isolated vertices, $\gamma^{e}_{ic}(G) + \alpha_0(G) \leq n$ equality holds if $G = K_n, n \geq 4$

Proof: Follows from theorem (2) and theorem (3).

For equality case, if $G = K_n, n \geq 4$ then by Proposition 1, $\gamma^{e}_{ic}(K_n) = 1$ and from the fact that $\alpha(K_n) = n - 1$, combining these two results, we get the required results.

Theorem 5: For any r-regular graph $G$, $\gamma^{e}_{ic}(G) = \gamma_{ic}(G)$.

Proof: Suppose $G$ is the regular graph. Then every vertex as the same degree $r$. Let $D$ be a minimum independent cototal dominating set of $G$, then $|D| = \gamma_{ic}(G)$. Let $u \in V - D$, then as $D$ is an independent cototal dominating set there exist a vertex $v \in D$ and $uv \in E(G)$. Also $\deg(u) = \deg(v) = r$.

Therefore $|\deg(u) - \deg(v)| = 0 < 1$. Hence $D$ is degree equitable independent cototal dominating set of $G$. So that $\gamma^{e}(G) \leq |D| \leq \gamma_{ic}(G)$. But $\gamma_{ic}(G) \leq \gamma^{e}_{ic}(G)$.

Hence $\gamma^{e}_{ic}(G) = \gamma_{ic}(G)$.

Theorem 6: For any graph without isolated vertices $\gamma^{e}_{ic}(G) \leq n - \Delta(G)$, where $\Delta(G)$is the maximum degree of G, equality holds if $G = K_n, n \geq 4$.

Proof: Let $G$ be a graph containing no isolated vertices. Let $v \in V(G)$ be a vertex of maximum degree that is $\deg(v) = \Delta(G)$. Since every vertex dominates at most the vertices in its neighborhood, that is $v \in D$ dominates $\Delta(G)$ if vertices. Further if $G$ contains vertex $u$ of minimum degree $\delta$. Such that
\[ \delta(G) \geq n - \Delta - 1 \text{ then every vertex in } V - D \text{ will be degree equitable to some vertex in } D. \text{ Further } \langle V - D \rangle \text{ contains no isolated vertices. Hence } \gamma_{ic}^{e}(G) \leq n - \Delta(G). \]

Equality follows from Proposition 1.

**Theorem 7:** For any graph \( G \) without isolated vertices, \( \frac{n}{\Delta(G)+1} \leq \gamma_{ic}^{e}(G) \).

**Proof:** We know that \( \frac{n}{\Delta(G)+1} \leq \gamma(G) \). Further the theorem follows from the fact that \( \gamma(G) \leq \gamma^{e}(G) \leq \gamma_{ic}^{e}(G) \).

Hence \( \frac{n}{\Delta(G)+1} \leq \gamma_{ic}^{e}(G) \).

**Theorem 8:** Every maximal equitable independent set is a minimal independent equitable cototal dominating set.

**Proof:** Let \( \{v_1, v_2, v_3, \ldots, v_n\} \) be the vertex set of a graph \( G \). Let \( M \) be a set of all independent vertices of \( G \) which are degree equitable to \( V - M \). That is for every vertex \( u \in M \) there exist a vertex \( v \in V - M \) such that \( |\text{deg}(u) - \text{deg}(v)| \leq 1 \).

Suppose \( M \) is a maximal independent equitable set then obviously \( M \) will be minimal independent equitable dominating set.

Further, if \( V - M \) contains no isolated vertices, then \( M \) will be equitable independent cototal dominating set of \( G \). Hence every maximal equitable independent set is a minimal independent equitable cototal dominating set.

**Nordhous and Gaddum Type results:**

**Theorem 9:** For any graph \( G \) without isolated vertices
i) \( \gamma_{ic}^{e}(G) + \gamma_{ic}^{e}(\overline{G}) \leq n + 1 \)
ii) \( \gamma_{ic}^{e}(G) \ast \gamma_{ic}^{e}(\overline{G}) \leq n. \)

Equality holds for \( G = K_n, n \geq 4 \).

**Proof:**
i) Let \( G \) be a graph without isolated vertices. Suppose \( \gamma_{ic}^{e}(G) + \gamma_{ic}^{e}(\overline{G}) \leq n + 1 \) then either \( \gamma_{ic}^{e}(G) = n \) or \( \gamma_{ic}^{e}(\overline{G}) = 1 \). If \( \gamma_{ic}^{e}(G) = n \) then the theorem (1). It’s not possible.

There fore \( \gamma_{ic}^{e}(\overline{G}) = 1 \)

By Proposition (1), \( G \) must be a complete graph contains no edges. Hence entire vertex set act as an independent equitable cototal dominating set. Hence \( \gamma_{ic}^{e}(\overline{G}) = n \).

Therefore \( \gamma_{ic}^{e}(G) + \gamma_{ic}^{e}(\overline{G}) \leq n + 1 \).

ii) Follows from (i).

**References**


