On Jordan \((\sigma,\tau)\)-Higher Homomorphisms of \(\Gamma\)M-Module

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Abstract: Let \(M\) be a \(\Gamma\)-ring and \(X\) be a left \(\Gamma\)M-module, in this paper proved that every Jordan \((\sigma,\tau)\)-higher homomorphism from a \(\Gamma\)-ring \(M\) into a prime left \(\Gamma\)M-module \(X\) is either \((\sigma,\tau)\)-higher homomorphism or \((\sigma,\tau)\)-higher anti homomorphism.

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I. Introduction

Let \(M\) and \(\Gamma\) be two additive belian groups, suppose that there is a mapping from \(M \times \Gamma \times M \longrightarrow M\)

\((\text{the image of } (a, \alpha, b)\) being denoted by \(a \cdot \alpha \cdot b\) satisfying for all \(a, b, c \in M\) and \(\alpha, \beta \in \Gamma\)

\((i)\) \((a + b) \alpha c = a \alpha c + b \alpha c\)

\((ii)\) \((a \alpha + \beta) c = a \alpha c + a \beta c\)

\((iii)\) \((a \alpha (b + c) = a \alpha b + a \alpha c\)

Then \(M\) is called \(\Gamma\)-ring . This definition is due to Barnes [1].

Let \(M\) be a \(\Gamma\)-ring and \(X\) be an additive belian group. \(X\) is a left \(\Gamma\)M-module if there exists a mapping

\(M \times \Gamma \times X \longrightarrow X\) (sending \((m, \alpha, x) \longrightarrow m \alpha x\), such that

\((i)\) \((m_1 + m_2) \alpha x = m_1 \alpha x + m_2 \alpha x\)

\((ii)\) \(m \alpha (x_1 + x_2) = m \alpha x_1 + m \alpha x_2\)

\((iii)\) \((m_1 \alpha m_2) \beta x = m_1 \alpha (m_2 \beta x)\)

\(X\) is prime if \(a \Gamma X b = (0)\) implies \(a = 0\) or \(b = 0\) , for all \(x \in X\) and \(X\) is semiprime if \(a \Gamma X \Gamma a = (0)\) implies \(a = 0\) , for all \(x \in X\) .

\(X\) is called a 2-torsion free if \(2x = 0\) implies \(x = 0\) for all \(x \in X\) .

Let \(X\) be a 2-torsion free semiprime \(\Gamma\)M-module \(X\) and suppose that \(a, b \in \Gamma\)M-module \(X\) if

\(a \Gamma X b + b \Gamma X a = 0\) for all \(x \in X\) , then \(a \Gamma X b = b \Gamma X a = 0\) .

Let \(M\) be \(\Gamma\)-ring , a mapping \(* : M \longrightarrow X\) is called an involution if for all \(a, b \in M\) and \(\alpha \in \Gamma\)

\((i)\) \((a *) = a\)

\((ii)\) \((a + b) * = a * + b *\)

\((iii)\) \((a b) * = b * a *\) .

Let \(\theta\) be an additive mapping of a ring \(R\) into a ring \(R'\), \(\theta\) is called a homomorphism if \(\theta(a b) = \theta(a) \theta(b)\).

And \(\theta\) is called a Jordan homomorphism if for all \(a \in R\)

\(\theta(a + b + a) = \theta(a) \theta(b) + \theta(b) \theta(a)\) for all \(a, b \in R\) .

Let \(\theta\) be an additive mapping of a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\), \(\theta\) is called Jordan homomorphism if

\(\theta(a \alpha + b \alpha a) = \theta(a) \alpha \theta(b) + \theta(b) \alpha \theta(a)\) for all \(a, b \in M\) and \(\alpha \in \Gamma\) .

Let \(\theta\) be an additive mapping of a ring \(R\) into a ring \(R'\) and \(\sigma, \tau\) be two endomorphism of \(R\), \(\theta\) is called

\((\sigma, \tau)\)-homomorphism if

\(\theta(ab) = \theta(\sigma(a)) \theta(\tau(b))\), for all \(a, b \in R\) .

And \(\theta\) is called Jordan \((\sigma, \tau)\)-homomorphism if

\(\theta(ab + ba) = \theta(\sigma(a)) \theta(\tau(b)) + \theta(\sigma(b)) \theta(\tau(a))\), for all \(a, b \in R\) .

Let \(\theta\) be an additive mapping of a \(\Gamma\)-ring \(M\) into a \(\Gamma\)-ring \(M'\) and \(\sigma, \tau\) be two endomorphism of \(M\), \(\theta\) is called

\((\sigma, \tau)\)-homomorphism if

\(\theta(ab) = \theta(\sigma(a)) \alpha \theta(\tau(b))\), for all \(a, b \in M\) and \(\alpha \in \Gamma\) .

And \(\theta\) is called Jordan \((\sigma, \tau)\)-homomorphism if

\(\theta(ab + ba) = \theta(\sigma(a)) \alpha \theta(\tau(b)) + \theta(\sigma(b)) \alpha \theta(\tau(a))\), for all \(a, b \in M\) and \(\alpha \in \Gamma\) .
Now, in this paper presented the definitions of \((\sigma, \tau)\)-higher homomorphism, Jordan \((\sigma, \tau)\)-higher homomorphism, Jordan triple \((\sigma, \tau)\)-higher homomorphism on a left \(\Gamma\)-module and prove that every Jordan \((\sigma, \tau)\)-higher homomorphism from a \(\Gamma\)-ring \(M\) into \(2\)-torsion free \(\Gamma\)-module \(X\), such that \(ao\beta c = a\beta bac\), for all \(a, b, c \in M\) and \(a, \beta \in \Gamma\), \(\sigma^{i} = \sigma^{i} \cdot \tau^{i} = \tau^{i} \cdot \sigma^{i+1} \) and \(\sigma^{i} \tau^{i} = \tau^{i} \sigma^{i}\) then \(\theta\) is a Jordan triple \((\sigma, \tau)\)-higher homomorphism.

II. Jordan \((\sigma, \tau)\)-Higher Homomorphism of \(\Gamma\)-Module

Definition (2.1): Let \(\theta = (\phi_{i})_{i \in N}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a left \(\Gamma\)-module \(X\) and \(\sigma, \tau\) be two endomorphisms of \(M\). \(\theta\) is called a \((\sigma, \tau)\)-higher homomorphism if

\[
\Phi_{\theta}(\alpha \alpha b + b \alpha \alpha) = \sum_{i=1}^{n} \Phi_{i}(\sigma^{i}(\alpha))a \Phi_{i}(\tau^{i}(b)) + \sum_{i=1}^{n} \Phi_{i}(\sigma^{i}(b))a \Phi_{i}(\tau^{i}(\alpha))
\]

for all \(a, b \in M\), \(\alpha \in \Gamma\) and \(n \in N\).

Example (2.2): Let \(\theta = (\phi_{i})_{i \in N}\) be a \((\sigma, \tau)\)-higher homomorphism of a ring \(R\) into a ring \(R'\). Let \(M = M_1 \oplus R\) and \(\Gamma = \{\begin{array}{l} n \in Z, \end{array}\}\). Then \(M\) is a \(\Gamma\)-ring.

Let \(\phi = (\phi_{i})_{i \in N}\) be a family of additive mappings from \(\Gamma\)-ring \(M\) into a left \(\Gamma\)-module \(X\) defined by:

\[
\phi_{n}(a) = \theta_{n}(a), \quad \phi_{n}(b) = \theta_{n}(b), \quad \text{for all } (a, b) \in M.
\]

Let \(\sigma_{\alpha}^{n}, \tau_{\alpha}^{n}\) be two endomorphisms of \(M\), such that

\[
\sigma_{\alpha}^{n}(a) = ((\sigma^{n}(a), \sigma^{n}(b)), \tau_{\alpha}^{n}((a, b)) = ((\tau^{n}(a), \tau^{n}(b)).
\]

Then \(\phi_{n}\) is a \((\sigma, \tau)\)-higher homomorphism.

Definition (2.3): Let \(\theta = (\phi_{i})_{i \in N}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a left \(\Gamma\)-module \(X\) and \(\sigma, \tau\) be two endomorphisms of \(M\). \(\theta\) is called Jordan \((\sigma, \tau)\)-higher homomorphism if

\[
\Phi_{\theta}(\alpha \alpha b + b \alpha \alpha) = \sum_{i=1}^{n} \Phi_{i}(\sigma^{i}(\alpha))a \Phi_{i}(\tau^{i}(b)) + \sum_{i=1}^{n} \Phi_{i}(\sigma^{i}(b))a \Phi_{i}(\tau^{i}(\alpha))
\]

for all \(a, b \in M\), \(\alpha \in \Gamma\) and \(n \in N\).

Remark (2.4): Clearly every \((\sigma, \tau)\)-higher homomorphism is Jordan \((\sigma, \tau)\)-higher homomorphism but the converse is not true in general, as shown by the following example.

Example (2.5): Let \(S\) be any \(\Gamma\)-ring with nontrivial involution \(*\) and \(\Gamma\) be the set of all integers.

Let \(M = S \oplus S\), such that \(a \in Z(S), s_{1}a = a s_{2} = 0, s_{1} \neq s_{2}\) and \(a^{2} = a\), for all \(s_{1}, s_{2} \in S\).

Let \(\theta = (\phi_{i})_{i \in N}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a left \(\Gamma\)-module \(X\) defined by:

\[
\Phi_{\theta}(s, t) = \begin{cases} ((2 - n)a \alpha s, (n-1) t^{*}), & n = 1, 2 \\ 0, & n \geq 3. 
\end{cases}
\]

for all \((s, t) \in M\). Let \(\sigma^{n}, \tau^{n}\) be two endomorphisms of \(M\), such that \(\sigma^{n}(s, t) = (ns, t), \tau^{n}(s, t) = (n^{2}s, t). \) Then \(\theta\) is Jordan \((\sigma, \tau)\)-higher homomorphism but not \((\sigma, \tau)\)-higher homomorphism.

Definition (2.6): Let \(\theta = (\phi_{i})_{i \in N}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a left \(\Gamma\)-module \(X\) and \(\sigma, \tau\) be two endomorphisms of \(M\). \(\theta\) is called Jordan triple \((\sigma, \tau)\)-higher homomorphism if

\[
\Phi_{\theta}(a \alpha b \beta a) = \sum_{i=1}^{n} \Phi_{i}(\sigma^{i}(a))a \Phi_{i}(\tau^{i}(\beta a)) \beta \Phi_{i}(\tau^{i}(\alpha))
\]

for all \(a, b \in M\), \(\alpha, \beta \in \Gamma\) and \(n \in N\).

Definition (2.7): Let \(\theta = (\phi_{i})_{i \in N}\) be a family of additive mappings of a \(\Gamma\)-ring \(M\) into a left \(\Gamma\)-module \(X\) and \(\sigma, \tau\) be two endomorphisms of \(M\). \(\theta\) is called a \((\sigma, \tau)\)-higher anti homomorphism if

\[
\Phi_{\theta}(a \alpha b) = \sum_{i=1}^{n} \Phi_{i}(\sigma^{i}(b))a \Phi_{i}(\tau^{i}(a))
\]

for all \(a, b \in M\), \(\alpha \in \Gamma\) and \(n \in N\).

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Lemma (2.8): Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan triple \((\sigma, \tau, \zeta)\)-higher homomorphism of a \( \Gamma \)-ring \( M \) into a left \( \Gamma M \)-module \( X \), then for all \( a, b, c \in M, \alpha, \beta \in \Gamma \) and \( n \in \mathbb{N} \\
(i) \text{ if } \sigma^i = \sigma^i, \tau^i = \tau^i, \sigma^i \tau^i = \sigma^i \tau^i \text{ and } \sigma^i = \tau^i \\
\phi_h(\alpha \beta b + a \beta b \alpha a) = \sum_{i=1}^n \phi_i(\sigma^i(\alpha)\alpha \phi_i(\sigma^i \tau^i(b))\beta \phi_i(\tau^i(\alpha)) + \\
\sum_{i=1}^n \phi_i(\sigma^i(\alpha)\beta \phi_i(\sigma^i \tau^i(b))\alpha \phi_i(\tau^i(\alpha))) \\
(ii) \phi_h(\alpha \beta b \beta c + c \alpha b \beta a) = \sum_{i=1}^n \phi_i(\sigma^i(\alpha)\alpha \phi_i(\sigma^i \tau^i(b))\beta \phi_i(\tau^i(\alpha)) + \\
\sum_{i=1}^n \phi_i(\sigma^i(\alpha)\beta \phi_i(\sigma^i \tau^i(b))\alpha \phi_i(\tau^i(\alpha))) \\
(iii) \text{ In particular, if } M \text{ is commutative and } \Gamma M \text{-module } X \text{ is a } 2\text{-torsion free , then} \\
\phi_h(\alpha \beta b \beta c) = \sum_{i=1}^n \phi_i(\sigma^i(\alpha)\alpha \phi_i(\sigma^i \tau^i(b))\beta \phi_i(\tau^i(\alpha))) \\
(iv) \phi_h(\alpha \beta b \beta c + c \alpha b \alpha a) = \sum_{i=1}^n \phi_i(\sigma^i(\alpha)\alpha \phi_i(\sigma^i \tau^i(b))\alpha \phi_i(\tau^i(\alpha))) + \\
\sum_{i=1}^n \phi_i(\sigma^i(\alpha)\beta \phi_i(\sigma^i \tau^i(b))\alpha \phi_i(\tau^i(\alpha))) \\
Proof: \\
(i) \text{ Replace } a \beta b + b \beta a \text{ for } b \text{ in Definition (2.3) , we get :} \\
\phi_h(\alpha \alpha(\beta b + b \beta a) = \sum_{i=1}^n \phi_i(\sigma^i(\alpha)\alpha \phi_i(\tau^i(a) + \\
\sum_{i=1}^n \phi_i(\sigma^i(\alpha)\beta \phi_i(\alpha \phi_i(\tau^i(a))) \\
(ii) \phi_h(\alpha \beta b \beta c + c \alpha b \beta a) = \sum_{i=1}^n \phi_i(\sigma^i(\alpha)\alpha \phi_i(\tau^i(a) + \\
\sum_{i=1}^n \phi_i(\sigma^i(\alpha)\beta \phi_i(\alpha \phi_i(\tau^i(a))) \\
On the other hand: \\
\phi_h(\alpha \beta b + b \beta a) = \sum_{i=1}^n \phi_i(\sigma^i(\alpha)\alpha \phi_i(\tau^i(a) + \\
\sum_{i=1}^n \phi_i(\sigma^i(\alpha)\beta \phi_i(\alpha \phi_i(\tau^i(a))) \\
Comparing (1) and (2), we get: \\
\phi_h(\alpha \beta b = \sum_{i=1}^n \phi_i(\sigma^i(\alpha)\alpha \phi_i(\tau^i(a) + \\
\sum_{i=1}^n \phi_i(\sigma^i(\alpha)\beta \phi_i(\alpha \phi_i(\tau^i(a))) \\
(ii) \text{ Replace } a + c \text{ for } a \text{ in Definition (2.6) , we get :} \\
\phi_h((a + c)\alpha b \beta (a + c) = \sum_{i=1}^n \phi_i(\sigma^i(a)\alpha \phi_i(\tau^i(a) + \\
\sum_{i=1}^n \phi_i(\sigma^i(a)\beta \phi_i(\alpha \phi_i(\tau^i(a))) \\
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\[
= \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{-i}(b)) \beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b)) \alpha \phi_i(\sigma^i \tau^{-i}(b)) \beta \phi_i(\tau^i(c))
\]

On the other hand:

\[
\phi_i((a+c)ab\tau(b+c)) = \phi_i(aab\tau a + aab\tau b + cab\tau a + cab\tau b)
\]

Comparing (1) and (2), we get:

\[
\phi_i(a \alpha b \beta c + c \alpha b \beta a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{-i}(b)) \beta \phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c)) \alpha \phi_i(\sigma^i \tau^{-i}(b)) \beta \phi_i(\tau^i(a))
\]

(ii) By (ii) and since \(M\) be a commutative and \(\GammaM\) - module \(X\) is a 2-torsion free

\[
\phi_i(a \alpha b \beta c + a b \tau(b+c)) = 2\phi_i(a \alpha b \beta c)
\]

(iv) Replace \(\beta\) for \(\alpha\) in (ii), we get:

\[
\phi_i(a \alpha b \alpha c + c \alpha b \alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{-i}(b)) \alpha \phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c)) \alpha \phi_i(\sigma^i \tau^{-i}(b)) \alpha \phi_i(\tau^i(a))
\]

**Definition (2.9):** Let \(0 = (\phi_i)_{i \in \mathbb{N}}\) be a Jordan \((\sigma, \tau)-\)higher homomorphism from a \(\Gamma\)-ring \(M\) into a left \(\GammaM\) - module \(X\), then for all \(a, b \in M, \alpha \in \Gamma\) and \(n \in \mathbb{N}\), we define

\[
G_n(a \alpha b_\beta)_\sigma = \phi_i(a \alpha b_\beta) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\tau^i(b))
\]

**Lemma (2.10):** Let \(0 = (\phi_i)_{i \in \mathbb{N}}\) be a Jordan \((\sigma, \tau)-\)higher homomorphism from a \(\Gamma\)-ring \(M\) into a left \(\GammaM\) - module \(X\), then for all \(a, b, c \in M, \alpha, \beta \in \Gamma\) and \(n \in \mathbb{N}\):

(i) \(G_n(a,b)_\alpha = -G_n(b,a)_\alpha\)

(ii) \(G_n(a+b,c)_\alpha = G_n(a,c)_\alpha + G_n(b,c)_\alpha\)

(iii) \(G_n(a \alpha b + c)_\alpha = G_n(a,b)_\alpha + G_n(a,c)_\alpha\)

(iv) \(G_n(a \alpha b + c)_\alpha \beta = G_n(a,b)_\alpha + G_n(a,b)_\beta\)

**Proof:**

(i) By Definition (2.3)

\[
\phi_i(aab) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\tau^i(b)) = -(\phi_i(baa) - \sum_{i=1}^{n} \phi_i(\sigma^i(b)) \alpha \phi_i(\tau^i(a)))
\]

\[
G_n(a,b)_\alpha = -G_n(b,a)_\alpha
\]

(ii) \(G_n(a+b,c)_\alpha = \phi_i((a+b) \alpha c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a+b)) \alpha \phi_i(\tau^i(c))
\]

\[
= \phi_i(a \alpha c + b \alpha c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\tau^i(c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(b)) \alpha \phi_i(\tau^i(c))
\]

\[
= G_n(a,c)_\alpha + G_n(b,c)_\alpha
\]

(iii) \(G_n(a \alpha b + c)_\alpha \beta = \phi_i(a \alpha (b+c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\tau^i(c))
\]

\[
= \phi_i(aab + ac) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\tau^i(b)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\tau^i(c))
\]
= \phi_n(aab) - \sum_{i=1}^{n} \phi_i(\sigma'(a)) \alpha \phi_i(\tau'(b)) + \phi_n(ac) - \sum_{i=1}^{n} \phi_i(\sigma'(a)) \alpha \phi_i(\tau'(c))

= G_a(ba) + G_a(ba) 

(iv) \ G_n(a,b)G_n(\alpha,\beta) = \phi_n(\alpha(\alpha + \beta)b) - \sum_{i=1}^{n} \phi_i(\sigma'(a))\alpha \phi_i(\tau'(b)) + \phi_n(\alpha \beta b) - \sum_{i=1}^{n} \phi_i(\sigma'(a))\beta \phi_i(\tau'(b))

= G_a(ba) + G_a(ba)

**Remark (2.11):** Note that \( \theta = (\phi_n)_{n=1}^\infty \) is a \((\sigma,\tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a left \( \Gamma M \) - module \( X \) if and only if for all \( a, b \in M, \alpha, \beta \in \Gamma \) and \( n \in N \).

**Lemma (2.12):** Let \( \theta = (\phi_n)_{n=1}^\infty \) be a \((\sigma,\tau)\)-higher homomorphism of a \( \Gamma \)-ring \( M \) into a left \( \Gamma M \) - module \( X \), such that \( \sigma^{n+1} = \sigma^n, \tau^{n+1} = \tau^n, \sigma^2 = \sigma, \tau^2 = \tau \) and \( \sigma \tau = \tau \sigma \) for all \( i \in N \), then for all \( a, b, m \in M, \alpha, \beta \in \Gamma \) and \( n \in N \)

(i) \( G_n(\alpha,\beta)G_n(\sigma,\tau)G_n(\sigma,\tau) = 0 \)

(ii) \( G_n(\alpha,\beta)G_n(\sigma,\tau)G_n(\sigma,\tau) = 0 \)

(iii) \( G_n(\alpha,\beta)G_n(\sigma,\tau)G_n(\sigma,\tau) = 0 \)

**Proof:**

(i) We prove by using the induction, if \( n = 1 \)

Let \( w = aabmbmab + baa(\alpha \beta \gamma)ab \), since \( \theta \) is a \((\sigma,\tau)\)-homomorphism

\[
\theta(w) = \theta(aabmbmab) + \theta(baa(\alpha \beta \gamma)ab) = \theta(aaabm) + \theta(baa(\alpha \beta \gamma))ab(ta) + \theta(ba(\alpha \beta \gamma))ab(ta)

= \theta(aaabm) + \theta(baa(\alpha \beta \gamma))ab(ta) + \theta(ba(\alpha \beta \gamma))ab(ta) - \theta(ta(aab)) + (\theta(ba(\alpha \beta \gamma))ab(ta) - \theta(ba(\alpha \beta \gamma))ab(ta)) + \theta(\sigma^{n+1}(ta)ab(ta)) = 0

On the other hand

\[
\theta(w) = \theta(aaabm) + \theta(baa(\alpha \beta \gamma))ab(ta) + \theta(ba(\alpha \beta \gamma))ab(ta) - \theta(ta(aab)) + (\theta(ba(\alpha \beta \gamma))ab(ta) - \theta(ba(\alpha \beta \gamma))ab(ta)) + \theta(\sigma^{n+1}(ta)ab(ta)) = 0

\]

Compare (1), (2) and since \( \sigma \tau = \tau \sigma \)

\[
0 = -\theta(aaabm)\theta(baa(\alpha \beta \gamma))\theta(ta(aab)) + \theta(\sigma^{n+1}(ta)ab(ta)) = 0

Since \( \sigma^2 = \sigma \) and \( \tau^2 = \tau \)

\[
0 = -\theta(aaabm)\theta(baa(\alpha \beta \gamma))\theta(ta(aab)) + \theta(\sigma^{n+1}(ta)ab(ta)) = 0

\]

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Let $w = \phi((aa)(bb))βaαa + ba(βmβα)αab$

Since $θ$ is a Jordan (σ, τ)-higher homomorphism, then

$$\phi_ω(w) = \phi_ω((aa)(bb))βaαa + ba(βmβα)αab$$

On the other hand:

$$\phi_ω(w) = \phi_ω((aa)(bb))βaαa + ba(βmβα)αab$$

Thus, we have:

$$G(σ(a), σ(b))_β G(τ(b), τ(a))_α + G(σ(b), σ(a))_β G(τ(a), τ(b))_α = 0$$

Now, we can assume that:

$$G_α(σ^α(a), σ^α(b))_β G_α(τ^α(b), τ^α(a))_α + G_α(σ^α(b), σ^α(a))_β G_α(τ^α(a), τ^α(b))_α = 0$$

for all $a, b, m \in M$, and $s, n \in N, s < n$.

Let $w = aaβmβbbaa + baβmβaaβ$
\[ \phi_n(\sigma(a b)) = \phi_n(\sigma(m))G_n(\tau(a), \tau(b)) + \sum_{j=1}^{n-1} \phi_n(\sigma(a b)) \phi_j(\sigma(m))G_n(\tau(a), \tau(b)) \]

\[ \phi_n(\sigma(a)) = \phi_n(\sigma(m))G_n(\tau(a), \tau(b)) + \sum_{j=1}^{n-1} \phi_n(\sigma(a)) \phi_j(\sigma(m))G_n(\tau(a), \tau(b)) \]

Compare (1), (2) and since \( \sigma^{2n} = \sigma^n \), \( \sigma^i \sigma^{-i} = \tau \sigma^i \), \( \sigma \tau = \tau' \sigma \)

\[ O = -\phi_n(\sigma(a b)) \phi_n(\sigma(m))G_n(\tau(a), \tau(b)) - \phi_n(\sigma(a b)) \phi_n(\sigma(m))G_n(\tau(a), \tau(b)) + \phi_n(\sigma(a)) \phi_n(\sigma(b)) \phi_n(\sigma(m))G_n(\tau(a), \tau(b)) \]

By our hypothesis, we have:

\[ G_n(\sigma(a), \sigma(a)) = \phi(\sigma(m))G_n(\tau(a), \tau(a)) \]

(ii) Replace \( \beta \) by \( \alpha \) in (i) proceeding in the same way as in the proof of (i) by the similar arguments, we get (ii).

(iii) Interchanging \( \alpha \) and \( \beta \) in (i), we get (iii).

**Lemma (2.13):** Let \( \theta = (\phi_{n})_{n \in \mathbb{N}} \) be a Jordan \((\sigma, \tau)\)-higher homomorphism of a \( \Gamma \)-ring \( M \) into a \( 2^n \)-torsion free prime left \( \Gamma \)-module \( X \), then for all \( a, b \in M, \alpha, \beta \in \Gamma \) and \( n \in N \)

(i) \( G_n(\sigma(a), \sigma(a), \sigma(a)) = \phi(\sigma(m))G_n(\tau(a), \tau(a), \tau(a)) \)

(ii) \( G_n(\sigma(a), \sigma(a), \sigma(a)) = \phi(\sigma(m))G_n(\tau(a), \tau(a), \tau(a)) \)

(iii) \( G_n(\sigma(a), \sigma(a), \sigma(a)) = \phi(\sigma(m))G_n(\tau(a), \tau(a), \tau(a)) \)

**Proof:**

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(i) By Lemma (2.12) (i), we have:
\[ G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(b), τ^n(a)) = 0 \]
\[ G_n(σ^n(b), σ^n(a)) \alpha \phi_n(σ^n(m)) G_n(τ^n(a), τ^n(b)) = 0 \]

And by Lemma (Let X be a 2-torsion free semiprime left ΓM-module X and suppose that a, b ∈ ΓM-module X if \( aXΓb + bXΓa = 0 \) for all \( x \in X \), and if \( aXb = bXa = 0 \), we get:
\[ G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(b), τ^n(a)) = 0 \]
\[ G_n(σ^n(b), σ^n(a)) \alpha \phi_n(σ^n(m)) G_n(τ^n(a), τ^n(b)) = 0 \]

(ii) Replace \( \beta \) for \( α \) in (i), we get (ii).

(iii) Interchanging \( α \) and \( β \) in (i), we get (iii).

**Theorem (2.14):** Let \( 0 = (φ_i)_{i \in \mathbb{N}} \) be a Jordan (σ, τ)-higher homomorphism of a Γ-ring M into a prime left ΓM-module X, then for all \( a, b, c, d, m \in M, α, β \in Γ \) and \( n \in N \)

(i) \( G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(d), τ^n(c)) = 0 \)

(ii) \( G_n(σ^n(c), σ^n(b)) \alpha \phi_n(σ^n(m)) G_n(τ^n(d), τ^n(c)) = 0 \)

Proof:

(i) Replacing \( a + c \) for \( a \) in Lemma (2.13) (i), we get:
\[ G_n(σ^n(a + c), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(d), τ^n(c)) = 0 \]
\[ G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(b), τ^n(a)) = 0 \]

Therefore, we get:
\[ G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(b), τ^n(c)) = 0 \]
\[ G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(b), τ^n(a)) = 0 \]

Hence, by the primness of left ΓM-module X:
\[ G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(b), τ^n(c)) = 0 \]

Now, replacing \( b + d \) for \( b \) in Lemma (2.13) (i), we get:
\[ G_n(σ^n(a), σ^n(b + d)) \beta \phi_n(σ^n(m)) G_n(τ^n(b + d), τ^n(a)) = 0 \]
\[ G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(b), τ^n(a)) = 0 \]

Therefore, we get:
\[ G_n(σ^n(a), σ^n(b + d)) \beta \phi_n(σ^n(m)) G_n(τ^n(b + d), τ^n(a)) = 0 \]

Thus, \( G_n(σ^n(a), σ^n(b)) \beta \phi_n(σ^n(m)) G_n(τ^n(b + d), τ^n(a)) = 0 \)
\[ G_n (\sigma^n(a), \sigma^n(b)) \beta \phi_h (\sigma^n(m)) \beta G_n (\tau^n(b), \tau^n(a)) \alpha + \]
\[ G_n (\sigma^n(a), \sigma^n(b)) \beta \phi_h (\sigma^n(m)) \beta G_n (\tau^n(b), \tau^n(c)) \alpha + \]
\[ G_n (\sigma^n(a), \sigma^n(b)) \beta \phi_h (\sigma^n(m)) \beta G_n (\tau^n(d), \tau^n(a)) \alpha + \]
\[ G_n (\sigma^n(a), \sigma^n(b)) \beta \phi_h (\sigma^n(m)) \beta G_n (\tau^n(d), \tau^n(c)) \alpha = 0 \]

By (1), (2) and Lemma (2.13)(i), we get:
\[ G_n (\sigma^n(a), \sigma^n(b)) \beta \phi_h (\sigma^n(m)) \beta G_n (\tau^n(d), \tau^n(c)) \alpha = 0. \]

(ii) Replace \( \beta \) for \( \alpha \) in (i), we get (ii).

(iii) Replacing \( \alpha + \beta \) for \( \alpha \) in (ii), we get:
\[ G_n (\sigma^n(a), \sigma^n(b)) \alpha \phi_h (\sigma^n(m)) \alpha G_n (\tau^n(d), \tau^n(c)) \beta = 0 \]
\[ G_n (\sigma^n(a), \sigma^n(b)) \alpha \phi_h (\sigma^n(m)) \alpha G_n (\tau^n(d), \tau^n(c)) \alpha = 0 \]
\[ G_n (\sigma^n(a), \sigma^n(b)) \alpha \phi_h (\sigma^n(m)) \alpha G_n (\tau^n(d), \tau^n(c)) \beta = 0 \]

By (i) and (ii), we get:
\[ G_n (\sigma^n(a), \sigma^n(b)) \alpha \phi_h (\sigma^n(m)) \alpha G_n (\tau^n(d), \tau^n(c)) \beta = 0 \]
\[ G_n (\sigma^n(a), \sigma^n(b)) \beta \phi_h (\sigma^n(m)) \beta G_n (\tau^n(d), \tau^n(c)) \alpha = 0 \]

Therefore, we have:
\[ G_n (\sigma^n(a), \sigma^n(b)) \alpha \phi_h (\sigma^n(m)) \alpha G_n (\tau^n(d), \tau^n(c)) \beta = 0 \]
\[ G_n (\sigma^n(a), \sigma^n(b)) \beta \phi_h (\sigma^n(m)) \beta G_n (\tau^n(c), \tau^n(c)) \alpha = 0 \]
\[ = - \sum_{i=1}^{G_n (\sigma^n(a), \sigma^n(b)) \alpha \phi_h (\sigma^n(m)) \alpha G_n (\tau^n(d), \tau^n(c)) \beta = 0} \]

Since left \( GM \)-module \( X \) is prime, then:
\[ G_n (\sigma^n(a), \sigma^n(b)) \alpha \phi_h (\sigma^n(m)) \alpha G_n (\tau^n(d), \tau^n(c)) \beta = 0. \]

III. The Main Result

**Theorem (3.1):** Every Jordan \((\sigma, \tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a prime left \( GM \)-module \( X \) is either \((\sigma, \tau)\)-higher homomorphism or \((\sigma, \tau)\)-higher anti homomorphism.

**Proof:** Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan \((\sigma, \tau)\)-higher homomorphism of a \( \Gamma \)-ring \( M \) into a prime left \( GM \)-module \( X \), then by Lemma (2.14)(i):
\[ G_n (\sigma^n(a), \sigma^n(b)) \beta \phi_h (\sigma^n(m)) \beta G_n (\tau^n(c), \tau^n(c)) \alpha = 0 \]

Since left \( GM \)-module \( X \) is prime, therefore either \[ G_n (\sigma^n(a), \sigma^n(b)) \alpha = 0 \]
\[ or \]
\[ G_n (\tau^n(d), \tau^n(c)) \alpha = 0, \text{ for all } a, b, c, d \in M, \alpha \in \Gamma \text{ and } n \in \mathbb{N}. \]

If \[ G_n (\tau^n(d), \tau^n(c)) \alpha \neq 0, \text{ for all } c, d \in M, \alpha \in \Gamma \text{ and } n \in \mathbb{N} \text{ then} \]
\[ G_n (\sigma^n(a), \sigma^n(b)) \alpha = 0, \text{ for all } a, b \in M, \alpha \in \Gamma \text{ and } n \in \mathbb{N}, \text{ hence, we get} \]
\[ \theta \text{ is a } (\sigma, \tau) \text{-higher homomorphism}. \]

But if \[ G_n (\tau^n(d), \tau^n(c)) \alpha = 0, \text{ for all } c, d \in M, \alpha \in \Gamma \text{ and } n \in \mathbb{N}, \text{then} \]
\[ \theta \text{ is a } (\sigma, \tau) \text{-higher anti homomorphism}. \]

**Proposition (3.2):** Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan \((\sigma, \tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a 2-torsion free left \( GM \)-module \( X \), such that \( ab\beta\phi = a\beta b\phi \), for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \), \( \sigma' = \sigma', \tau' = \tau', \sigma' = \sigma' \tau^{n-1} \), and \( \sigma' = \sigma' \tau' \), then \( \theta \) is a Jordan triple \((\sigma, \tau)\)-higher homomorphism.

**Proof:** Replace \( \beta \) by \( a\beta b + b\beta \alpha \) in Definition (2.3), we get:
\[ \Phi_i (a \alpha (a\beta b + b\beta \alpha) + (a\beta b + b\beta \alpha) a\alpha) \]
\[ = \sum_{i=1}^{\Phi_i (\sigma'(a)) \alpha \Phi_i (\tau'(a)) \beta (a\beta b + b\beta \alpha)} + \sum_{i=1}^{\Phi_i (\sigma'(a)) \beta \Phi_i (\tau'(a)) \alpha (a\beta b + b\beta \alpha) \alpha} \]
\[ = \sum_{i=1}^{\Phi_i (\sigma'(a)) \alpha \Phi_i (\tau'(a)) \beta (a\beta b + b\beta \alpha) + (a\beta b + b\beta \alpha) a\alpha} \]
\[ = \sum_{i=1}^{\Phi_i (\sigma'(a)) \beta \Phi_i (\tau'(a)) \alpha (a\beta b + b\beta \alpha) + (a\beta b + b\beta \alpha) a\alpha} \]

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\[ \sum_{i=1}^{n} \phi_i (\sigma^i(a)) \alpha \left( \sum_{j=1}^{n} \phi_j (\sigma^j(z_i(a))) \beta \phi_j (z^j(b)) + \sum_{j=1}^{n} \phi_j (\sigma^j(z_i(b))) \beta \phi_j (z^j(a)) \right) + \\ + \sum_{i=1}^{n} \phi_i (\sigma^i(a)) \alpha \phi_i (\sigma^i(z_i(a))) \beta \phi_i (z^i(a)) = \sum_{i=1}^{n} \phi_i (\sigma^i(a)) \alpha \phi_i (\sigma^i(z_i(a))) \beta \phi_i (z^i(a)) + \\ + \sum_{i=1}^{n} \phi_i (\sigma^i(a)) \alpha \phi_i (\sigma^i(z_i(b))) \beta \phi_i (z^i(a)) + \sum_{i=1}^{n} \phi_i (\sigma^i(a)) \alpha \phi_i (\sigma^i(z_i(b))) \beta \phi_i (z^i(a)) \]

Since X is left ΓM-module, \( \sigma^i = \sigma^i \), \( \tau^i = \tau^i \), \( \sigma^i \tau^i = \sigma^{i+n-i} \) and \( \tau^i \tau^i = \tau^{i+1} \), we get

\[ \sum_{i=1}^{n} \phi_i (\sigma^i(a)) \alpha \phi_i (\sigma^i(z_i^{-1}(a))) \beta \phi_i (z^i(b)) + 2 \sum_{i=1}^{n} \phi_i (\sigma^i(a)) \alpha \phi_i (\sigma^i(z_i^{-1}(b))) \beta \phi_i (z^i(a)) + \\ + \sum_{i=1}^{n} \phi_i (\sigma^i(b)) \beta \phi_i (\sigma^i(z_i^{-1}(a))) \alpha \phi_i (z^i(a)) \]

On the other hand:
\[ \phi_{a+b}(a+b\alpha + b\alpha b + b\alpha b\alpha + b\alpha b\alpha a) = \phi_{a+b}(a+b\alpha b + a\alpha b\alpha + a\alpha b\alpha a + b\alpha b\alpha a) \]

Since \( a+b\alpha b = a\alpha b \alpha \), for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \)

\[ = \phi_{a+b}(a\alpha b \alpha b + b\alpha b \alpha a) + 2 \phi_{a+b}(a\alpha b \alpha b) \]

Compare (1) and (2), we get:

\[ 2\phi_{a+b}(a\alpha b \alpha b) = 2 \sum_{i=1}^{n} \phi_i (\sigma^i(a)) \alpha \phi_i (\sigma^i(z_i^{-1}(b))) \beta \phi_i (z^i(a)) \]

Since X is a 2-torsion free left ΓM-module, we obtain that 0 is a Jordan triple (σ,τ)-higher homomorphism.

References