Twin Edge Colourings of Wheel Graphs

S.Lakshmi 1, V.Kowsalya 2

1 (Assistant Professor, Department of Mathematics, PSGR Krishnammal College for women, Coimbatore-641004, India)
2 (Research scholar, Department of Mathematics, PSGR Krishnammal College for women, Coimbatore-641004, India)

Abstract: A proper edge colouring of a graph G with the elements of \( \mathbb{Z} \) is said to be a twin edge k-colouring of G if the induced vertex colouring is a proper vertex colouring in which the colour of a vertex v in G is defined as the sum (in \( \mathbb{Z} \)) of the colours of the edges incident with v. Twin chromatic index of G is the minimum k for which G has a twin edge k-colouring. Twin chromatic index of \( W_n \) is determined, where \( W_n \) is the wheel graph of order n.

Keywords: edge colouring, vertex colouring.

I. Introduction

Vertex labelling was introduced by Rosa [14] in 1968, which induces an edge-distinguishing labelling defined by subtracting labels. A vertex labelling \( f : V(G) \rightarrow \{0,1,\ldots,m\} \) for a graph G of size m was called \( \beta \)-valuation by Rosa if the induced edge labelling \( f' : E(G) \rightarrow \{1,2,\ldots,m\} \) defined by \( f'(uv) = f(u) - f(v) \) was bijective. A \( \beta \)-valuation was called as a graceful labelling and a graph which possessing a graceful labelling was called a graceful graph [9]. A popular conjecture in graph theory, due to Anton Kotzig and Gerhard Ringel, is the following.

The Graceful Tree Conjecture: Every nontrivial tree is graceful.

Definition 1.1. For a connected graph G of order \( n \geq 3 \), let \( f : E(G) \rightarrow \mathbb{Z}_m \) be an edge labelling of G that induces a bijective function \( f' : V(G) \rightarrow \mathbb{Z}_m \) defined by \( f'(v) = \sum_{e \in E(v)} f(e) \) for each vertex v of G, where \( E_v \) is the set of edges of G incident with a vertex v. Such a labelling \( f \) is called a modular edge-graceful labelling, while a graph possessing such a labelling is called modular edge-graceful.

Definition 1.2. For the set \( \mathbb{N} \) of positive integers, an edge colouring \( c : E(G) \rightarrow \mathbb{N} \), where adjacent edges may be coloured the same, is said to be vertex-distinguishing if the colouring \( c' : V(G) \rightarrow \mathbb{N} \) induced by \( c \) and defined by \( c'(v) = \sum_{e \in E(v)} c(e) \) has the property that \( c'(x) \neq c'(y) \) for every two distinct vertices x and y of G.

Definition 1.3. A neighbour-distinguishing colouring of a graph G is a colouring in which every pair of adjacent vertices of G are coloured differently. Such a colouring is more commonly called a proper vertex colouring. The minimum number of colours needed in a proper vertex colouring of a graph G is the chromatic number \( \chi(G) \) and denoted by \( \chi(G) \).

Definition 1.4. For \( k \in \mathbb{N} \), let \( c : E(G) \rightarrow \{1,2,\ldots,k\} \) be an edge colouring of G (where adjacent edges may be assigned the same colour). A vertex colouring \( c' : V(G) \rightarrow \mathbb{N} \) is defined where \( c'(v) \) is the sum of the colours of the edges incident with v. If \( c' \) is a proper vertex colouring of G, then \( c \) is called a neighbour-distinguishing edge colouring of G.

The 1-2-3 Conjecture. For every connected graph G of order at least 3, there exists a neighbour-distinguishing edge colouring of G using only the colours 1, 2, 3.

Definition 1.5. In a proper edge colouring of a graph G, each edge of G is assigned a colour from a given set of colours where adjacent edges are coloured differently. The minimum number of colours needed in a proper edge colouring of G is called the chromatic index of G and is denoted by \( \chi'(G) \).

Observation 1.6. For every nonempty graph G, \( \Delta (G) \leq \chi'(G) \leq \Delta(G)+1 \). (This was proved by Vizing [15]).

Definition 1.7. Total colouring of a graph G that assigns colours to both the vertices and edges of G so that not only the vertex colouring and edge colouring are proper but no vertex and an incident edge are assigned the
same colour. The minimum number of colours required for a total colouring of \( G \) is the total chromatic number of \( G \), denoted by \( \chi''(G) \).

**The Total Colouring Conjecture.** For every graph \( G \), \( \chi''(G) \leq 2+\Delta(G) \).

## II. Twin Chromatic Index

For a connected graph \( G \) of order at least 3, a proper edge colouring \( c:E(G)\rightarrow\mathbb{Z}_k \) for some integer \( k \geq 2 \) is sought for which the induced vertex colouring \( c':V(G)\rightarrow\mathbb{Z}_k \) defined by

\[
c'(v) = \sum_{e \in E(v)} c(e) \mod k
\]

(where the indicated sum is computed in \( \mathbb{Z}_k \)) results in a proper vertex colouring of \( G \). We refer to such a colouring as a twin edgecolouring or simply twin edge colouring of \( G \). The minimum \( k \) for which \( G \) has a twin edge \( k \)-colouring is called the twin chromatic index of \( G \) and is denoted by \( \chi'(G) \). Since a twin edge colouring is not only a proper edge colouring of \( G \) but induces a proper vertex colouring of \( G \), it follows that

\[
\chi'(G) \geq \max\{\chi(G), \chi(G)\}.
\]

Since \( \chi(G) \) does not exist if \( G \) is the connected graph of order 2, every connected graph of order at least 3 has a twin edge colouring. To see this, let \( G \) be a connected graph of size \( m \geq 2 \). If \( m=2 \), then assign the colours 1 and 2 in \( \mathbb{Z}_2 \) to the two edges of \( G \). If \( m \geq 3 \), then assign the \( m \) elements \( 0,1,2,4,...,2^{m-1} \in \mathbb{Z}_2^{m-1} \) to the \( m \) edges of \( G \) in a one-to-one manner so that the colour 0 is assigned to a pendant edge if \( G \) has such an edge. Hence the sets of edges coloured by nonzero elements in \( \mathbb{Z}_2^{m-1} \) that are incident with every two adjacent vertices are distinct. Since the base 2 representations of the colours of these vertices are different, it follows that adjacent vertices are assigned distinct colours in \( \mathbb{Z}_2^{m-1} \). Thus, this colouring is a twin edge colouring. This observation yields the following.

**Proposition 2.1.** If \( G \) is a connected graph of order at least 3 and size \( m \), then \( \chi'(G) \) exists. Furthermore, \( \chi'(G) \leq 2^{m/2} \) if \( m \geq 3 \).

**Proposition 2.2.** If \( P_n \) is a path of order \( n \geq 3 \), then \( \chi'(P_n) = 3 \).

**Observation 2.3.** If a connected graph \( G \) contains two adjacent vertices of degree \( \Delta(G) \), then \( \chi'(G) \geq 1+\Delta(G) \).

**Proposition 2.4.** If \( C_n \) is a cycle of order \( n \geq 3 \), then

\[
\chi'(C_n) = \begin{cases} 
3, & \text{if } n \equiv 0 \mod 3 \\
4, & \text{if } n \equiv 0 \mod 2 \text{ and } n \neq 5 \\
5, & \text{if } n = 5 
\end{cases}
\]

### III. Wheel Graphs

We now investigate twin edge colourings of wheel graphs \( W_n \).

**Proposition 3.1.** If \( W_n \) is a wheel of order \( n \geq 5 \), then \( \chi'(W_n) = n(n-1)/2 \).

**Proof:** Let \( W_n = (v_1,v_2,...,v_n) \) be a wheel of order \( n \geq 5 \) and let \( v_1 \) be the mid vertex of the wheel, where

\[
e_i = v_{i+1}v_i, \quad i = 1,2,...,n-1
\]
\[
e_{n,i+1} = v_{n+1}v_i, \quad i = 1,2,...,n-1
\]
\[
e_{n,1} = v_{n+1}v_1, \quad \text{since the number of edges in a wheel graph is } 2n-2.
\]

Now \( \Delta(W_n) = n-1 \). From the observation 1.6, we have \( \Delta(W_n) \leq \chi'(W_n) \leq \Delta(W_n) + 1 \). Therefore \( n-1 \leq \chi'(W_n) \leq n \). We show that \( \chi'(W_n) = n-1 \). Let \( c \) be a proper edge colouring of \( W_n \) and defined as follows.

- For even \( n \), define an edge colouring \( c : E(W_n) \rightarrow \mathbb{Z}_{n-1} \) as follows.
  \[
c(e) = \begin{cases} 
0, & i = n,n+2,n+4,...,2n-4 \\
1, & i = n+1,n+3,...,2n-3 \\
3, & i = 2n-2
\end{cases}
\]

- For odd \( n \), define an edge colouring \( c : E(W_n) \rightarrow \mathbb{Z}_{n-1} \) as follows.
  \[
c(e) = \begin{cases} 
0, & i = n,n+2,n+4,...,2n-4 \\
1, & i = n+1,n+3,...,2n-3 \\
2, & i = 2n-2
\end{cases}
\]

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- For odd \( n \), define an edge colouring 
  
  \[ c : E(W_n) \to \mathbb{Z}_{n+1} \] 
  
  as follows.

  \[
  i, \ i = 0,1,\ldots,n-1 \]

  \[
  c(e_i) = \begin{cases}
  1, & i = 0, \ldots, n-1 \\
  2, & i = n+1, \ldots, 2n-3 \\
  3, & i = 2n-2 
  \end{cases}
  \]

  Thus \( c \) is a \((n-1)\)-edge colouring. Hence \( \chi'(W_n) = n-1 \). It remains to show that \( W_n \) has a twin edge \( n(n-1)/2 \) colouring. A colouring \( c : E(W_n) \to \mathbb{Z}_{n+1} \) for both even and odd \( n \) is defined as above.

- If \( n \) is even, then

  \[
  c'(v_i) = \begin{cases}
  i+1, & i = 1, \ldots, n-2 \\
  n+3, & i = n-1 \\
  n(n+1)/2, & i = n 
  \end{cases}
  \]

  For example, if \( n = 6 \), then \( (c(e_1), c(e_2), \ldots, c(e_{10})) = (1, 2, 3, 4, 5, 0, 1, 0, 1, 3) \) and \( (c'(v_1), c'(v_2), \ldots, c'(v_6)) = (4, 3, 4, 5, 9, 15) \)

- If \( n \) is odd, then

  \[
  c'(v_i) = \begin{cases}
  i+1, & i = 1, \ldots, n-2 \\
  n+3, & i = n-1 \\
  n(n+1)/2, & i = n 
  \end{cases}
  \]

  For example, if \( n = 7 \), then \( (c(e_1), c(e_2), \ldots, c(e_{12})) = (1, 2, 3, 4, 5, 6, 0, 1, 0, 1, 0, 3) \) and \( (c'(v_1), c'(v_2), \ldots, c'(v_7)) = (4, 3, 4, 5, 6, 9, 21) \). Hence \( \chi'(W_n) = n(n-1)/2 \).

IV. Conclusion

In this paper we determined the twin chromatic index of a wheel graph \( W_n \). We may proceed this concept of finding twin chromatic index for some more graphs.

References