Degree Regularity on Edges of S – Valued Graph

S.Mangala Lavanya*, S.Kiruthiga Deepa** And
M.Chandramouleeswaran***
*Sree Sowdambika College Of Engineering Aruppukottai-626101. Tamilnadu. India.
**Bharath Niketan College Of Engineering Aundipatti - 625536. Tamilnadu. India.
***Saiva Bhanu Kshatriya College Aruppukottai - 626101. Tamilnadu. India.

Abstract: Recently in [6], the authors have introduced the notion of semiring valued graphs, which is a generalization of both the crisp graph and fuzzy graph. In [3] the authors have studied the regularity conditions on S - valued graphs. In [7] the authors have studied the notion of vertex degree regularity on S - valued graphs. In this paper, we study the edge degree regularity of S - valued graphs.

Keywords: Semirings, Graphs, S-valued graphs, \( d_x \) - edge regular graph.

I. Introduction

The concept of semiring was studied by several mathematicians such as Dedekind [2], Krull [5] and H.S.Vandiver [8]. Jonathan Golan [4] in his book, has introduced the notion of S - graph where S is a semiring. However, the theory was not developed further. In [6], the authors have introduced the notion of semiring valued graphs, which is a generalisation of both the crisp graph and fuzzy graph theory. In [3], the authors have studied the regularity conditions on S - valued graphs. In [7] the authors have studied the notion of vertex degree regularity on S - valued graphs. In this paper we study the edge degree regularity of S-valued graphs.

II. Preliminaries

In this section, we recall some basic definitions that are needed for our work.

Definition 2.1: [4] A semiring \((S, +, \cdot)\) is an algebraic system with a non-empty set \(S\) together with two binary operations \(+\) and \(\cdot\) such that
1. \((S, +, 0)\) is a monoid.
2. \((S, \cdot)\) is a semigroup.
3. For all \(a, b, c \in S\), \((b+c) = a \cdot b + a \cdot c\) and \((a+b) \cdot c = a \cdot c + b \cdot c\)
4. \(0 \cdot x = x \cdot 0 = 0 \quad \forall \ x \in S\).

Definition 2.2: [4] Let \((S, +, \cdot)\) be a semiring. \(\preceq\) is said to be a Canonical Pre-order if for \(a, b \in S\), \(a \preceq b\) if and only if there exists an element \(c \in S\) such that \(a + c = b\).

Definition 2.3: [1] A set \(F\) of edges in a graph \(G = (V, E)\) is called an edge dominating set in \(G\) if for every edge \(e \in E - F\) there exist an edge \(f \in F\) such that \(e\) and \(f\) have a vertex in common.

Definition 2.4: [1] A dominating set \(S\) is a minimal edge dominating set if no proper subset of \(S\) is an edge dominating set in \(G\).

Definition 2.5: [6] Let \(G = (V, E \subseteq V \times V)\) be a given graph with \(V, E \neq \emptyset\). For any semiring \((S, +, \cdot)\), a semiring-valued graph (or a \(S\) - valued graph), \(G^S\) is defined to be the graph
\(G^S = (V, E, \sigma, \psi)\) where \(\sigma: V \rightarrow S\) and \(\psi: E \rightarrow S\) is defined to be
\[
\psi(x, y) = \begin{cases} 
\min \{\sigma(x), \sigma(y)\}, & \text{if } \sigma(x) \preceq \sigma(y) \text{ or } \sigma(y) \preceq \sigma(x) \\
0, & \text{otherwise}
\end{cases}
\]
for every unordered pair \((x, y)\) of \(E \subseteq V \times V\). We call \(\sigma,\ A S\ -\ vertex\ set\ and \psi, A S\ -\ edge\ set\ of\ S\ -\ valued\ graph\ G^S\). Henceforth, we call a \(S\) – valued graph simply as a \(S\) - graph.

Definition 2.6: [6] If \(\sigma(x) = a, \forall x \in V\) and some \(a \in S\) then the corresponding \(S\) - graph \(G^S\) is called a vertex regular \(S\) - graph (or simply vertex regular). An \(S\) - graph \(G^S\) is said to be an edge regular \(S\) - graph (or
simply edge regular) if \( \psi(x, y) = a \) for every \((x, y) \in E \) and \( a \in S \). A \( S \)-valued graph is said to be \( S \)-regular if it is both vertex and edge regular.

**Definition 2.7:** [3] Let \( G^S \) be a \( S \)-graph corresponding to an underlying graph \( G \), and let \( a \in S \). \( G^S \) is said to be \((a, k)\)-vertex regular if the following conditions are true.
1. The crisp graph \( G \) is \( k \)-regular.
2. \( \sigma(v) = a, \forall v \in V \)

**Definition 2.8:** [7] The Order of a \( S \)-valued graph \( G^S \) is defined as
\[
P_S = \left( \sum_{v \in V} \sigma(v), n \right)
\]
where \( n \) is order of the underlying graph \( G \).

**Definition 2.9:** [7] The Size of the \( S \)-valued graph \( G^S \) is defined as
\[
q_S = \left( \sum_{(u,v) \in E} \psi(u,v), m \right)
\]
where \( m \) is the number of edges in the underlying graph \( G \).

**Definition 2.10:** [7] The Degree of the vertex \( v \) of the \( S \)-valued graph \( G^S \) is defined as
\[
\deg_S(v) = \left( \sum_{(v,i) \in E} \psi(v), v, j, \ell \right), \text{where} \ \ell \text{ is the number of edges incident with } v.
\]

**Definition 2.11:** A subset \( D \subseteq V \) is said to be a weight dominating vertex set if for each \( v \in D \), \( \sigma(u) \preceq \sigma(v), \forall u \in N_S[v] \).

### III. Degree Regularity on Edges of \( S \) – Valued Graph

In this section, we introduce the notion of the degree of an edge in \( S \)-valued graph \( G^S \), analogous to the notion in crisp graph theory, and discuss the regularity conditions on such edges of \( G^S \). We start with the definition.

**Definition 3.1:** Let \( G^S = (V, E, \sigma, \psi) \) be a \( S \)-valued graph. Let \( e \in E \). The open neighbourhood of \( e \), denoted by \( N_S(e) \), is defined to be the set \( N_S(e) = \{e_j, \psi(e_j)\} e and e_j \in E are adjacent\} \)

The closed neighbourhood of \( e \), denoted by \( N_S[e] \), is defined to be the set \( N_S[e] = N_S(e) \cup \{e, \psi(e)\} \)

**Definition 3.2:** Let \( G^S = (V, E, \sigma, \psi) \) be a \( S \)-valued graph. The degree of the edge \( e \) is defined as
\[
\deg_S(e) = \left( \sum_{e_i \in N_S(e)} \psi(e_i), m \right), \text{where} \ m \text{ is the number of edges adjacent to } e.
\]

**Example 3.3:** Let \( S = \{0, a, b, c\} \), \( +, \cdot \) be a semiring with the following Cayley Tables:

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>\cdot</th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>0</td>
<td>b</td>
<td>b</td>
</tr>
</tbody>
</table>

Let \( \leq \) be a canonical pre-order in \( S \), given by
\[
0 \leq 0, 0 \leq a, 0 \leq b, 0 \leq c, a \leq a, b \leq b, b \leq a, c \leq c, c \leq a, c \leq b
\]

Consider the \( S \)-graph \( G^S \).
Degree Regularity on Edges of $S$ – Valued Graph

Define $\sigma : V \to S$ by $\sigma(v_1) = \sigma(v_3) = a, \sigma(v_2) = \sigma(v_4) = b, \sigma(v_5) = c$ and
\[ \psi : E \to S \text{ by } \psi(e_1) = \psi(e_2) = \psi(e_6) = b, \psi(e_3) = \psi(e_4) = \psi(e_7) = c, \psi(e_5) = a \]

Here $N^S_3(e_1) = \{(e_2, b), (e_5, a), (e_6, b), (e_7, c)\}, \deg^S_3(e_1) = (a, 4)$
\[ N^S_3(e_2) = \{(e_1, b), (e_3, c), (e_6, b), (e_7, c)\}, \deg^S_3(e_2) = (b, 4) \]
\[ N^S_3(e_3) = \{(e_2, b), (e_4, c), (e_7, c)\}, \deg^S_3(e_3) = (b, 3) \]
\[ N^S_3(e_4) = \{(e_3, c), (e_5, a), (e_6, b)\}, \deg^S_3(e_4) = (a, 4) \]
\[ N^S_3(e_5) = \{(e_1, b), (e_4, c), (e_6, b)\}, \deg^S_3(e_5) = (b, 3) \]
\[ N^S_3(e_6) = \{(e_1, b), (e_2, b), (e_4, c), (e_5, a), (e_7, c)\}, \deg^S_3(e_6) = (a, 5) \]
\[ N^S_3(e_7) = \{(e_1, b), (e_2, b), (e_3, c), (e_4, c), (e_6, b)\}, \deg^S_3(e_7) = (b, 5) \]

**Definition 3.4:** If $D \subseteq E$ in $G^S$ then the scalar cardinality of $D$ is defined by
\[ |D|_S = \sum_{e \in D} \psi(e) \]

**Definition 3.5:** Let $G^S = (V, E, \sigma, \psi)$ be a $S$ - valued graph. For any $e \in E$, the neighbourhood degree of $e$ is defined as
\[ N^S \deg(e) = \left[ |N^S_3(e)|, |N^S_5(e)| \right] \]

**Remark 3.6:** From definition 3.2 and 3.4 we observe that the degree of an edge is the same as its neighbourhood degree.

**Remark 3.7:** The scalar cardinality of $N_5[e]$ will be given by
\[ (1) |N^S_5[e]| = |N^S_3(e)| + 1 \]
\[ (2) |N^S_5[e]|_S = |N^S_3(e)|_S + \psi(e) \]

**Definition 3.8:** An $S$ - valued graph $G^S$ is said to be $d_S$ - edge regular if for any $e \in E$, $\deg^S_5(e) = \left[ |N^S_5(e)|, |N^S_5(e)| \right]$

**Example 3.9:** Consider the semiring $(S = \{0, a, b, c\}, +, \cdot)$ with canonical pre-order given in example 3.3 Consider the $S$ - graph $G^S$. 

DOI: 10.9790/5728-1205072227 www.iosrjournals.org 24 | Page
Define $\sigma : V \rightarrow S$ by $\sigma(v_1) = \sigma(v_2) = b, \sigma(v_4) = \sigma(v_3) = a$ and $\psi : E \rightarrow S$ by $\psi(e_1) = \psi(e_3) = \psi(e_4) = c, \psi(e_2) = \psi(e_5) = \psi(e_6) = b$. Here degree of every edge $e_i \in E$ is $(b, 3)$.

$\therefore G^S$ is an $d_S$-edge regular graph, $d_S(e) = (b, 3)$.

**Remark 3.10:** In terms of neighbourhood of an edge, definition 2.9 can be modified as $q_S = \left( \sum_{e \in E} \psi(e), q \right)$, where $q$ is the number of edges in the underlying graph $G$.

**Theorem 3.11:** If $S$ is an additively idempotent semiring and $G^S$ is $S$-regular then $\deg_S(e) \leq q_S, \forall e \in E$.

**Proof:**

Let $G^S = (V, E, \sigma, \psi)$ be $S$-regular.

$\therefore \sigma(v_i) = a, \forall i$ and for some $a \in S$

$\Rightarrow \psi(e_i) = a, \forall i$ and for some $a \in S$

Since $S$ is additively idempotent, $a + a = a, \forall a \in S$

Let $e \in E$

Now, $q_S = \left( \sum_{e \in E} \psi(e), q \right) = (a, q)$, where $q$ is the number of edges in $G$.

$= (a; q)$ where $q$ is the number of edges in $G$.

And $\deg_S(e) = \left( \sum_{e_i \in N_S(e)} \psi(e_i), m \right) = (a, m)$, where $m$ is number of edges adjacent with $e$.

Since $S$ is a semiring, it possess a canonical pre-order.

$\therefore a \leq a, \forall a \in S$

Clearly $q \geq m, \therefore \deg_S(e) \leq q_S$ for all $e \in E$.

Since every $(a, k)$- regular $S$- valued graph is $S$-regular, the above theorem holds good for $(a, k)$-regular $S$-valued graphs on an additively idempotent semiring. Thus we have the following

**Corollary 3.12:** An $(a, k)$-regular $S$-valued graph $G^S$ on an additively idempotent semiring $S$ satisfies $\deg_S(e) \leq q_S, \forall i$.

**Theorem 3.13:** Let $a \in S$ be an additively idempotent element in $S$. Then every $(a, k)$-regular $S$-valued graph $G^S$ is $d_S$-edge regular iff $\deg_S(e) \leq (a, k)$ for some $a$ and $\forall e \in E$.

**Proof:**

Let $G^S = (V, E, \sigma, \psi)$ be a $(a, k)$-regular $S$-valued graph.

Assume that $G^S$ is $d_S$-edge regular.

Then $\deg_S(e) = (b, k), \forall i$ and for some $b \in S$

That is
\[
\begin{aligned}
&\sum_{e_i \in N_S(e)} \psi(e_i), k = (b, k) \\
&\left( a + a + a + \ldots + a, k \right) = (b, k) \\
&(a, k) = (b, k) \Rightarrow a = b
\end{aligned}
\]

Let the pole \( v \) be a (a, k)-regular and a be an additively idempotent element in \( S \), and \( \deg_S(e) = (a, k) \) for some a and for each \( e \in E \).

Let \( v_1, v_2 \in V \) be such that \( e = (v_1, v_2) \in E \).

Since \( G^S \) is (a,k) regular, \( \sigma(v_1) = \sigma(v_2) = a \).

Then \( \psi(e) = \min \{ \sigma(v_1), \sigma(v_2) \} = a \).

This true for every edge \( e_i = (v_j, v_k) \in E, \psi(e_i) = a, \forall i \).

Now \( \deg_S(e) = \left( \sum_{e_i \in N_S(e)} \psi(e_i), k \right) = (a, k) \).

Since \( e \in E \) is arbitrary, \( G^S \) is a \( d_k \)-edge regular.

**Theorem 3.14:** Let \( G^S \) be a complete bipartite graph with \( V = (V_1, V_2) \). If \( |V_1| < |V_2| \) and \( V_1 \) is a weight dominating vertex set then \( G^S \) is a \( d_k \)-edge regular graph.

**Proof:**

Let \( G^S \) be a complete bipartite graph with \( V = (V_1, V_2) \).

Let \( V_1 \) be a weight dominating vertex set and \( |V_1| < |V_2| \).

Then \( \sigma(v) \leq \sigma(v), \forall v \in V_1, v_j \in V_2 \).

Consider \( \deg_S(e_{ij}) = \left( \sum_{e_{ij} \in N_S[e_{ij}]} \psi(e_{ij}), |V_2| \right) = \left( \sum_{v_s \in N_S[v_1]} \sigma(v_s), |V_2| \right) \)

And \( \deg_S(e_{ik}) = \left( \sum_{e_{ik} \in N_S[e_{ik}]} \psi(e_{ik}), |V_2| \right) = \left( \sum_{v_r \in N_S[v_1]} \sigma(v_r), |V_2| \right) \).

Here \( \deg_S(e_{ij}) = \deg_S(e_{ik}) \) iff \( \sum \sigma(v_s) = \sum \sigma(v_r) = a, \) for some \( a \in S \).

Therefore for any edge \( e_{ij}, e_{ik} \in N_S[V_1], \deg_S(e_{ij}) = \deg_S(e_{ik}) = (a, |V_2|) \).

Therefore \( G^S \) is a \( d_k \)-edge regular graph.

**Corollary 3.15:** Let \( G^S \) be a complete bipartite graph with \( V = (V_1, V_2) \). If \( |V_1| = |V_2| \) and either \( V_1 \) or \( V_2 \) is a weight dominating vertex set then \( G^S \) is a \( d_k \)-edge regular graph.

**Theorem 3.16:** Let \( G^S \) be a star with \( n \) vertices. If its pole has the maximum weight then \( G^S \) is a \( d_k \)-edge regular graph.

**Proof:**

Let \( G^S \) be a star with \( n \) vertices.

Let the pole \( v_1 \) have the maximum weight.

Then \( \sigma(v_j) \leq \sigma(v_1), \forall v_j \in V - \{v_1\} \).
Consider \( \deg_S(e_{ij}) = \left( \sum_{e_{1s} \in N_S[v_{ij}]} \psi(e_{1s}), n-2 \right) = \left( \sum_{v_s \in N_S[v_1]} \sigma(v_s), n-2 \right) \)

And \( \deg_S(e_{ik}) = \left( \sum_{e_{1r} \in N_S[v_{1k}]} \psi(e_{1r}), n-2 \right) = \left( \sum_{v_r \in N_S[v_1]} \sigma(v_r), n-2 \right) \)

Here \( \deg_S(e_{ij}) = \deg_S(e_{ik}) \) iff \( \sum_{v_s} \sigma(v_s) = \sum_{v_r} \sigma(v_r) = a \) for some \( a \in S \)

Therefore for any edge \( e_{ij}, e_{ik} \in N_s[v_1], \deg_S(e_{ij}) = \deg_S(e_{ik}) = (a, n-2) \)

Therefore \( G \) is a \( d_s \)-edge regular graph.

### IV. Conclusion

Unlike the crisp graph theory, in \( S^- \) valued graphs, in this paper we have introduced the notion of degree of an edge in \( S^- \) valued graph. Also we have studied the degree regularity conditions on the edges of \( S^- \) valued graph. In our future work, we would like to extend the study of \( S^- \) valued graphs with irregularity conditions.

### Reference


