Cubic fuzzy H-ideals in BF-Algebras

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Abstract: In this paper, we introduce the notion of cubic fuzzy H-ideals in BF-algebras and prove some interesting properties.

I. Introduction

Zadeh [8] has introduced the concept of fuzzy subsets in 1965. This concept has been widely adopted and applied to many disciplines. Zhan and Tan [11] introduced the notion of fuzzy H-ideals in BCK-algebras and Satyanarayana et.al ([5], [6]) studied intuitionistic fuzzy H-ideals in BCK-algebras. Jun. et.al. [2] introduced the notion of cubic sets. In this paper we introduce the notion of cubic fuzzy H-ideals in BF-algebras and investigate some of its properties.

Definition 1.1([4], [7]). A BF-algebra is a non-empty set X with a constant 0 and a binary operation * satisfying the following axioms:

(i) x * x = 0,
(ii) x * 0 = x,
(iii) 0 *(x * y) = (y * x) for all x, y ∈ X.

Definition 1.2. A subset I of a BF-algebra (X, *, 0) is called an ideal of X, if for any x, y ∈ X

1. 0 ∈ I
2. x * y and y ∈ I ⇒ x ∈ I

Definition 1.3. [11] A non-empty subset I of X is called an H-ideal of X if

1. 0 ∈ I
2. x *(y * z) ∈ I and y ∈ I ⇒ x * z ∈ I

Since x * 0 = x, It is clear that every H-ideal is an ideal.

Definition 1.4. [11] A fuzzy subset  in a BF-algebra X is called a fuzzy H-ideal of X if

(i)  μ(0) ≥ μ(x)
(ii) μ(x * z) ≥ min{μ(x *(y * z)), μ(y)}, for all x, y, z ∈ X.

Since x * 0 = x, It is clear that every fuzzy H-ideal is an fuzzy ideal.

The determination of maximum and minimum between two real numbers is very simple, but it is not simple for two intervals. Biswas [1] described a method to find max/sup and min/inf between two intervals and set of intervals. By an interval number  on [0, 1], we mean an interval [a−, a+] where 0 ≤ a− ≤ a+ ≤ 1. The set of all closed subintervals of [0, 1] is denoted by D[0, 1]. The interval [a, a] is identified with the number a ∈ [0, 1].

For an interval numbering  of D[0, 1], we define

\[ \inf \bar{a}_i = \left[ \min_{i \in I} a_i^-, \min_{i \in I} b_i^+ \right] \quad \text{and} \quad \sup \bar{a}_i = \left[ \max_{i \in I} a_i^-, \max_{i \in I} b_i^+ \right] \]

And put

\[ \bar{a}_1 \wedge \bar{a}_2 = \min(\bar{a}_1, \bar{a}_2) = \min(\left[ a_1^-, b_1^+ \right], \left[ a_2^-, b_2^+ \right]) \]
\[ \begin{align*}
\mu, \lambda & \text{ is fuzzy set in } X. \\
\mu, \lambda & \text{ be a non-fuzzy set in } X. \\
\mu, \lambda & \text{ in } X. \\
\mu, \lambda & \text{ is a cubic BF subalgebra in } X. \\
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\mu, \lambda & \text{ is a cubic BF subalgebra in } X.
\end{align*} \]

Let X be a given nonempty set. An interval valued fuzzy set (briefly, i-v fuzzy set) \( B \) on X is defined by \( B = \{(x, [\mu^-(x), \mu^+(x)]) : x \in X\} \), where \( \mu^- \) and \( \mu^+ \) are fuzzy sets of X such that \( \mu^- (x) \leq \mu^+ (x) \) for all \( x \in X \). Let \( B = \{(x, [\mu^-(x), \mu^+(x)]) : x \in X\} \), then \( B = \{(x, [\mu^-(x), \mu^+(x)]) : x \in X\} \). Where \( \mu^- : X \rightarrow [0, 1] \).

**Definition 1.5.** Consider two elements \( D_1, D_2 \in D[0, 1] \). If \( D_1 = [a^-_1, a^+_1] \) and \( D_2 = [a^-_2, a^+_2] \), then \( r \min(D_1, D_2) = [\min(a^-_1, a^-_2), \min(a^+_1, a^+_2)] \) which is denoted by \( D_1 \land' D_2 \). Thus if \( D_1 = [a^-_1, a^+_1] \in D[0, 1] \) for \( i = 1, 2, 3, 4, \ldots \) then we define \( r \sup_i(D_i) = [\sup_i(a^-_i), \sup_i(a^+_i)], \) i.e. \( \sup_i(D_i) = [\sup_i(a^-_i), \sup_i(a^+_i)] \). Now we call \( D_1 \geq D_2 \) if and only if \( a^-_1 \geq a^-_2 \) and \( a^+_1 \geq a^+_2 \). Similarly, the relations \( D_1 \leq D_2 \) are defined.

Based on the (interval-valued fuzzy sets, Jun et al. [2] introduced the notion of cubic sets, and investigated several properties.

**Definition 1.6.** Let X be a non-empty set. A cubic set \( A \) in X is a Structure which is briefly denoted by \( A = (\mu, \lambda) \) where \( \mu = [\mu^-_1, \mu^+_1] \) is an interval valued fuzzy set in X and \( \lambda = \mu^- \) is fuzzy set in X.

**Definition 1.7.** Let \( A = (\mu, \lambda) \) be cubic set in X, where X is BF subalgebra, then the set A is cubic BF subalgebra over the binary operation * if it satisfies the following conditions

(i) \( \mu (x \ast y) \geq r \min \{\mu (x), \mu (y)\} \)

(ii) \( \lambda (x \ast y) \leq \max \{\lambda (x), \lambda (y)\} \).

**Proposition 1.8.** If \( A = (\mu, \lambda) \) is a cubic BF subalgebra in X, then for all \( x \in X \), \( \mu_A (0) \geq \mu_A (x) \) and \( \lambda_A (0) \leq \lambda_A (x) \). Thus \( \mu_A (0) \) and \( \lambda_A (0) \) are the upper bounds and lower bounds of \( \mu_A (x) \) and \( \lambda_A (x) \) respectively.

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Cubic fuzzy H-ideals in BF-algebras

In this section, we apply the concept of cubic fuzzy set to H-ideal of BF-algebras and introduced the notions of cubic fuzzy H-ideals of BF-algebras and investigate some of its related properties.

**Definition 2.1.** Let \( A = (X, \lambda_A, \mu_A) \) be cubic fuzzy set in \( X \), where \( X \) is a BF algebra, then the set \( A \) is cubic fuzzy ideal over the binary operation \(*\) if it satisfies the following conditions:

(CF1) \( \mu_A(0) \geq \mu_A(x) \) and \( \lambda_A(0) \leq \lambda_A(x) \)

(CF2) \( \mu_A(x) \geq r \min \{ \mu_A(x+y), \mu_A(y) \} \)

(CF3) \( \lambda_A(x) \leq \max \{ \lambda_A(x+y), \lambda_A(y) \} \) for all \( x, y \in X \).

**Definition 2.2.** A non empty sub set \( I \) of BF-algebra \( X \) is called an H-ideal of \( X \), if

(i) \( 0 \in I \)

(ii) \( x \ast (y \ast z) \in I \) and \( y \in I \Rightarrow x \ast z \in I \)

**Definition 2.3.** A cubic fuzzy set \( A = (X, \lambda_A, \mu_A) \) in a BF-algebra \( X \) is called a cubic fuzzy H-ideal of \( X \), if

(CFH1) \( \mu_A(0) \geq \mu_A(x) \) and \( \lambda_A(0) \leq \lambda_A(x) \)

(CFH2) \( \mu_A(x \ast z) \geq r \min \{ \mu_A(x \ast (y \ast z)), \mu_A(y) \} \) and \( \lambda_A(x \ast z) \leq \max \{ \lambda_A(x \ast (y \ast z)), \lambda_A(y) \} \) for all \( x, y, z \in X \).

**Proposition 2.4.** Every cubic fuzzy H-ideal \( A = (X, \lambda_A, \mu_A) \) is a cubic fuzzy ideal.

**Proof:** By setting \( Z = 0 \), in (CFH 2) and (CFH 3) we get

\[
\mu_A(x) \geq r \min \{ \mu_A(x \ast y), \mu_A(y) \} \quad \text{and} \quad \lambda_A(x) \leq \max \{ \lambda_A(x \ast y), \lambda_A(y) \}
\]

for all \( x, y \in X \). Therefore \( A = (X, \lambda_A, \mu_A) \) is a cubic fuzzy ideal of \( X \).

**Theorem 2.5.** Let \( A = (X, \lambda_A, \mu_A) \) be a cubic fuzzy H-ideal of \( X \), if there is a sequence \( \{ x_n \} \) in \( X \) such that

(i) \( \lim_{n \to \infty} \lambda_A(x_n) = [1,1] \) then \( \lambda_A(0) = [1,1] \)

(ii) \( \lim_{n \to \infty} \lambda_A(x_n) = 0 \) when \( \lambda_A(0) = 0 \).

**Proof:** Since \( \mu_A(0) \geq \mu_A(x) \) for all \( x \in X \),

Therefore, \( \mu_A(0) \geq \mu_A(x_n) \) for every positive integer \( n \).

Consider \( [1,1] \geq \mu_A(0) \geq \lim_{n \to \infty} \mu_A(x_n) = [1,1] \)

Hence \( \mu_A(0) = [1,1] \). Since \( \lambda_A(0) \leq \lambda_A(x) \) for all \( x \in X \),

Thus \( \lambda_A(0) \leq \lambda_A(x_n) \) for every positive integer \( n \). Now \( 0 \leq \lambda_A(0) \leq \lim_{n \to \infty} \lambda_A(x_n) = 0 \)

Hence \( \lambda_A(0) = 0 \).

**Theorem 2.6.** A cubic fuzzy set \( A = (X, \lambda_A, \mu_A) \) in \( X \) is a cubic fuzzy H-ideal of \( X \) if and only if \( \mu_A, \mu_A^+ \) and \( \lambda_A \) are fuzzy H-ideals of \( X \).

**Proof:** Let \( \mu_A, \mu_A^+ \) and \( \lambda_A \) are fuzzy H-ideals of \( X \) and \( x, y, z \in X \).

Then by definition \( \mu_A^-(0) \geq \mu_A^-(x), \mu_A^+(0) \geq \mu_A^+(x) \),

\[
\mu_A^-(x \ast z) \geq \min \{ \mu_A^-(x \ast (y \ast z)), \mu_A^-(y) \} \]

\[
\mu_A^+(x \ast z) \geq \min \{ \mu_A^+(x \ast (y \ast z)), \mu_A^+(y) \} \]

\[
\lambda_A(x \ast z) \leq \max \{ \lambda_A(x \ast (y \ast z)), \lambda_A(y) \} \]

Now \( \mu_A(x \ast z) = [\mu_A^-(x \ast z), \mu_A^+(x \ast z)] \)

\[
\geq \min \{ \mu_A^-(x \ast (y \ast z)), \mu_A^+(y) \}, \min \{ \mu_A^+(x \ast (y \ast z)), \mu_A^-(y) \} \]

\[
= \min \{ \mu_A^-(x \ast (y \ast z)), \mu_A^+(x \ast (y \ast z)), [\mu_A^-(y), \mu_A^+(y)] \}
\]

\[
= \min \{ \mu_A(x \ast (y \ast z)), \mu_A(y) \}
\]

Therefore \( A \) is cubic fuzzy H-ideal of \( X \).
Conversely assume that $A = (X, \tilde{\mu}_A, \lambda_A)$ is cubic fuzzy H-ideal of X. For any $x, y, z \in X$,
\[
[\mu_A^-(x \ast z), \mu_A^+(x \ast z)] = \tilde{\mu}_A(x \ast z) \\
\geq r \min\{\tilde{\mu}_A(x \ast (y \ast z)), \tilde{\mu}_A(y)\} \\
= r \min\{[\mu_A^-(x \ast (y \ast z)), \mu_A^+(x \ast (y \ast z))], [\mu_A^-(y), \mu_A^+(y)]\} \\
= [\min\{\mu_A^-(x \ast (y \ast z)), \mu_A^-(y)\}, \min\{\mu_A^+(x \ast (y \ast z)), \mu_A^+(y)\}] \\
\]
Thus
\[
\mu_A^-(x \ast z) \geq \min\{\mu_A^-(x \ast (y \ast z)), \mu_A^-(y)\}, \\
\mu_A^+(x \ast z) \geq \min\{\mu_A^+(x \ast (y \ast z)), \mu_A^+(y)\}, \\
\lambda_A(x \ast z) \leq \max\{\lambda_A(x \ast (y \ast z)), \lambda_A(y)\}. \\
\]
Hence $\mu_A^-, \mu_A^+$ and $\lambda_A$ are fuzzy H-ideals of X.

**Theorem 2.7.** Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy H-ideal of BF-algebra X and let $n \in \mathbb{N}$ (the set of natural numbers). Then

(i) $\tilde{\mu}_A\left(\prod^n x \ast x\right) \geq \tilde{\mu}_A(x)$, for any odd number n.

(ii) $\tilde{\lambda}_A\left(\prod^n x \ast x\right) \leq \tilde{\lambda}_A(x)$, for any odd number n.

(iii) $\tilde{\mu}_A\left(\prod^n x \ast x\right) = \tilde{\mu}_A(x)$, for any even number n.

(iv) $\tilde{\lambda}_A\left(\prod^n x \ast x\right) = \tilde{\lambda}_A(x)$, for any even number n.

**Theorem 2.8.** If $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy H-ideal of BF-algebra X, then the non empty upper s-level cut $U(\tilde{\mu}_A; \tilde{s})$ and non-empty lower t-level cut $L(\lambda_A; \tilde{t})$ are H-ideals of X, for any $\tilde{s} \in \mathbb{D}[0,1]$ and $\tilde{t} \in [0,1]$.

**Proof:** proof is straight forward.

**Corollary 2.9.** Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be cubic fuzzy set. If A is a cubic fuzzy H-ideal of BF-algebra X then the sets $J = \{x \in X/ \tilde{\mu}_A(x) = \tilde{\mu}_A(0)\}$ and $K = \{x \in X/ \lambda_A(x) = \lambda_A(0)\}$ are H-ideals of X.

**Proof:** Since $0 \in X$, $\tilde{\mu}_A(0) = \tilde{\mu}_A(0)$ and $\lambda_A(0) = \lambda_A(0)$ implies $0 \in J$ and $0 \in K$, So $J \neq \emptyset$ and $K \neq \emptyset$. Let $x \ast (y \ast z) \in J$, $y \in J \Rightarrow \tilde{\mu}_A(x \ast (y \ast z)) = \tilde{\mu}_A(0)$ and $\tilde{\mu}_A(y) = \tilde{\mu}_A(0)$.

Since $\tilde{\mu}_A(x \ast z) \geq \min\{\tilde{\mu}_A(x \ast (y \ast z)), \tilde{\mu}_A(y)\} = \min\{\tilde{\mu}_A(0), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(0)$ it follows that $x \ast z \in J$, for all $x, y, z \in X$. Thus J is H-ideal of X.

Let $x \ast (y \ast z) \in K$, $y \in K \Rightarrow \lambda_A(x \ast (y \ast z)) = \lambda_A(0)$ and $\lambda_A(y) = \lambda_A(0)$.

Since $\lambda_A(x \ast z) \leq \max\{\lambda_A(x \ast (y \ast z)), \lambda_A(y)\} = \max\{t, t\} = t$ but $\lambda_A(0) \leq \lambda_A(x \ast z)$.

Therefore $x \ast z \in K$, for all $x, y, z \in X$. Thus K is H-ideal of X.

**Theorem 2.10.** Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy ideal of BF-algebra X. If $\tilde{\mu}_A(x \ast y) \geq \tilde{\mu}_A(x)$ and $\lambda_A(x \ast y) \leq \lambda_A(x)$ for any $x, y \in X$, then $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy H-ideal of BF-algebra X.

**Proof:** Let $A = (X, \tilde{\mu}_A, \lambda_A)$ be a cubic fuzzy ideal of BF-algebra X. If $\tilde{\mu}_A(x \ast y) \geq \tilde{\mu}_A(x)$ and $\lambda_A(x \ast y) \leq \lambda_A(x)$ for any $x, y \in X$. We have by hypothesis
\[
r \min\{\tilde{\mu}_A((x \ast z) \ast (y \ast z)), \tilde{\mu}_A((y \ast z) \ast (x \ast z))\} \leq \min\{\tilde{\mu}_A((x \ast z) \ast (y \ast z)), \tilde{\mu}_A((y \ast z) \ast (x \ast z))\} \\
\]

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\[ \leq \tilde{\mu}_A(y^*z) \]
\[ \tilde{\mu}_A(y^*z) \geq \min \{\tilde{\mu}_A(x^*(y^*z)),\tilde{\mu}_A(y)\} \], and
\[ \max \{\tilde{\lambda}_A(x^*(y^*z)),\tilde{\lambda}_A(y)\} \geq \max \{\tilde{\lambda}_A((x^*z)^*(y^*z)),\tilde{\lambda}_A(y^*z)\} \]
\[ \geq \tilde{\lambda}_A(y^*z) \]
\[ \tilde{\lambda}_A(y^*z) \leq \max \{\tilde{\lambda}_A(x^*(y^*z)),\tilde{\lambda}_A(y)\} \].

Hence \( A = (X,\tilde{\mu}_A,\tilde{\lambda}_A) \) is a cubic fuzzy \( H \)-ideal of BF-algebra \( X \).

**Definition 2.11.** Let \( f \) be a mapping from a set \( X \) into a set \( Y \). Let \( B = (\tilde{\mu}_B,\tilde{\lambda}_B) \) be cubic fuzzy set in \( Y \). Then the inverse image of \( B \) is defined as \( f^{-1}(B) = \{ (x, f^{-1}(\tilde{\mu}_B), f^{-1}(\tilde{\lambda}_B)) \mid x \in X \} \) with the membership function and non-membership function respectively given by \( f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_B(f(x)) \) and \( f^{-1}(\tilde{\lambda}_B)(x) = \tilde{\lambda}_B(f(x)) \). It can be shown that \( f^{-1}(B) \) is cubic fuzzy set.

**Theorem 2.12.** Let \( f : X \rightarrow Y \) be a homomorphism of BF-algebras. If \( B = (\tilde{\mu}_B,\tilde{\lambda}_B) \) is a cubic fuzzy \( H \)-ideal of \( Y \), then the pre image \( f^{-1}(B) = \{ (x, f^{-1}(\tilde{\mu}_B), f^{-1}(\tilde{\lambda}_B)) \mid x \in X \} \) of \( B \) under \( f \) is a cubic fuzzy \( H \)-ideal of \( X \).

**Proof:** Assume that \( B = (\tilde{\mu}_B,\tilde{\lambda}_B) \) is a cubic fuzzy \( H \)-ideal of \( Y \). Let \( x,y,z \in X \Rightarrow f(x), f(y), f(z) \in Y \) Consider \( f^{-1}(\tilde{\mu}_B)(0) = \tilde{\mu}_B(f(0)) \geq \tilde{\mu}_B(f(x)) = f^{-1}(\tilde{\mu}_B)(x) \) and
\[ f^{-1}(\tilde{\lambda}_B)(0) = f^{-1}(\tilde{\lambda}_B)(x) \leq f^{-1}(\tilde{\lambda}_B)(x). \]
Thus \( f^{-1}(\tilde{\mu}_B)(x^*z) = \tilde{\mu}_B(f(x^*z)) \geq \min \{\tilde{\mu}_B(f(x^*y^*z)),\tilde{\mu}_B(f(y))\} \]
\[ = \min \{f^{-1}(\tilde{\mu}_B)(x^*(y^*z)),f^{-1}(\tilde{\mu}_B)(y)\} \]
And \( f^{-1}(\tilde{\lambda}_B)(x^*z) = \tilde{\lambda}_B(f(x^*z)) \leq \max \{\tilde{\lambda}_B(f(x^*(y^*z)),\tilde{\lambda}_B(f(y))\} \]
\[ = \max \{f^{-1}(\tilde{\lambda}_B)(x^*(y^*z)),f^{-1}(\tilde{\lambda}_B)(y)\} \]
Therefore \( f^{-1}(B) = \{ (x, f^{-1}(\tilde{\mu}_B), f^{-1}(\tilde{\lambda}_B)) \mid x \in X \} \) is a cubic fuzzy \( H \)-ideal of \( Y \).

**References**