Fixed point theorems in Banach spaces using three steps iteration

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Abstract: Suppose K is a nonempty closed convex nonexpansive retraction of real uniformly convex Banach space E with P as a nonexpansive retraction. Let T: K → E be a nonexpansive non-self map with $F(T) = \{ x \in K : Tx = x \} \neq \emptyset$, suppose \( \{ x_n \} \) is generated iteratively by

$$x_{n+1} = P((1-\alpha_n)x_n + \alpha_n Ty_n + \beta_n TTx_n),$$

\( n \geq 1 \) where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are real sequences \( [\varepsilon, 1 - \varepsilon] \) for some \( \varepsilon \in (0,1) \). (1) If the dual $E^*$ of E has the Kadec-Klee property, then weak convergence of \( \{ x_n \} \) to some \( x^* \in F(T) \) is proved. (2) If T satisfies condition (A), then strong convergence of \( \{ x_n \} \) to some \( x^* \in F(T) \) is obtained.

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I. Introduction

Let K be a nonempty subset of a real normed linear space E. Let T be a self-mapping of K. Then T is said to be nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|$$

For all \( x, y \in K \)

Definition 1.1:

Let X be a real Banach space. A subset K of X is said to be a retract of X if there exists a continuous map P: X → K such that Px = x for all x ∈ K. A map P: X → X is said to be a retraction if P\(^2\) = P. It follows that if P is a real retraction map, then Py = y for all y in the range of P.

Let K be a nonempty convex subset of X and T: K → K. For \( x_i \in K \) and some \( \{ \alpha_n \} \), \( \{ \beta_n \} \), \( \{ \gamma_n \} \subseteq [0,1] \).

The Mann iteration formula is given as

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n TTx_n \quad n \geq 1$$

The Ishikawa iterative scheme is defined by

$$\begin{align*}
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n \\
y_n &= (1-\beta_n)x_n + \beta_n TTx_n \\
n \geq 1
\end{align*}$$

The three step Noor iteration scheme is defined by

$$\begin{align*}
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n TTy_n \\
y_n &= (1-\beta_n)x_n + \beta_n Tz_n \\
z_n &= (1-\gamma_n)x_n + \gamma_n TTx_n
\end{align*}$$

Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors (see e.g. [5, 13, 15, 17] using the Mann iteration method (see e.g. [10]) or the Ishikawa iteration method (see e.g. [6]) and Noor [18].

In [15], Tan and Xu introduced a modified Ishikawa process to approximate fixed points of nonexpansive mapping defined on nonempty closed convex bounded subsets of a uniformly convex Banach space E. More precisely, they proved the following theorem.

**Theorem Xu** (Tan and Xu[15, Theorem1, p. 305]) Let E be a uniformly convex Banach space which satisfies Opial’s condition or has a Fréchet differentiable norm and K a nonempty closed convex bounded subset of E.
Let $T: K \to K$ be a nonexpansive mapping. Let $T: K \to K$ be a nonexpansive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$, such that $\sum_{n=1}^{\infty} \alpha_n (1-\alpha_n) = \infty$, $\sum_{n=1}^{\infty} \beta_n (1-\alpha_n) < \infty$ and $\limsup_{n \to \infty} \beta_n < 1$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in K$ by

$$x_{n+1} = (1-\alpha_n) x_n + \alpha_n T((1-\beta_n) x_n + \beta_n T x_n), \quad n \geq 1$$

(1.2) Converges weakly to some fixed point of $T$.

In the above result, $T$ remains self-mapping of a nonempty closed convex subset $K$ of a uniformly convex Banach space. If, however, the domain $K$ of $T$ is a proper subset of $E$ (and this is the case in several applications), and $T$ maps $K$ into $E$ then, the iteration formula (1.2) may fail to be well defined.

II. Preliminaries

Let $E$ be a real Banach space. A subset $K$ of $E$ is said to be a retract of $E$ if there exists a continuous map $P: E \to K$ such that $P x = x$ for all $x \in K$. A map $P: E \to E$ is said to be a retraction if $P^2 = P$. If $\{x_n\}$ is a sequence in $E$ such that $\{x_n\}$ converges weakly to a point $x \in E$, then $P x = x$. A set $K$ is optimal if each point outside $K$ can be moved to be closer to all points of $K$.

Let $E$ be a strictly convex, smooth, reflexive Banach space and $K \subset E$ an optimal set with interior, then $K$ is a nonexpansive retract of $E$.

Note that every nonexpansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a nonexpansive retract.

A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is said to be a nonexpansive mapping if it is a continuous, strictly increasing and convex function $g: [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and $g(x) \leq g(y)$ for all $x \leq y$.

Let $p \geq 1$ and $R > 1$ be two fixed numbers and $E$ a Banach space. Then $E$ is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g: [0, \infty) \to [0, \infty)$ with

$$g(0) = 0 \quad \text{and} \quad g(x) \leq g(y) \quad \text{for all} \quad x \leq y \in B_1(0) = \{ x \in E : \|x\| \leq R \}.$$
Lemma 2.4 (Kaczor [7]) Let E be a real reflexive Banach space such that its dual E* has the Kadec-Klee property. Let \{x_n\} be a bounded sequence in E and x*, y* ∈ ω_ω(x_n); here ω_ω(x_n) denotes the weak w-limit set of \{x_n\}. Suppose \lim_{n→∞} \|x_n + (1-t)x^* - y^*\| exists for all t ∈ [0,1]. Then x^* = y^*.

Lemma 2.5 (Browder [11]) Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E and T: K→ E a nonexpansive mapping. Then I- T is semi closed at zero.

III. Main Results

In this section, we prove our main theorems. Let K be a nonempty closed convex subset of a real uniformly convex Banach space E, which is also a nonexpansive retract of E. Let T: K→ E be a nonexpansive mapping. The following iteration scheme is studied:

\[ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(y_n)) - x^* \]

With \( x_1 \in K, n \geq 1 \) where \{\alpha_n\}, \{\beta_n\} and \{\gamma_n\} are sequences in [0, 1] and P is a nonexpansive retraction of E onto K.

Theorem 3.1 Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let T: K→ E be a nonexpansive mapping with x* ∈ F(T) = \{x ∈ K: Tx = x\}. Let \{\alpha_n\} and \{\beta_n\} be sequences in \([\epsilon, 1-\epsilon]\) for some \( \epsilon \in (0,1) \) starting from arbitrary \( x_1 \in K \), define the sequence \{x_n\} by the recursion (3.1). Then \( \lim_{n→∞} \|x_n - x^*\| \) exists.

Proof: We observe that

\[ \|x_{n+1} - x^*\| = \|P((1-\alpha_n)x_n + \alpha_n T(y_n)) - x^*\| \]

\[ \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|P((1-\beta_n)x_n + \beta_n T(z_n)) - x^*\| \]

\[ \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n (1-\beta_n)\|x_n - x^*\| + \alpha_n \beta_n\|Tz_n - x^*\| \]

\[ \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n (1-\beta_n)\|x_n - x^*\| + \alpha_n \beta_n\|Tz_n - x^*\| \]

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\[ \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n (1-\beta_n)\|x_n - x^*\| + \alpha_n \beta_n\|Tz_n - x^*\| \]

Consequently, we have

\[ \|x_{n+1} - x^*\| \leq \|x_1 - x^*\| \]

This implies that \{x_n\} is bounded and Lemma 2.3 guarantees that \( \lim_{n→∞} \|x_n - x^*\| \) exists.

This completes the proof.

Theorem 3.2 Let E be a real uniformly convex Banach space and K a nonempty closed convex subset of E which is also a nonexpansive retract of E. Let T: K→ E be a nonexpansive mapping with \( F(T) \neq \emptyset \). Let \{\alpha_n\}, \{\beta_n\} and \{\gamma_n\} be sequences in \([\epsilon, 1-\epsilon]\) for some \( \epsilon \in (0,1) \). From arbitrary \( x_1 \in K \), define the sequence \{x_n\} by the recursion (3.1). Then

\[ \lim_{n→∞} \|x_n - Tz_n\| = 0 \]

Proof: Let \( x^* \in F(T) \), then, by theorem 3.1, \( \lim_{n→∞} \|Tz_n - x_n\| = 0 \) exists. Let \( \lim_{n→∞} \|x_n - x^*\| = 0 \). If \( r = 0 \), Then by the continuity of T the conclusion follows. Now suppose \( r > 0 \). We claim

\[ \lim_{n→∞} \|Tz_n - x_n\| = 0 \]

Now set \( y_n = P((1-\beta_n)x_n + \beta_n Tz_n) \). Since \{x_n\} is bounded, there exists \( R > 0 \) such that \( x_n - x^* \), \( y_n - x^* \) and \( z_n - x^* \in B_R(0) \) for all \( n \geq 1 \). Using Lemma 2.2, we have that
\[
\begin{align*}
\|z_n - x^*\|^2 &= \|P((1-\gamma_n)x_n + \gamma_n Tx_n) - x^*\|^2 \\
&= \|P((1-\gamma_n)x_n + \gamma_n Tx_n - (1-\gamma_n)x^* - \gamma_n x^*)\|^2 \\
&\leq (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 \\
&\leq (1-\gamma_n)\|x_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 - W_2(\gamma_n)g(\|Tx_n - x_n\|) \\
&\leq \gamma_n\|x_n - x^*\|^2 + (1-\lambda_n)\|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2
\end{align*}
\]
From Lemma 2.2, it follows that
\[
\|x_{n+1} - x^*\|^2 = \|P((1-\alpha_n)x_n + \alpha_n Ty_n) - Px^*\|^2 \\
\leq \alpha_n(Ty_n - x^*) + (1-\alpha_n)(x_n - x^*)\|^2 \\
\leq \alpha_n\|y_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - W_2(\alpha_n)g(\|Ty_n - x_n\|) \\
\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 - W_2(\alpha_n)g(\|Ty_n - x_n\|)
\]
\[
\leq \|x_n - x^*\|^2 - W_2(\alpha_n)g(\|Ty_n - x_n\|)
\]
(3.2)

Observe that \( W_2(\alpha_n) \geq \varepsilon^2 \). Now (3.2) implies that
\[
\varepsilon^2 \sum_{n=1}^{\infty} g(\|Tz_n - x_n\|) \leq \|x_1 - x^*\|^2 < \infty
\]
Therefore, we have \( \lim_{n \to \infty} g(\|Tz_n - x_n\|) = 0 \); since \( g \) is strictly increasing continuous at 0. It follows that
\[
\lim_{n \to \infty} \|Tz_n - x_n\| = 0.
\]
Since \( T \) is nonexpansive, we can get that
\[
\|z_n - x^*\| \leq \|z_n - Ty_n\| + \|y_n - x^*\|
\]
Which on taking \( \lim \inf \) gives \( r \leq \lim \inf \|z_n - x^*\| \). On the other hand, we have
\[
\|z_n - x^*\| \leq \|(1-\gamma_n)x_n + \gamma_n Tx_n - (1-\gamma_n)x^* + \gamma_n x^*\| \\
\leq \|(1-\gamma_n)(x_n - x^*) + \gamma_n(x_n - x^*)\| \\
\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\
= \|x_n - x^*\|
\]
Which implies \( \lim \sup_{n \to \infty} \|z_n - x^*\| \leq r \). Therefore, \( \lim_{n \to \infty} \|z_n - x^*\| = r \) and so
\[
\lim_{n \to \infty} \beta_n(Tz_n - x^*) + (1-\gamma_n)(z_n - x^*) = r
\]
Since \( \lim \sup \|Tx_n - x^*\| \leq r \), it follows from Lemma 2.1 that
\[
\lim_{n \to \infty} \|Tx_n - x_n\| = 0.
\]
This completes the proof.
Reference