On the Diophantine equation $5(x^2 + y^2) - 9xy = 35z^2$

M.A.Gopalan 1, Nirmala Rebecca Paul 2, P. Suganya 3

1Professor, Department of Mathematics, Shrimati Inddira Gandhi College, Trichy, Tamil Nadu, India
2Department of Mathematics, Lady Doak College, Madurai, Tamil Nadu, India
3Department of Mathematics, Lady Doak College, Madurai, Tamil Nadu, India

Abstract: The ternary quadratic Diophantine equation $5(x^2 + y^2) - 9xy = 35z^2$ representing cone is analyzed for its non-zero distinct integer points on it.

Keywords: Ternary quadratic, integral solutions, homogeneous cone.

Mathematics Subject classification: 11D09

I. Introduction

The ternary quadratic Diophantine equations offer an unlimited field for research by reason of their variety [1,2,3]. In particular, one may refer [4-12] for finding points in integers on some specific three dimensional surfaces. This communication concern with yet another ternary quadratic Diophantine equation $5(x^2 + y^2) - 9xy = 35z^2$ representing cone for determining its infinitely many integer solutions.

II. Method of Analysis

Consider the equation

$$5(x^2 + y^2) - 9xy = 35z^2$$

(1)

The transformed equation of (1) after using the linear transformations

$$x = u + v, \quad y = u - v(u \neq v \neq 0)$$

is

$$u^2 + 19v^2 = 35z^2$$

(3)

The above equation is solved through different methods and employing (2), different sets of distinct integer solutions to (1) are obtained which are illustrated below:

Method: 1

Write 35 as $35 = (4 + i\sqrt{19})(4 - i\sqrt{19})$ (4)

Assume $z = a^2 + 19b^2$ (5)

where a and b are non zero distinct integers

Using (4) & (5) in (3) and employing the method of factorization, define

$$u + i\sqrt{19}v = (4 + i\sqrt{19})(a + i\sqrt{19}b)^2$$

from which, on equating the real and imaginary parts

$$u = 4(a^2 - 19b^2) - 38ab$$

$$v = (a^2 - 19b^2) + 8ab$$

Substituting the above values of u and v in (2), the values of x and y are given by

$$x = 5(a^2 - 19b^2) - 30ab$$

(6)

$$y = 3(a^2 - 19b^2) - 46ab$$

(7)

Thus, (5), (6) and (7) represent non zero distinct integer solutions to (1) in two parameters.

Note: In addition to (4), one may write 35 as $35 = \frac{(11 + i\sqrt{19})(11 - i\sqrt{19})}{4}$

For this choice, the corresponding integer solutions to (1) are given by

$$x = 6(a^2 - 19b^2) - 8ab$$

$$y = 5(a^2 - 19b^2) - 30ab$$

$$z = a^2 + 19b^2$$
On the Diophantine equation \( 5(x^2 + y^2) - 9xy = 35z^2 \)

Method: 2

Consider (3) as \( u^2 - 16z^2 = 19(z^2 - v^2) \) \hspace{1cm} (8)

Write (8) in the form of ratio as

\[
\frac{u + 4z}{z - v} = \frac{19(z + v)}{u - 4z} = \frac{a}{b}, \quad b > 0
\]

Which is equivalent to the system of double equations

\[
(a - 4b)z - av - bu = 0 \]
\[
(-4a - 19b)z - 19bv + au = 0
\]

Applying the method of cross multiplication to the above equations, we have

\[
u = (a^2 - 19b^2) - 8ab
\]
\[
z = a^2 + 19b^2
\]

Substituting the above values of \( u \) and \( v \) in (2), the values of \( x \) and \( y \) are given by

\[
x = 5(a^2 - 19b^2) + 30ab
\]
\[
y = 3(a^2 - 19b^2) + 46ab
\]

Thus, (9) and (10) represent non zero distinct integer solutions to (1) in two parameters.

Note: (8) can also be expressed in the form of ratio in three different ways as follows:

(i) \[
\frac{u + 4z}{19(z - v)} = \frac{(z + v)}{u - 4z} = \frac{a}{b}, \quad b > 0
\]

(ii) \[
\frac{u + 4z}{19(z + v)} = \frac{(z - v)}{u - 4z} = \frac{a}{b}, \quad b > 0
\]

(iii) \[
\frac{u + 4z}{z + v} = \frac{19(z - v)}{u - 4z} = \frac{a}{b}, \quad b > 0
\]

Repeating the analysis as above, we get three different sets of integer solutions to (1) and they are presented below:

Solutions of (i):

\[
x = 95a^2 - 5b^2 + 30ab
\]
\[
y = 57a^2 - 3b^2 + 46ab
\]
\[
z = 19a^2 + b^2
\]

Solutions of (ii):

\[
x = -57a^2 + 3b^2 - 46ab
\]
\[
y = -95a^2 + 5b^2 - 30ab
\]
\[
z = -a^2 - 19b^2
\]

Solutions of (iii):

\[
x = -3a^2 + 57b^2 - 46ab
\]
\[
y = -5a^2 + 95b^2 - 30ab
\]
\[
z = -a^2 - 19b^2
\]

Method: 3

Write (3) as \( 19v^2 = 35z^2 - u^2 \) \hspace{1cm} (11)

Write 19 as \( 19 = (\sqrt{35} + 4)(\sqrt{35} - 4) \) \hspace{1cm} (12)

Assume \( v = 35a^2 - b^2 \) \hspace{1cm} (13)

Where \( a \) and \( b \) are non zero distinct integers

Using (12) & (13) in (11) and employing the method of factorization, define
On the Diophantine equation \( 5(x^2 + y^2) - 9xy = 35z^2 \)

\[ \sqrt{35}z + u = (\sqrt{35} + 4)(35a + b)^2 \]

Equating the rational and irrational parts, we get

\begin{align*}
u &= 4(35a^2 + b^2) + 70ab \\
z &= (35a^2 + b^2) + 8ab
\end{align*}

(14)

Substituting the above values of \( u \) and \( v \) in (2), the values of \( x \) and \( y \) are obtained as

\begin{align*}
x &= 175a^2 + 3b^2 + 70ab \\
y &= 105a^2 + 5b^2 + 70ab
\end{align*}

(15)

Thus, (14) and (15) represent the integer solutions of (1).

**Method: 3**

Introducing the linear transformations

\[ z = \alpha \pm 19\beta, \quad v = \alpha \pm 35\beta, \quad u = 4U \]

in (3), it leads to

\[ \alpha^2 = U^2 + 665\beta^2 \]

(16)

which is satisfied by

\[ \beta = 2pq, U = 665p^2 - q^2, \quad \alpha = 665p^2 + q^2 \]

Substituting the above values of \( \alpha, \beta, U \) in (16) and (2), the corresponding non-zero integer solutions to (1) are given by

\begin{align*}
x &= 3325p^2 - 3q^2 \pm 70pq \\
y &= 1995p^2 - 5q^2 \pm 70pq \\
z &= 665p^2 + q^2 \pm 38pq
\end{align*}

It is worth to mention here that, (17) may be expressed as the system of double equations as shown in the table below:

**Table 1: system of equations**

<table>
<thead>
<tr>
<th>system</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha + U )</td>
<td>( \beta^2 )</td>
<td>5( \beta^2 )</td>
<td>7( \beta^2 )</td>
<td>19( \beta^2 )</td>
<td>35( \beta^2 )</td>
<td>95( \beta^2 )</td>
<td>133,( \beta^2 )</td>
<td>665( \beta^2 )</td>
<td>35( \beta )</td>
<td>95( \beta )</td>
<td>133( \beta )</td>
<td>665( \beta )</td>
</tr>
<tr>
<td>( \alpha - U )</td>
<td>665</td>
<td>133</td>
<td>95</td>
<td>35</td>
<td>19</td>
<td>7</td>
<td>5</td>
<td>1</td>
<td>19( \beta )</td>
<td>7( \beta )</td>
<td>5( \beta )</td>
<td>( \beta )</td>
</tr>
</tbody>
</table>

Solving each of the above system for \( \alpha, \beta, U \) and using (16) and (2), the corresponding non-zero integer solutions satisfying (1) are exhibited in the table below:

**Table 2: integer solutions**

<table>
<thead>
<tr>
<th>system</th>
<th>( x )</th>
<th>( y )</th>
<th>( z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10( k^2 + 80k - 960 ) ( k^2 - 64k - 1696 ) ( 2k^2 + 40k + 352 ) ( 2k^2 - 36k + 314 )</td>
<td>6( k^2 + 76k - 1626 )</td>
<td>2( k^2 + 48k + 88 )</td>
</tr>
<tr>
<td>2</td>
<td>50( k^2 + 120k - 152 ) ( 30k^2 - 40k - 360 ) ( 10k^2 + 48k + 88 ) ( 10k^2 - 28k + 50 )</td>
<td>50( k^2 - 20k - 222 ) ( 30k^2 + 100k - 290 )</td>
<td>14( k^2 + 52k + 70 )</td>
</tr>
<tr>
<td>3</td>
<td>70( k^2 + 140k - 90 ) ( 42k^2 - 28k - 262 ) ( 14k^2 - 24k + 32 )</td>
<td>70( k^2 - 160 ) ( 42k^2 + 112k - 192 )</td>
<td>14( k^2 + 52k + 70 )</td>
</tr>
<tr>
<td>4</td>
<td>190( k^2 + 260k + 30 ) ( 114k^2 + 44k - 94 ) ( 38k^2 + 76k + 46 ) ( 38k^2 + 8 )</td>
<td>190( k^2 + 120k - 40 ) ( 114k^2 + 184k - 24 )</td>
<td>38k^2 + 8</td>
</tr>
<tr>
<td>5</td>
<td>350( k^2 + 420k + 30 ) ( 210k^2 + 140k - 94 ) ( 70k^2 + 108k + 46 ) ( 70k^2 + 32k + 8 )</td>
<td>350( k^2 + 280k - 40 ) ( 210k^2 + 280k - 24 )</td>
<td>70k^2 + 32k + 8</td>
</tr>
<tr>
<td>6</td>
<td>950( k^2 + 1020k + 262 ) ( 570k^2 + 500k + 90 ) ( 190k^2 + 228k + 70 ) ( 190k^2 + 152k + 32 )</td>
<td>950( k^2 + 880 + 192 ) ( 570k^2 + 640k + 160 )</td>
<td>190k^2 + 228k + 70</td>
</tr>
</tbody>
</table>
On the Diophantine equation \(5(x^2 + y^2) - 9xy = 35z^2\)

<table>
<thead>
<tr>
<th>Method: 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider (3) as (u^2 + 19v^2 = 35z^2 + 1)</td>
</tr>
<tr>
<td>Write 1 as (1 = (\frac{5 + i3\sqrt{19}}{14}(5 - i3\sqrt{19})))</td>
</tr>
<tr>
<td>Using (4), (5) and (19) in (18) and employing the method of factorization, define</td>
</tr>
<tr>
<td>(u + i\sqrt{19}v = (4 + i\sqrt{19})(a + i\sqrt{19b})^2 \frac{(5 + i3\sqrt{19})}{14})</td>
</tr>
<tr>
<td>Equating the real and imaginary parts, we have</td>
</tr>
<tr>
<td>(u = \frac{1}{14}[-37(a^2 - 19b^2) - 646ab])</td>
</tr>
<tr>
<td>(v = \frac{1}{14}[17(a^2 - 19b^2) - 74ab])</td>
</tr>
<tr>
<td>Substituting the above values of (u) and (v) in (2), the values of (x) and (y) are given by</td>
</tr>
<tr>
<td>(x = \frac{1}{7}[10(a^2 - 19b^2) + 360ab])</td>
</tr>
<tr>
<td>(y = \frac{1}{7}[27(a^2 - 19b^2) + 286ab])</td>
</tr>
</tbody>
</table>

Replacing \(a\) by \(7A\) and \(b\) by \(7B\) in (20) and (5), the corresponding non-zero integer solutions to (1) are given by

\[ x = -[70(A^2 - 19B^2) + 2520AB] \]
\[ y = -[189(A^2 - 19B^2) + 2002AB] \]
\[ z = 49(A^2 + 19B^2) \]

**Note:** In addition to (19), one may write 1 as \(1 = (\frac{3 + i5\sqrt{19}}{484}(3 - i5\sqrt{19}))\)

For this choice, a different set of solutions to (1) are obtained.

**III. Generation of solutions**

**Illustration 1:**

Let \((x_0, y_0, z_0)\) be the given initial solution of (1). \(x_i = 10x_{i-1} - 3h, y_i = 10y_{i-1}, z_i = 10z_{i-1} + h\) \(21\) be the second solution of (1) where \(h\) is any non-zero integer to be determined.

Substituting (21) in (1) and simplifying, we have \(h = 30x_0 - 27y_0 + 70z_0\)

Therefore, the second solution \((x_1, y_1, z_1)\) of (1) expressed in the matrix form is

\[
\begin{pmatrix} x_1, y_1, z_1 \end{pmatrix} = M \begin{pmatrix} x_0, y_0, z_0 \end{pmatrix} \text{ where } M = \begin{pmatrix} -80 & -210 & 0 \\ -30 & -27 & 80 \end{pmatrix}
\]

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The repletion of the above process leads to the general solution of (1) represented as follows:

\[
\begin{align*}
(x_{2n-1}, y_{2n-1}, z_{2n-1}) &= 10^{2(n-1)} M(x_n, y_n, z_n) \\
(x_{2n}, y_{2n}, z_{2n}) &= 10^n M(x_0, y_0, z_0)
\end{align*}
\]

**Illustration 2:**

Let \((u_0, v_0, z_0)\) be the given initial solution of (3).

Let \(u_1 = 6h - u_0\), \(v_1 = v_0\), \(z_1 = z_0 + h\)

be the second solution of (3) where \(h\) is any non-zero integer to be determined.

Substituting (22) in (3) and simplifying, we get \(h = 12u_0 + 70z_0\)

Therefore, the second solution \((x_1, y_1, z_1)\) of (3) expressed in the matrix form is

\[
(u_1, z_1) = M(u_0, z_0)' \quad \text{where } M = \begin{pmatrix} 71 & 420 \\ 12 & 71 \end{pmatrix}
\]

Repeating the above process, we have, in general

\[
(u_n, z_n)' = M^n(u_0, z_0)' \quad \text{where } M = \frac{1}{\alpha - \beta} \begin{pmatrix} \alpha^n + \beta^n & \sqrt{35}((\alpha^n - \beta^n) \\ \frac{2}{\sqrt{35}} \frac{\alpha^n - \beta^n}{2} & \frac{2}{\sqrt{35}} \frac{\alpha^n + \beta^n}{2} \end{pmatrix}
\]

In view of (2), the general solution \((x_n, y_n, z_n)\) of (1) is given by

\[
\begin{align*}
x_n &= \frac{1}{2\sqrt{35}}((\alpha^n - \beta^n)u_0 + \frac{1}{2}(\alpha^n + \beta^n)z_0) \\
y_n &= \frac{1}{2\sqrt{35}}((\alpha^n + \beta^n)u_0 + \frac{1}{2}(\alpha^n - \beta^n)z_0)
\end{align*}
\]

**Illustration 3:**

Let \(u_1 = 8u_0\), \(v_1 = 8v_0 + h\), \(z_1 = h - 8z_0\) be the second solution of (3).

Following the analysis presented above, the corresponding integer solutions to (1) are given by

\[
\begin{align*}
x_n &= 8^n u_0 + \frac{\alpha^n + \beta^n}{2} v_0 + \frac{\sqrt{35}}{2\sqrt{19}}((\alpha^n - \beta^n)z_0) \\
y_n &= 8^n u_0 - \frac{\alpha^n + \beta^n}{2} v_0 - \frac{\sqrt{35}}{2\sqrt{19}}((\alpha^n - \beta^n)z_0)
\end{align*}
\]
On the Diophantine equation $5(x^2 + y^2) - 9xy = 35z^2$

$$z_n = \frac{\sqrt{19}}{2\sqrt{35}} (\alpha^n - \beta^n)v_0 + \frac{1}{2}(\alpha^n + \beta^n)z_0$$

where $\alpha = 27 + \sqrt{665}, \beta = 27 - \sqrt{665}$

IV. Conclusion

To conclude, one may search for other patterns of general solutions to ternary quadratic Diophantine equation in the title and obtain their corresponding properties.

References