Reciprocal Series of \( k \)-Fibonacci Numbers with Subscripts in Linear Form

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Abstract: The aim of this paper is to find a formula that allows to find the infinite sum of reciprocals of certain \( k \)-Fibonacci numbers whose subscripts are in linear form. Particularizing this formula, we are able to obtain other formulas previously found by other authors for the case of both classical Fibonacci and Pell sequences.

Keywords: Binet identity, \( k \)-Fibonacci numbers, \( k \)-Lucas numbers.

I. Introduction

One of the more studied sequences is the Fibonacci sequence \([1, 2]\), and it has been generalized in many ways [3]. Here, we use the following one-parameter generalization of the Fibonacci sequence [4, 5].

1.1 Definition. For any integer number \( k \geq 1 \), the \( k \)-Fibonacci sequence, say \( \{ F_{k,n} \}_{n \in \mathbb{N}} \), is defined recurrently by

\[
F_{k,0} = 0, \quad F_{k,1} = 1 \quad \text{and} \quad F_{k,n+1} = k F_{k,n} + F_{k,n-1}
\]

Note for \( k = 1 \) the classical Fibonacci sequence \( F = \{ F_n \} = \{0,1,2,3,5,8...\} \) is obtained and for \( k = 2 \) it is the Pell sequence \( P = \{ P_n \} = \{0,1,2,5,12,29,70...\} \).

The characteristic equation of the definition is \( r^2 = kr + 1 \) whose solutions are \( \sigma_1 = \frac{k + \sqrt{k^2 + 4}}{2} \) and \( \sigma_2 = \frac{k - \sqrt{k^2 + 4}}{2} \), that verify \( \sigma_1 + \sigma_2 = k \), \( \sigma_1 - \sigma_2 = \sqrt{k^2 + 4} \), \( \sigma_1^2 = k \sigma_1 + 1 \), \( \sigma_1 > 0 \), \( \sigma_2 < 0 \).

For the properties of the \( k \)-Fibonacci numbers, see [4, 5]. In particular, the Binet Identity is

\[
F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}
\]

and the convolution formula,

\[
F_{k,n+m} = F_{k,n+m-1}F_{k,n} + F_{k,n+m-1}F_{k,n-1}
\]

Finally, we define the \( k \)-Fibonacci numbers of negative index as \( F_{k,-n} = (-1)^{n+1} F_{k,n} \).

1.2 Definition. For any integer number \( k \geq 1 \), the \( k \)-Lucas sequence \([6]\), say \( \{ L_{k,n} \}_{n \in \mathbb{N}} \), is defined recurrently by

\[
L_{k,0} = 2, \quad L_{k,1} = k \quad \text{and} \quad L_{k,n+1} = k L_{k,n} + L_{k,n-1}
\]

Note for \( k = 1 \) the classical Lucas sequence is obtained \( L = \{ L_n \} = \{2,1,3,4,7,11...\} \) and for \( k = 2 \) it is the Pell-Lucas sequence \( L = PL = \{ PL_n \} = \{2,2,6,14,34,82...\} \).

The Binet Identity for the \( k \)-Lucas numbers is \( L_{k,n} = \frac{k^n \sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \). The \( k \)-Lucas numbers are related to the \( k \)-Fibonacci numbers by the relation \( L_{k,n} = F_{k,n-1} + F_{k,n+1} \).

II. Preliminary

In this section we will prove some lemmas that we will need for the proof of the main theorem of this paper.

2.1 Lemma 1. For \( m \geq 1 \),

\[
F_{k,2m} = F_{k,m}L_{k,m}
\]

Proof. Applying the Binet formula,

\[
F_{k,m}L_{k,m} = \frac{\sigma_1^m - \sigma_2^m}{\sigma_1 - \sigma_2} (\sigma_1^m + \sigma_2^m) = \frac{\sigma_1^m - \sigma_2^{2m}}{\sigma_1 - \sigma_2} = F_{k,2m}
\]
2.2 Lemma 2.
It is
\[ F_{k,a+2b} + k_{a} = F_{k,a+2b} L_{k,2b} \] \hspace{1cm} (2)

Proof. Applying the convolution formula and \( F_{k,a} = (-1)^{a+1} F_{k,a} \), the Left Hand Side (LHS) of this equation
is
\[ \text{LHS} = F_{k,a+2b} + k_{a} = F_{k,a+2b} + k_{a} = F_{k,a+2b}(F_{k,2b} + k_{2b} + 1) + F_{k,a+2b} + F_{k,a+2b} + F_{k,a+2b} \]

2.3 Lemma 3.
\[ F_{k,2(a+1)+(2,p+1)}, + F_{k,2p+1} = F_{k,2(a+1)+L_{k,2(a+1)+1}} \] \hspace{1cm} (3)

Proof. Using the Binet formulas for \( F_{k,a} \) and \( L_{k,a} \), and taking into account \( \sigma, \sigma_{2} = -1 \), it
is
\[ \text{RHS} = \frac{1}{\sqrt{k^2 + 4}} \left( \sigma_{2}^{2a+2b} - \sigma_{2}^{2a+2b+2} \right) \]
\[ = \frac{1}{\sqrt{k^2 + 4}} \left( \sigma_{1}^{2a+2b+1} + \sigma_{2}^{2a+2b+1} + \sigma_{1}^{2a+2b+1} - \sigma_{2}^{2a+2b+1} \right) \]
\[ = \frac{1}{\sqrt{k^2 + 4}} \left( \sigma_{1}^{2a+2b+1} - \sigma_{2}^{2a+2b+1} - \sigma_{1}^{2a+2b+1} + \sigma_{2}^{2a+2b+1} \right) = \text{LHS} \]

2.4 Lemma 4.
\[ L_{k,a} L_{k,2b} - (k^2 + 4) F_{k,2a} F_{k,2b} = 2L_{k,a+2b} \] \hspace{1cm} (4)

It is enough to apply the Binet formulas for \( F_{k,a} \) and \( L_{k,a} \)

2.5 Lemma 5.
\[ L_{k,2a} L_{k,a} - L_{k,2a} L_{k,2b+2a} = -2(-1)^{a} F_{k,2b} \] \hspace{1cm} (5)

Proof.
\[ \text{LHS} = \frac{1}{\sqrt{k^2 + 4}} \left( \sigma_{1}^{2a+2b} - \sigma_{2}^{2a+2b} \right) \]
\[ = \frac{1}{\sqrt{k^2 + 4}} \left( \sigma_{1}^{2a+2b} - \sigma_{2}^{2a+2b} - \sigma_{1}^{2a+2b} + \sigma_{2}^{2a+2b} \right) \]
\[ = \frac{2(-1)^{a}}{\sqrt{k^2 + 4}} = -2(-1)^{a} F_{k,2b} \]

III. Main Result

In this section we will prove the main theorem of this paper.

3.1 Main theorem.
Let us suppose \( p \) and \( r \) are integer numbers. Then
\[ \sum_{j=0}^{2n} \frac{1}{F_{k,2(p+1)(2j+1)+4r} + F_{k,2p+1}} = \frac{L_{k,2(p+1)(2n+1)}}{L_{k,2} L_{k,2r} F_{k,2(p+1)(2n+1)+2r}} \] \hspace{1cm} (6)

First we prove the following equality:
\[ \sum_{j=0}^{2n} \frac{1}{F_{k,2(p+1)(2j+1)+4r} + F_{k,2p+1}} = 1 \frac{L_{k,2(p+1)(2n+1)+2r}}{L_{k,2} (F_{k,2(p+1)(2n+1)+2r})} \] \hspace{1cm} (7)

Proof. We will proceed by induction on 2n. 

If \( n = 0 \), we must prove that 
\[ \frac{1}{F_{k,2p+1+4r} + F_{k,2p+1}} = \frac{1}{2L_{k,2}} \left( F_{k,2} L_{k,2r} - (k^2 + 4) F_{k,2r} \right) \] \hspace{1cm} (8)

From Lemma 2, with \( a = 2p + 1 \) and \( b = r \), LHS = \[ F_{k,2p+1+2} L_{k,2r} \]

RHS = \[ \frac{L_{k,2p+1+2} L_{k,2r} - (k^2 + 4) F_{k,2r}}{2F_{k,2p+1+2} L_{k,2r} L_{k,2p+1}} \] \hspace{1cm} (9)

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From Lemma 4 with $a = 2p + 1 + 2r$ and $b = r$, $L_{k,2p+1+2r}K_{k,2r} - (k^2 + 4)F_{k,2p+1+2r}F_{k,2r} = 2L_{k,2p+1}$. Hence

$$RHS = \frac{1}{F_{k,2p+1} + L_{k,2r}} = LHS$$

Assuming Main Theorem is true for $n$, we must prove it is true for $n + 1$. If $S_n$ is the sum of (7) and $a_n$ the summands, then for $n + 1$ it is

$$S_{2n+2} = S_{2n} + a_{2n+1} + a_{2n+2}$$

$$= \frac{1}{2L_{k,2p+1}} \left( \frac{L_{k,2(2p+1)(2n+1)+2r}}{F_{k,2(2p+1)(2n+1)+2r}} - (k^2 + 4) \frac{F_{k,2r}}{L_{k,2r}} \right) + \frac{1}{F_{k,2p+1} + F_{k,2r}} + \frac{1}{F_{k,2p+1} + F_{k,2r}}$$

Therefore, we must prove

$$\frac{1}{F_{k,2p+1} + F_{k,2r}} + \frac{1}{F_{k,2p+1} + F_{k,2r}} = \frac{1}{2L_{k,2p+1}} \left( \frac{L_{k,2q+2(2p+1)}}{F_{k,q+2(2p+1)}} - \frac{L_{k,q}}{F_{k,q}} \right)$$

(8)

From Lemma 3 with $2a+1 = q$ it is $F_{k,2p+1} + F_{k,2r} = L_{k,q+2(2p+1)}L_{k,q}$.

From Lemma 2 with $a = 2p + 1$, and $b = 2q + 2p + 1$ it is $F_{k,a} = F_{k,2(2p+2)q}$.

Then, in the equation (8) $LHS = \frac{1}{F_{k,q+2(2p+1)} + F_{k,q} + F_{k,q+2(2p+1)}} = \frac{1}{2L_{k,2p+1}} F_{k,q+2(2p+1)} + F_{k,q}$.

From Lemma 3, $F_{k,2q+2(2p+1)} + F_{k,2q+1} = F_{k,2p+1} + L_{k,2(a+1)}$ with $2a+1 = q$.

On the other hand, in the equation (8) it is $RHS = \frac{L_{k,q+2(2p+1)}F_{k,q} - L_{k,q}F_{k,q+2(2p+1)}}{2L_{k,2p+1}F_{k,q+2(2p+1)}F_{k,q}}$.

From Lemma 5, with $a = q$, $c = 2p + 1$, it is $L_{k,q+2(2p+1)}F_{k,q} - L_{k,q}F_{k,q+2(2p+1)} = F_{k,a+2r}F_{k,a} - L_{k,a}F_{k,a+2r} = 2F_{k,2(2p+1)}$.

Taking into account $F_{k,2m} = F_{k,m}L_{k,m}$, finally $RHS = \frac{L_{k,2(2p+1)}F_{k,q+2(2p+1)}F_{k,q}}{2L_{k,2p+1}F_{k,q+2(2p+1)}F_{k,q}} = \frac{F_{k,2p+1}}{L_{k,2p+1}F_{k,q+2(2p+1)}F_{k,q}}$ and $LHS = \frac{F_{k,2p+1}}{L_{k,2p+1}F_{k,q+2(2p+1)}F_{k,q}}$.

The formula (7) is proven.

If $a = (2p+1)(2n+1)+2, b = r$ in the equation (4), the numerator of the RHS of the equation (6) is $L_{k,2(2p+1)(2n+1)+2r} - (k^2 + 4)F_{k,2(2p+1)(2n+1)+2r}F_{k,2r} = 2L_{k,2p+1}(2n+1)$, and Main theorem of this paper is at last proven.

3.2 Lemma 6.

It is verified

$$\lim_{n \to \infty} \frac{F_{k,n}}{L_{k,n}} = \sqrt{k^2 + 4}$$

(9)

Proof. In [3, 4] it is proven \( \lim_{n \to \infty} \frac{F_{k,n}}{F_{k,a}} = \sigma_i \). Then \( \lim_{n \to \infty} \frac{F_{k,n}}{F_{k,a}} + \frac{F_{k,n+1}}{F_{k,a}} = \sigma_i + \frac{1}{\sigma_i} = \sigma_1 - \sigma_2 = \sqrt{k^2 + 4} \)

Consequently

$$\sum_{j=0}^{k} \frac{1}{F_{k,2p+1}(2n+1)+a} + F_{k,2p+1} = \frac{1}{2L_{k,2p+1}} \left( \sqrt{k^2 + 4} - (k^2 + 4) \frac{F_{k,2r}}{L_{k,2r}} \right)$$

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3.3 Corollary 1.
In this Corollary, we will use the equation (7) or the (6), as appropriate.
Giving integer values to \( r, p, \) and \( k \), these formulas are particularized as follows:

1. \( r = 0: \) \[ \sum_{j=0}^{2n} \frac{1}{F_{1,2j+1} + F_{1,2j+1} + 1} = \frac{L_{k,2p+1}(2k+1)}{2L_{k,2p+1}(2k+1)} \]
   \[ \sum_{j=0}^{2n} \frac{1}{F_{1,2j+1} + 1} = \frac{\sqrt{k^2 + 4}}{2k} \]
   \[ k = 1: \sum_{j=0}^{n} \frac{1}{F_{1,2j+1} + 1} = \frac{\sqrt{2}}{2} \quad [7] \]
   \[ k = 2: \sum_{j=0}^{n} \frac{1}{P_{1,2j+1} + 1} = \frac{\sqrt{10}}{2} \]

2. \( r = 1: \) \[ \sum_{j=0}^{2n} \frac{1}{F_{1,2p+1}(2k+1) + F_{1,2p+1} + 1} = \frac{L_{k,2p+1}(2k+1)}{2L_{k,2p+1}(2k+1)} \]
   \[ \sum_{j=0}^{2n} \frac{1}{F_{1,2p+1} + 1} = \frac{\sqrt{k^2 + 4} - (k^2 + 4)k}{k^2 + 2} \]
   \[ k = 1: \sum_{j=0}^{n} \frac{1}{F_{1,2p+1} + 1} = \frac{1}{2} \left( \sqrt{5} - \frac{5}{3} \right) \]
   \[ k = 2: \sum_{j=0}^{n} \frac{1}{P_{1,2p+1} + 1} = \frac{1}{2} \left( \sqrt{8} - \frac{8}{3} \right) \]

3.4 Corollary 2.
If \( S_n(k, r, p) = \sum_{j=0}^{n} \frac{1}{F_{k,(2j+1)(2j+1)+p} + F_{k,2j+1}} \), then \( S_n(k, r, p + h) = \frac{L_{k,2p+1}}{L_{k,2p+1}+1} S_n(k, r, p) \)
For instance: from Corollary 1, \( S_n(2, 1, 1) = \sum_{j=0}^{n} \frac{1}{P_{1,2j+1}+1} = \frac{1}{14} \left( \sqrt{5} - \frac{4}{3} \right) \) therefore
\[ S_n(2, 1, 4) = \frac{L_{2,3}}{L_{2,9}} S_n(2, 1, 1) = \sum_{j=0}^{n} \frac{1}{P_{1,2j+1}+1} + 985 = \frac{1}{2786} \left( \sqrt{5} - \frac{4}{3} \right) \]
3.5 Corollary 3.

If \( p = 0 \) then
\[
\sum_{j=0}^{n} \frac{1}{F_{2j+1} + 1} = \frac{1}{2k} \sqrt{k^2 + 4 - (k^2 + 4)} \left( \frac{F_{k,2r}}{L_{2,2r}} \right)
\]
and
\[
\sum_{j=0}^{n} \frac{1}{F_{2j+1+4r} + 1} = \frac{1}{2k} \sqrt{k^2 + 4 - (k^2 + 4)} \left( \frac{F_{k,2(r+4)}}{L_{2,2(r+4)}} \right).
\]

So
\[
\sum_{j=0}^{2n} \left( \frac{1}{F_{2j+1} + 1} - \frac{1}{F_{2j+1+4r} + 1} \right) = \frac{k^2 + 4}{2k} \left( \frac{F_{k,2(r+4)}}{L_{2,2(r+4)}} - \frac{F_{k,2r}}{L_{2,2r}} \right)
\]
and then
\[
\sum_{j=0}^{2n} \frac{1}{F_{2j+1} + 1} = \frac{k^2 + 4}{2k} \left( \frac{F_{k,2(r+4)}}{L_{2,2(r+4)}} - \frac{F_{k,2r}}{L_{2,2r}} \right)
\]
Finally, applying Lemma 5 with \( a = 2r \) and \( c = n \), it is
\[
\sum_{j=0}^{2n} \frac{1}{F_{2j+1} + 1} = \frac{k^2 + 4}{2k} \frac{F_{2n}}{L_{2,2(n)}} + 1
\]
In particular, for \( k = 1 \) and \( r = 0 \) it is
\[
\sum_{j=0}^{2n} \frac{1}{F_{2j+1} + 1} = \frac{5F_{2n}}{2L_{2n}}
\]

In this form we have found the formula for the sum of an odd number of terms of the indicated form, while from the formula (1) (a) of Corollary 1 it is
\[
\sum_{j=0}^{2n} \frac{1}{F_{2j+1} + 1} = \frac{L_{2n+1}}{2F_{2n+1}}
\]
From the difference of these last two formulas and applying the Binet formulas, we get
\[
L_{2n}L_{2n+1} = F_{4n+1} + 1
\]

References