Continued fraction expansion of the relative operator entropy and the Ts all is relative entropy

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Abstract: The aim of this paper is to provide some results and applications of continued fractions with matrix arguments. First, we recall some properties of matrix functions with real coefficients. Afterwards, we give a continued fraction expansion of the relative operator entropy and for the Ts all is relative operator entropy. At the end, we study some metrical equations.

Keywords: Continued fraction expansion, positive definite matrix, relative operator entropy, Ts all is relative operator entropy.

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I. Introduction and Motivation

Over the last two hundred years, the theory of continued fractions has been a topic of extensive study. The basic idea of this theory over real numbers is to give an approximation of various real numbers by the rational ones. One of the main reasons why continued fractions are so useful in computation is that they often provide representation for transcendental functions that are much more generally valid than the classical representation by, say, the power series. Further; in the convergent case, the continued fractions expansions have the advantage that they converge more rapidly than other numerical algorithms. Recently, the extension of continued fractions theory from real numbers to the matrix case has seen several developments and interesting applications (see [6],[8],[13]). The real case is relatively well studied in the literature. However, in contrast to the theoretical importance, one can find in mathematical literature only a few results on the continued fractions with matrix arguments. There have been some reasons why all this attention has been devoted to what is, in essence, a very humble idea. Since calculations involving matrix valued functions with matrix arguments are feasible with large computers, it will be an interesting attempt to develop such matrix theory.

The main difficulty arises from the fact that the algebra of square matrices is not commutative.

In 1850, Clausius, introduced the notion of entropy in thermodynamics. Since then several extensions and reformulations have been developed in various disciplines [11,12,14,15]. There have been investigated the so-called entropy inequalities by some mathematicians, see [2,3,10] and references therein. A relative operator entropy of strictly positive operators $A$, $B$ was introduced in noncommutative information theory by Fujii and Kamei [9] by

$$S(A|B) = A^{1/2} \ln(A^{-1/2}BA^{-1/2})A^{1/2},$$

as a generalization of the operator entropy

$$H(A) = S(A|I) = -A \ln A.$$

In the present paper, we also study a parametric extension of the relative operator entropy which is called Tsallis relative operator entropy. It is firstly introduced in [18] in the following manner.

Definition 1.1 For two invertible positive operators $A$ and $B$ on Hilbert space, and any real number $\lambda \in [0,1]$, the Tsallis relative operator entropy is defined by

$$T_\lambda(A|B) = \frac{A^{1/2}(A^{-1/2}BA^{-1/2})^\lambda A^{1/2} - A}{\lambda}.$$

For simplicity and clearness, we restrict ourselves to positive definite matrices, but our results can be, without special difficulties, projected to the case of positive definite operators from an infinite dimensional Hilbert space into itself.
This article is organized as follows: The section 2 contains some basic notions and results about matrix continued fractions that are needed later. In Section 3, we give a continued fractions expansion of the relative operator entropy and the Tsallis relative operator entropy. However, the last result of this paper is devoted to provide the solution of a matrix algebraic equation.

II. Preliminary an notations

The functions of matrix arguments play a widespread role in science and engineering, with applications areas ranging from nuclear magnetic resonance [1]. So for any scalar polynomial \( p(z) = \sum_{i=0}^{k} \alpha_i z^i \) gives rise to a matrix polynomial with scalar coefficients by simply substituting \( A \) for \( z \):

\[
P(A) = \sum_{i=0}^{k} \alpha_i A^i
\]

More generally, for a function \( f \) with a series representation on an open disk containing the eigenvalues of \( A \), we are able to define the matrix function \( f(A) \) via the Taylor series for \( f \) [7].

Alternatively, given a function \( f \) that is analytic inside a closed contour \( \Gamma \) which encloses the eigenvalues of \( A \), \( f(A) \) can be defined, by analogy with Cauchy’s integral theorem by

\[
f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(zI - A)^{-1} dz.
\]

The definition is known as the matrix version of Cauchy’s integral theorem. Let \( \mathcal{M}_m \) be the algebra of real square matrices, we now mention an important result of matrix functions.

Lemma 2.1 Let \( f \) be an analytic function in a domain \( D \).

(i) If two matrices \( A \in \mathcal{M}_m \) and \( B \in \mathcal{M}_m \) are similar, with \( A = ZBZ^{-1} \), and \( \text{sp}(A) \subseteq D \), then the matrices \( f(A) \) and \( f(B) \) are also similar, with \( f(A) = Zf(B)Z^{-1} \).

(ii) If \( A \in \mathcal{M}_m \) is a block diagonal matrix \( A = \text{diag}(A_1, A_2, \ldots, A_r) \) then \( f(A) = \text{diag}(f(A_1), f(A_2), \ldots, f(A_r)) \).

Proof. Its proof is obvious.

Let \( A \in \mathcal{M}_m \), \( A \) is said to be positive semidefinite (resp. positive definite) if \( A \) is symmetric and

\[
\forall x \in \mathbb{R}^m, \quad <Ax, x> \geq 0 \quad \text{(resp. } \forall x \in \mathbb{R}^m, \quad x \neq 0 \quad <Ax, x> > 0)\]
where $< , , >$ denotes the standard scalar product of $\mathbb{R}^m$.

We observe that positive semidefiniteness induces a partial ordering on the space of symmetric matrices: if $A$ and $B$ are two symmetric matrices, we write $A \preceq B$ if $B - A$ is positive semidefinite.

Henceforth, whenever we say that $A \in \mathcal{M}_m$ is positive semidefinite (or positive definite), it will be assumed that $A$ is symmetric.

For any matrices $A, B \in \mathcal{M}_m$ with $B$ invertible, we write $A/B = B^{-1}A$, in particular, if $A = I$, the matrix identity, then $I/B = B^{-1}$. It is easy to verify that for any invertible matrix $X$ we have

$$\frac{A}{B} = \frac{XA}{XB} \neq \frac{AX}{BX}.$$

**Definition 2.2** Let $\{A_n\}_{n \geq 0}$ and $\{B_n\}_{n \geq 1}$ be two sequences of matrices in $\mathcal{M}_m$. We denote the continued fraction expansion by

$$A_0 + \frac{B_1}{A_1 + \frac{B_2}{A_2 + \ldots}} := \left[ A_0; \frac{B_1}{A_1}, \ldots, \frac{B_n}{A_n} \right].$$

Sometimes, we denote this continued fraction by $\left[ A_0; \frac{B_n}{A_n} \right]_{n=1}^{+\infty}$ or $A_0 + K(B_n/A_n)$.

The fractions $\frac{B_n}{A_n}$ and $\frac{P_n}{Q_n} := \left[ A_0; \frac{B_1}{A_1}, \ldots, \frac{B_n}{A_n} \right]$ are called, respectively, the $n^{th}$ partial quotient and the $n^{th}$ convergent of the continued fraction $A_0 + K(B_n/A_n)$.

We note that the evaluation of $n^{th}$ convergent according to the definition 2.1 is not practical because we have to repeatedly inverse matrices. The following proposition gives an adequate method to calculate $A_0 + K(B_n/A_n)$.

**Proposition 2.3** [16]. For the continued fraction $A_0 + K(B_n/A_n)$, define

$$\begin{cases}
P_{-1} = I, & P_0 = A_0 \\
Q_{-1} = 0, & Q_0 = I
\end{cases}
\quad \text{and} \quad
\begin{cases}
P_n = A_n P_{n-1} + B_n P_{n-2} \\
Q_n = A_n Q_{n-1} + B_n Q_{n-2}
\end{cases}
\quad n \geq 1. \quad (2.1)

Then $Q_n^{-1}P_n$ is the $n^{th}$ convergent of the continued fraction $A_0 + K(B_n/A_n)$.

The proof of the next proposition is elementary and we left it to the reader.

**Proposition 2.4** For any two matrices $C$ and $D$ with $C$ invertible, we have

$$C \left[ A_0; \frac{B_k}{A_k} \right]_{k=1}^{n} D = \left[ C A_0 D; \frac{B_1 D}{A_1 C^{-1}}, \ldots, \frac{B_k D}{A_k C^{-1}} \right]_{k=3}^{n}.$$
Definition 2.5 Let \( \{A_n\}, \{B_n\}, \{C_n\} \) and \( \{D_n\} \) be four sequences of matrices. We say that the continued fractions \( A_0 + K(B_n/A_n) \) and \( C_0 + K(D_n/C_n) \) are equivalent if we have \( F_n = G_n \) for all \( n \geq 1 \), where \( F_n \) and \( G_n \) are the \( n^{th} \) convergents of \( A_0 + K(B_n/A_n) \) and \( C_0 + K(D_n/C_n) \) respectively.

In order to simplify the statements on some partial quotients of continued fractions with matrices arguments, we need the following proposition which is an example of equivalent continued fractions.

Proposition 2.6 Let \( \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty} \) be a given continued fraction. Then

\[
\frac{P_n}{Q_n} = \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{n} = \left[A_0; \frac{X_kB_kX_{k-1}^{-1}}{X_kA_kX_{k-1}^{-1}}\right]_{k=1}^{n},
\]

where \( X_{-1} = X_0 = I \) and \( X_1, X_2, ..., X_n \) are arbitrary invertible matrices.

Proof. Let \( \frac{P_n}{Q_n} \) and \( \frac{\tilde{P}_n}{Q_n} \) be the \( n^{th} \) convergents of the continued fractions \( \left[A_0; \frac{B_k}{A_k}\right]_{k=1}^{+\infty} \) and \( \left[A_0; \frac{X_kB_kX_{k-1}^{-1}}{X_kA_kX_{k-1}^{-1}}\right]_{k=1}^{+\infty} \) respectively. By proposition 2, for all \( n \geq 1 \), we can write

\[
\tilde{P}_n = X_nA_nX_{n-1}^{-1}\tilde{P}_{n-1} + X_nB_nX_{n-2}^{-1}\tilde{P}_{n-2},
\]

which is equivalent to

\[
X_n^{-1}\tilde{P}_n = A_n(X_{n-1}^{-1}\tilde{P}_{n-1}) + B_n(X_{n-2}^{-1}\tilde{P}_{n-2}).
\]

This last result joined to the initial conditions prove that for all \( n \geq 1 \),
\[
X_n^{-1}\tilde{P}_n = P_n.
\]

A similar result can be obtained for \( Q_n \). Consequently, both continued fractions have the same convergents and the proof of proposition 2.6 follows. We also recall the following proposition in real case.

Proposition 2.7 Let \( (r_n) \) be a non-zero sequence of real numbers. We prove easily that the following continued fractions

\[
\left[a_0; \frac{b_1}{a_1}, \frac{b_2}{a_2}, ..., \frac{b_n}{a_n}, ...\right] \quad \text{and} \quad \left[a_0; \frac{r_1b_1}{r_1a_1}, \frac{r_2b_2}{r_2a_2}, ..., \frac{r_{n-1}r_nb_n}{r_na_n}, ...\right]
\]

are equivalent.
Definition 2.8 (Contraction of a continued fraction)
Let $B_n, A_n$ and $f_n$ denote the $n^{th}$ numerator, denominator and approximant, respectively of a continued fraction $a_0 + K(b_n/a_n)$ and we let $D_n, C_n$ and $g_n$ denote the $n^{th}$ numerator, denominator and approximant, respectively of a continued fraction $c_0 + K(d_n/c_n)$. Then $c_0 + K(d_n/c_n)$ is called an even contraction or even part of $a_0 + K(b_n/a_n)$ if and only if

$$g_n = f_{2n} \text{ for all } n \geq 1.$$

Proposition 2.9 [13] i) The even canonical contraction of $a_0 + K(b_n/a_n)$ is given by

$$[c_0; \frac{d_1}{c_1}; \frac{d_2}{c_2}; \frac{d_n}{c_n}]_{n=3}^{+\infty} =$$

$$[a_0; \frac{b_1a_2}{a_1a_2 + b_2}; \frac{-b_3b_4a_4}{(a_2a_3 + b_3)a_4 + a_2b_4}; \frac{-b_{2n-2}b_{2n-1}a_{2n-4}a_{2n}}{(a_{2n-2}a_{2n-1} + b_{2n-1})a_{2n} + a_{2n-2}b_{2n}}]_{n=3}^{+\infty}.$$

ii) The odd canonical contraction of $a_0 + K(b_n/a_n)$ is given by

$$[c_0; \frac{d_1}{c_1}; \frac{d_2}{c_2}; \frac{d_n}{c_n}]_{n=3}^{+\infty} =$$

$$[a_0a_1 + b_1; \frac{-b_1b_2a_3/a_1}{a_1b_2a_3 + a_1(b_3 + a_2a_3)}; \frac{-b_{2n-1}b_{2n}a_{2n+1}a_{2n-3}}{b_{2n}a_{2n+1} + a_{2n-1}(b_{2n+1} + a_{2n}a_{2n+1})}]_{n=2}^{+\infty}.$$

We end this section by introducing some topological notions of continued fractions with matrix arguments. We provide $\mathcal{M}_m$ with the topology induced by the following classical norm:

$$\forall A \in \mathcal{M}_m, \|A\| = \sup_{x \neq 0} \frac{|Ax|}{|x|} = \sup_{|x|=1} |Ax|.$$

The continued fraction $\left[ \frac{A_0}{A_k} \right]_{k=1}^{+\infty}$ is said to be convergent in $\mathcal{M}_m$ if the sequence $(F_n) = (P_n/Q_n)$ converges in $\mathcal{M}_m$ in the sense that there exists a matrix $F \in \mathcal{M}_m$ such that

$$\lim_{n \to +\infty} \|F_n - F\| = 0.$$

III. Main Result

Our aim in this section is to give the continued fraction expansions of the relative operator entropy and of the Tsallis relative operator entropy for two invertible and positive definite matrices $A$ and $B$.

3.1 Continued fractions expansion of a relative operator entropy.

For simplicity, we start with the real case and we begin by recalling La- guerre’s continued fraction of $\ln x$; where $x$ is a strictly positive real number in the following lemma.
Lemma 3.1.1 Let $x$ be a real number such that $x > 0$. A continued fraction expansion of $\ln x$ is given by:

$$\ln x = \left[ 0; \frac{2}{x+1}, -\frac{2}{5}, -\frac{n^2}{2n+1} \right]_{n=3}^{+\infty}.$$  \hfill (3.1)

Proof of lemma 3.1.1 Let $z$ be a real number such $|z| < 1$. We know that (see [13]) a continued fraction expansion of $\ln(1 + z)$ is

$$\ln(1 + z) = \left[ 0; z, -\frac{z}{1}, -\frac{z}{1} \right]_{n=2}^{+\infty},$$

where for $k \geq 1$

$$a_{2k} = \frac{k}{2(2k - 1)}, \quad a_{2k+1} = \frac{k}{2(2k + 1)}.$$  \hfill (3.2)

The even canonical contraction of a previous continued fraction expansion of $\ln(1 + z)$ below is given by

$$\ln(1 + z) = \left[ 0; \frac{2z}{2 + z}, -\frac{1}{3}, -\frac{n^2 z^2}{(2n + 1)(2 + z)} \right]_{n=2}^{+\infty}.$$  \hfill (3.3)

Let $z$ be a real number such that $|z| < 1$, according to the relationship (3.2), we have

$$\ln \left( \frac{1 + z}{1 - z} \right) = \left[ 0; \frac{2z}{1}, -\frac{1}{3}, -\frac{n^2 z^2}{2n + 1} \right]_{n=2}^{+\infty}.$$  \hfill (3.3)

Let $x$ be a real number such that $x > 0$, in order to conclude the proof it suffices to put $z = \frac{x-1}{x+1}$ which is equivalent to $x = \frac{1+z}{1-z}$.

The next lemma is a matrix version of the previous lemma 3.1.1.

Lemma 3.1.2 Let $A \in \mathcal{M}_m$ be a positive definite matrix. Then a continued fraction expansion of $\ln(A)$ is

$$\ln(A) = \left[ 0; \frac{2(\frac{A-I}{A+I})}{I}, -\frac{(\frac{A-I}{A+I})^2}{3I}, -\frac{(\frac{A-I}{A+I})^2}{5I}, -\frac{n^2 (\frac{A-I}{A+I})^2}{(2n + 1)I} \right]_{n=3}^{+\infty}.$$  \hfill (3.4)

Now we establish a main theorem which gives a continued fraction expansions of the relative operator entropy $S(A|B)$. 
Theorem 3.1.3 Let $A$ and $B$ be two invertible and positive definite matrices in $\mathcal{M}_m$. A continued fraction expansion of the relative operator entropy $S(A|B)$ is given by

$$S(A|B) = \left[ 0; \frac{2A(\frac{B-A}{B+A})}{I}, \frac{-A(\frac{B-A}{B+A})^2 A^{-1}}{3I}, \frac{-2A(\frac{B-A}{B+A})^2 A^{-1}}{5I}, \frac{-n^2 A(\frac{B-A}{B+A})^2 A^{-1}}{(2n+1)I} \right]_{n=3}^{+\infty}.$$  

Proof of Lemma 3.1.2 Let $A \in \mathcal{M}_m$ be a positive definite matrix. Then there exists an invertible matrix $X$ such that $A = XDX^{-1}$, where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\lambda_i > 0$.

As the function $z \to \ln(z)$ is analytic in the open halfplane $\{z \in \mathbb{C}, \ Re(z) > 0\}$, then

$$\ln(A) = X \ln(D) X^{-1} = X \text{diag}(\ln(\lambda_1), \ln(\lambda_2), \ldots, \ln(\lambda_m)) X^{-1}.$$  

Let us define the sequences $\{P_n\}$ and $\{Q_n\}$ by:

$$\begin{align*}
P_{-1} &= I, \quad P_0 = 0, \quad P_1 = 2\phi(D), \\
Q_{-1} &= 0, \quad Q_0 = I, \quad Q_1 = I
\end{align*}$$

and for $n \geq 2$,

$$\begin{align*}
P_n &= (2n+1)P_{n-1} - n^2(\phi(D))^2P_{n-2}, \\
Q_n &= (2n+1)Q_{n-1} - n^2(\phi(D))^2Q_{n-2},
\end{align*}$$

where $\phi(D) = \frac{D-I}{D+I}$.

We see that $P_n$ and $Q_n$ are diagonal matrices. By setting $p_n = \text{diag}(p_{n,1}^1, p_{n,2}^2, \ldots, p_{n,m}^m)$ and $q_n = \text{diag}(q_{n,1}^1, q_{n,2}^2, \ldots, q_{n,m}^m)$, we obtain for each $i$ where $1 \leq i \leq m$,

$$\begin{align*}
p_{i-1}^i &= 1, \quad p_0^i = 0, \quad p_1^i = 2\phi(\lambda_i) \\
q_{i-1}^i &= 0, \quad q_0^i = 1, \quad q_1^i = 1
\end{align*}$$

and for $n \geq 2$,

$$\begin{align*}
p_n^i &= (2n+1)p_{n-1}^i - n^2(\phi(\lambda_i))^2p_{n-2}^i, \\
q_n^i &= (2n+1)q_{n-1}^i - n^2(\phi(\lambda_i))^2q_{n-2}^i.
\end{align*}$$

By Lemma 3.1.1, the convergent $(p_n^i/q_n^i)$ converges to $\ln(\lambda_i)$. It follows that $P_n/Q_n$ converges to the matrix $\ln(D)$, so that

$$\ln D = \left[ 0; \frac{2\phi(D)}{I}, \frac{-1^2(\phi(D))^2}{3I}, \frac{-n^2(\phi(D))^2}{(2n+1)I} \right]_{n=2}^{+\infty}.$$
By proposition 2.4, we get
\[
\ln A = X(\ln D)X^{-1} = X \left[ \begin{array}{ccc}
O; & 2\phi(D) & -1^2(\phi(D))^2 \\
1 & 3I & (2n+1)I \\
2 & 3I & (2n+1)I
\end{array} \right]_{n=2}^{+\infty} X^{-1}
\]

Let us define the sequence \((X_n)_{n \geq -1}\) by
\[
\begin{cases}
X_{-1} = X_0 = I \\
X_n = X, \text{ for } n \geq 1.
\end{cases}
\]

Then
\[
\frac{X_1B_1X_{-1}}{X_1A_1X_0^{-1}} = \frac{2X_1\phi(D)X^{-1}X_{-1}}{X_0X^{-1}X_0^{-1}} = \frac{2\phi(A)}{I},
\]
\[
\frac{X_2B_2X_{-1}}{X_2A_2X_1^{-1}} = \frac{X_2(-1^2(\phi(D))^2)X^{-1}X_{-1}}{X_2(3I)X_1^{-1}} = \frac{-1^2(\phi(A))^2}{3I}.
\]

For \(n \geq 3\), we have
\[
\frac{X_nB_nX_{-2}}{X_nA_nX_{-1}} = \frac{-(n-1)^2(\phi(A))^2}{(2n-1)I}.
\]

By applying the result of proposition 2.6 to the sequence \((X_n)_{n \geq -1}\), we finish the proof of lemma 3.1.2

**Proof of theorem 3.1.3** Let \(A\) and \(B\) be two invertible and positive definite matrices in \(\mathcal{M}_m\). In order to apply lemma 3.1.2, we have
\[
\phi(A^{-1/2}BA^{-1/2}) = \frac{A^{-1/2}BA^{-1/2} - I}{A^{-1/2}BA^{-1/2} + I} = \frac{A^{-1/2}(B - A)A^{-1/2}}{A^{-1/2}(B - A)A^{-1/2}} = A^{1/2}B - A \frac{B + A}{B + A} A^{-1/2}.
\]

So,
\[
(\phi(A^{-1/2}BA^{-1/2}))^2 = A^{1/2} \left( \frac{B - A}{B + A} \right)^2 A^{-1/2}.
\]

Then, according to lemma 3.1.2, we obtain
\[
\ln(A^{-1/2}BA^{-1/2}) = \left[ 0; \frac{2A^{1/2} \left( \frac{B - A}{B + A} \right) A^{-1/2}}{I}, -A^{1/2} \left( \frac{B - A}{B + A} \right)^2 A^{-1/2}, -n^2A^{1/2} \left( \frac{B - A}{B + A} \right)^2 A^{-1/2} \right]_{n=2}^{+\infty}.
\]

Due to proposition 2.4, we deduce that
\[
S(A|B) = A^{1/2} \ln(A^{-1/2}BA^{-1/2})A^{1/2} = \left[ 0; \frac{2A^{1/2} \left( \frac{B - A}{B + A} \right)}{A^{-1/2}}, -A^{1/2} \left( \frac{B - A}{B + A} \right)^2 A^{-1/2}, -n^2A^{1/2} \left( \frac{B - A}{B + A} \right)^2 A^{-1/2} \right]_{n=2}^{+\infty}.
\]

In order to achieve the proof of theorem 3.1.3, let us take
\[
\begin{cases}
X_{-1} = X_0 = I, \\
X_n = A^{1/2}, \forall n \geq 1.
\end{cases}
\]
Thanks to proposition 2.6, we see that
\[
S(A|B) = \left\{ \begin{array}{c}
0; \\
\frac{2A \left( \frac{B-A}{B+\lambda A} \right)}{I} \\
-A \left( \frac{B-A}{B+\lambda A} \right)^2 A^{-1} \\
-n^2 A \left( \frac{B-A}{B+\lambda A} \right)^2 A^{-1} \\
\end{array} \right\}_{n=2}^{+\infty}. 
\]

### 3.1.4 Examples of applications

This section is devoted to illustrate our above theoretical result (3.4) with some examples.

**Example 1.** Consider the matrix \( A \) such that
\[
A = \begin{pmatrix}
80/99 & -8/99 \\
-8/99 & 80/99
\end{pmatrix}.
\]

\( A \) is a diagonal matrix and we have \( A = PD P^{-1} \) where
\[
P = \begin{pmatrix}
80/99 & -8/99 \\
-8/99 & 80/99
\end{pmatrix}, \quad D = \begin{pmatrix}
8/9 & 0 \\
0 & 8/9
\end{pmatrix}.
\]

The exact value of \( S(A|I) = -A \ln(A) \) is
\[
S(A|I) = \begin{pmatrix}
-(4/9) \ln(8/9) - (4/11) \ln(8/11) & (4/9) \ln(8/9) - (4/11) \ln(8/11) \\
(4/9) \ln(8/9) - (4/11) \ln(8/11) & -(4/9) \ln(8/9) - (4/11) \ln(8/11)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
0.168149372 & 0.06345334 \\
0.06345334 & 0.168149372
\end{pmatrix}.
\]

Applying the theorem 3.1.3, the first convergents of \( S(A|B) \) are given by:
\[
F_1 = \begin{pmatrix}
19/99 & 8/99 \\
8/99 & 19/99
\end{pmatrix} = \begin{pmatrix}
0.191919191 & 0.08080808 \\
0.08080808 & 0.191919191
\end{pmatrix},
\]
\[
F_2 = \begin{pmatrix}
5344/31977 & 2000/31977 \\
2000/31977 & 5344/31977
\end{pmatrix} = \begin{pmatrix}
0.167120117 & 0.062544954 \\
0.062544954 & 0.167120117
\end{pmatrix},
\]
\[
F_3 = \begin{pmatrix}
4331/25740 & 409/6435 \\
409/6435 & 4331/25740
\end{pmatrix} = \begin{pmatrix}
0.168259518 & 0.063558663 \\
0.063558663 & 0.168259518
\end{pmatrix},
\]
\[
F_4 = \begin{pmatrix}
0.168142779 & 0.063446859 \\
0.063446859 & 0.168142779
\end{pmatrix},
\]
\[
F_5 = \begin{pmatrix}
0.168150016 & 0.063453981 \\
0.063453981 & 0.168150016
\end{pmatrix},
\]
\[
F_6 = \begin{pmatrix}
0.168149330 & 0.063453298 \\
0.063453298 & 0.168149330
\end{pmatrix}.
\]
We observe that very good approximations of $S(A|B)$ are obtained from the first iterations. And this example explains the fast convergence of the continued fraction expansion of $S(A|I)$ given in (3.4).

Example 2: Now, let us consider the matrices $A$ and $B$ such that

\[
A = \begin{pmatrix}
5 & -4 & 0 \\
-4 & 5 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
24 & -12 & -3 \\
-12 & 15 & 0 \\
-3 & 0 & 4
\end{pmatrix}.
\]

We will compute a relative operator entropy $S(A|B)$. It is not hard to see that

\[
S(A|B) = \begin{pmatrix}
\frac{13}{2} \ln 3 & -4 \ln 3 & \frac{-1}{2} \ln 3 \\
-4 \ln(3) & 5 \ln 3 & 0 \\
\frac{-1}{2} \ln 3 & 0 & \frac{7}{6} \ln 3
\end{pmatrix}
= \begin{pmatrix}
7.14097978 & -4.394449156 & -0.5493061445 \\
-4.394449156 & 5.493061445 & 0 \\
-0.5493061445 & 0 & 1.281714338
\end{pmatrix},
\]

Applying the theorem 3.1.3 the first convergents of $S(A|B)$ are given by:

\[
F_2 = \begin{pmatrix}
6.8690029276 & -4.3636364 & -0.47149461 \\
-4.3636364 & 5.490196078 & 0 \\
-0.47149461 & 0 & 1.248073960
\end{pmatrix},
\]

\[
F_3 = \begin{pmatrix}
7.074306086 & -4.392156863 & -0.5283066692 \\
-4.392156863 & 5.4930196078 & 0 \\
-0.5283066692 & 0 & 1.274051439
\end{pmatrix},
\]

\[
F_4 = \begin{pmatrix}
7.124146853 & -4.394281415 & -0.5437650283 \\
-4.394281415 & 5.492851768 & 0 \\
-0.5437650283 & 0 & 1.279825363
\end{pmatrix},
\]

\[
F_5 = \begin{pmatrix}
7.136715652 & -4.394436967 & -0.5478898144 \\
-4.394436967 & 5.493046209 & 0 \\
-0.5478898144 & 0 & 1.281239180
\end{pmatrix},
\]

\[
F_6 = \begin{pmatrix}
7.139901896 & -4.394448272 & -0.5489471762 \\
-4.394448272 & 5.493060341 & 0 \\
-0.5489471762 & 0 & 1.281684124
\end{pmatrix},
\]

\[
F_7 = \begin{pmatrix}
7.140708024 & -4.394449091 & -0.5492155533 \\
-4.394449091 & 5.493061364 & 0 \\
-0.5492155533 & 0 & 1.281684124
\end{pmatrix},
\]

\[
F_8 = \begin{pmatrix}
7.140911453 & -4.394449150 & -0.5492833386 \\
-4.394449150 & 5.493061438 & 0 \\
-0.5492833386 & 0 & 1.281706734
\end{pmatrix},
\]
Continued fraction expansion of the relative operator entropy and the Tsallis relative entropy

\[
F_0 = \begin{pmatrix}
7.140962679 & -4.394449154 & -0.5493004121 \\
-4.394449154 & 5.493061443 & 0 \\
-0.5493004121 & 0 & 1.281712426
\end{pmatrix},
\]

\[
F_{10} = \begin{pmatrix}
7.140975559 & -4.394449155 & -0.5493047051 \\
-4.394449155 & 5.493061443 & 0 \\
-0.5493047051 & 0 & 1.281713857
\end{pmatrix}.
\]

We see that \(F_{10}\) is approximately the exact value of \(S(A|B)\). This example justifies the importance of our approach.

**Theorem 3.2.2** Let \(A\) and \(B\) be two invertible and positive definite matrices in \(\mathcal{M}_m\). Then a continued fraction expansion of Tsallis relative operator entropy is defined by

\[
T_\lambda(A|B) = \left[ 0; \frac{B_n}{A_n} \right]_{n=1}^{+\infty},
\]

where

\[
\begin{align*}
B_1 &= 2\lambda A^{1/2} \frac{A - B}{A + B}, \\
B_2 &= \lambda (\lambda^2 - 1) A^{1/2} \left( \frac{A - B}{A + B} \right)^2 A^{-1}, \\
A_1 &= -\lambda A^{-1/2} - \lambda^2 A^{1/2} \frac{A - B}{A + B} A^{-1}, \\
B_n &= \lambda (\lambda^2 - (n - 1)^2) A^{1/2} \left( \frac{A - B}{A + B} \right)^2 A^{-1/2} \text{ for all } n \geq 3, \\
A_n &= -(2n - 1) I \text{ for all } n \geq 2.
\end{align*}
\]

In order to prove theorem 3.2.2, we recall the following lemma.

**Lemma 3.2.3** [17]. Let \(A\) and \(B\) be two invertible and positive matrix in \(\mathcal{M}_m\), \(\lambda\) a real number such that \(0 < \lambda < 1\). The continued fraction expansion of \((A^{-1/2}BA^{-1/2})^\lambda\) is given by

\[
(A^{-1/2}BA^{-1/2})^\lambda = \left[ I; \frac{B_n}{A_n} \right]_{n=1}^{+\infty},
\]

where we set

\[
\begin{align*}
B_1' &= 2\lambda A^{1/2} \frac{A - B}{A + B} A^{-1/2}, \\
A_1' &= -I - \lambda A^{1/2} \frac{A - B}{A + B} A^{-1/2}, \\
B_n' &= (\lambda^2 - (n - 1)^2) A^{1/2} \left( \frac{A - B}{A + B} \right)^2 A^{-1/2}, \quad n \geq 2, \\
A_n' &= -(2n - 1) I, \quad n \geq 2.
\end{align*}
\]
Proof of theorem 3.2.2 With the same notations as in lemma 3.2.3, we have
\[(A^{-1/2}BA^{-1/2})^\lambda = \left[ I; \frac{B_1'}{A_1'}, \frac{B_2'}{A_2'}, \frac{\lambda B_k'}{A_k'} \right]_{k=1}^{+\infty}, \]
By adding \((-I)\), dividing by lambda from the both sides and using the proposition 2.7, we get
\[\frac{(A^{-1/2}BA^{-1/2})^\lambda - I}{\lambda} = \left[ 0; \frac{B_1'}{\lambda A_1'}, \frac{\lambda B_2'}{A_2'}, \frac{B_k'}{A_k'} \right]_{k=3}^{+\infty} \]
(3.9)
Multiplying \(A^{1/2}\) from both sides of (3.9), we have
\[A^{1/2} \left( \frac{(A^{-1/2}BA^{-1/2})^\lambda - I}{\lambda} \right) A^{1/2} = A^{1/2} \left[ 0; \frac{B_1'}{\lambda A_1'}, \frac{\lambda B_2'}{A_2'}, \frac{B_k'}{A_k'} \right]_{k=3}^{+\infty} A^{1/2}. \]
By propositions 2.6 and 2.7, we get the result of theorem 3.2.2.

3.3 The solution of a matrix algebraic equation and its continued fraction expansion.

Let \(A\) and \(B\) be two positive definite matrices in \(\mathcal{M}_m\). As it is known, the explicit form of the geometric mean of \(A\) and \(B\) is given by
\[g_2(A, B) = f_{1/2}(A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.\]

Definition 3.3.1 Let \(m \geq 2\) be an integer, the geometric mean of \(m\) positive definite matrices \(A_1, A_2, ..., A_m\) is recursively defined by the relationship:
\[g_m(A_1, A_2, ..., A_m) = f_{1/m}(A_1, g_{m-1}(A_2, A_3, ..., A_m))\]
where
\[f_{1/m}(A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/m}A^{1/2}.\]

We consider the following matrix equation: Find a positive matrix \(X\) such that
\[X(AX)^3 = B.\]  \hspace{1cm} (3.10)
It is well known that equation (3.10) has a unique solution given by
\[X = g_4(B, A^{-1}, A^{-1}, A^{-1}) = A^{-1/2}(A^{1/2}BA^{1/2})^{1/4}A^{-1/2}.\]
With the appearance of the term \(A^{1/2}\) and \((A^{1/2}BA^{1/2})^{1/4}\), it is hard to calculate \(g_4(B, A^{-1}, A^{-1}, A^{-1})\) directly.

The following theorem approximates the solution of (3.10) in term of continued fraction.
Theorem 3.3.3  Let $A$ and $B$ be two positive definite matrices in $\mathcal{M}_m$, the solution of equation (8) has the following continued fraction expansion:

$$X = \left[ A^{-1}, \frac{B_n}{A_n} \right]_{n=1}^{+\infty},$$

where

$$B_1 = \frac{1}{2} I - AB, \quad B_2 = \frac{-15}{16} \left( \frac{I - AB}{I + AB} \right)^2 A,$$

$$A_1 = -A - \frac{1}{4} \frac{I - AB}{I + AB} A,$$

$$B_n = \left( \frac{1}{16} - (n - 1)^2 \right) \left( \frac{I - AB}{I + AB} \right)^2, \quad \text{for } n \geq 3,$$

$$A_n = -(2n - 1) I, \quad \text{for all } n \geq 2.$$

Proof. From lemma 3.2.3 and classical transformation, we easily find this result.

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References


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