Effects of MHD on the Unsteady Thin Flow of Non-Newtonian Oldroyd–B Fluid over an Oscillating Inclined Belt Through a Porous Medium

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Abstract: The unsteady (MHD) thin film flow of an incompressible Oldroyd–B fluid over an oscillating inclined belt making a certain angle with the horizontal through a porous medium is analyzed. The analytical solution of velocity field is obtained by a semi-numerical technique optimal homotopy asymptotic method (OHAM). Finally, the influence of various dimensionless parameters emerging in the model on velocity field is analyzed by graphical illustrations.

I. Introduction

In everyday life and in engineering flow of non-Newtonian fluid are frequently occurred. It is ubiquitous in nature and technologies. Therefore, to understand these mechanics is essential in most applications. Most of the problems appeared in several examples are polymer solutions, paints, certain oils, exotic lubricants, colloidal and suspension solutions, clay coatings and cosmetic products. In non-Newtonian fluids thin film flows have a large varying of practical applications in nonlinear science and engineering industries.

For modeling of non-Newtonian fluid flow problem we used several model like (second grade, third grade, Maxwell fluid, Oldroyd-B) exact fluid model by which we got linear or nonlinear ordinary differential equation or partial differential equations which may be solved exactly, analytically or numerically. For solution of these problems different varieties of methods are used, in which (ADM, VAM, HAM, HPM, OHAM) are frequently used. In this work we have modeled a partial differential equations by using oldroyd-B fluid model. Solution of the problem is obtained by using OHAM. Fetecau et al. [3] studied the exact solutions of an Oldroyd-B fluid over a flat plate on constantly accelerating flow. In the following year, Fetecau et al. [4] obtained the exact solutions of the transient oscillating motion of an Oldroyd-B fluids in cylindrical domains.

II. Governing Equation

The continuity and momentum equations for an unsteady magnetic hydrodynamic (MHD) incompressible flow over an inclined belt through porous medium, defined by the following equations

\[ \text{div} \ V = 0 \]

\[ \rho \frac{\partial V}{\partial t} = \text{div} \ T + \rho g \sin \theta + J \times B + R \]

Where \( T \) denotes the Cauchy stress, \( V \) is the velocity vector of fluid, \( \rho \) is the fluid density, \( g \) is the external body force, \( B \) is the magnetic field, and \( J \) is current density (or conduction current).

The Cauchy stress tensor \( T \) for incompressible Oldroyd–B viscous fluid is defined by the constitutive equation

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\[ T = -pl + S \] (3)

where \( p \) is the pressure, \( I \) the identity tensor, the extra stress tensor \( S \) satisfies

\[ S + \lambda_1 \frac{DS}{Dt} = \mu \left(1 + \lambda_2 \frac{D}{Dt} \right) A \] (4)

Where \( \mu \) the dynamic viscosity, \( \lambda_1, \lambda_2 \) are the relaxation and retardation times respectively, and \( A \) is the first Rivlin–Ericksen tensor and \( \frac{D}{Dt} \) the upper convected derivative defined as follows.

\[ A = L + L^T, \quad L = \text{grad} \, V \] (5)

\[ \frac{DS}{Dt} = \frac{\partial S}{\partial t} + V \cdot \nabla S - \dot{S}L - \ddot{S}L^T \] (6)

\[ \frac{D}{Dt} A = \frac{\partial A}{\partial t} + V \cdot \nabla A - \dot{A}L - \ddot{A}L^T \] (7)

Where \( \nabla \) is the gradient operator.

We assume that the velocity field and the shear stress of the form

\[ V = (u(y, t), 0, 0) \quad \text{and} \quad S = S(y, t) \] (8)

Where \( u(y, t) \) is the velocity. Since \( V \) is dependent of \( y \) and \( t \), we also assume that \( S \) dependent only on \( y \) and \( t \).

If the fluid being at rest up to the moment \( t=0 \), then

\[ S(y, 0) = 0, \quad y > 0 \] (9)

We can get

\[ S_{xx} + \lambda_1 \left( \frac{\partial S_{xx}}{\partial t} - 2 \frac{\partial u}{\partial y} S_{yx} \right) = -2 \mu \lambda_2 \left( \frac{\partial u}{\partial y} \right)^2 \] (10)

\[ S_{xy} + \lambda_1 \left( \frac{\partial S_{xy}}{\partial t} - \frac{\partial u}{\partial y} S_{yy} \right) = \mu \frac{\partial u}{\partial y} + \mu \lambda_2 \frac{\partial^2 u}{\partial t \partial y} \] (11)

\[ S_{yy} + \lambda_1 \frac{\partial S_{yy}}{\partial t} = 0 \] (12)

then Eq.(12) reduces to

\[ S_{yy} = C(y)e^{-\frac{y}{\delta_1}} \] (13)

where \( C(y) \) is arbitrary function. From Eq.(9), \( C(y) = 0 \).

where \( S_{xx} = S_{yy} = S_{xx} = 0, S_{xy} = S_{yx} \).

### III. Statement of the problem

This paper consider a thin film flow of a non-Newtonian Oldroyd –B fluid on an oscillating inclined belt. We assumed the flow is unsteady, laminar, incompressible, and the pressure gradient is zero. The force of gravity has been initiated the motion of a layer of liquid in the downward direction. Thickness, \( \delta \), of the liquid layer is assumed to be uniform. A magnetic field is applied to the belt in the direction perpendicular to fluid motion, as shown in Fig. (1). The boundary conditions are given by

\[ u(0, t) = V \cos \omega t, \quad t > 0 \] (14)

\[ \frac{\partial u(y, t)}{\partial y} = 0 \quad \text{as} \quad y \to \infty, \quad t > 0 \] (15)

where \( \omega \) is oscillating belt

![Fig.(1): Geometry of the problem](image)

### IV. Momentum and continuity Equation

By using the above argument we will write the formula of the momentum equations which governing the magnetohydrodynamic in \( x \)-direction as follows:

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\[
\frac{\partial u}{\partial t} = \frac{\partial \rho}{\partial x} + \rho g \sin \theta - \sigma B_0^2 u - \frac{\mu}{K} u
\]  
(16)

Differential Eq.(11) with respect to \( y \), we get
\[
\frac{\partial S_{xy}}{\partial y} + \lambda_1 \frac{\partial}{\partial y} \left( \frac{\partial S_{xy}}{\partial x} - \frac{\partial S_{yx}}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} + \mu \lambda_2 \frac{\partial^3 u}{\partial t \partial y^2}
\]
\( \left( 1 + \lambda_1 \frac{\partial}{\partial y} \right) S_{xy} - \lambda_1 \frac{\partial^2 u}{\partial y^2} S_{xy} + \frac{\partial v}{\partial y} S_{yy} = \mu \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} \)  
(17)

Multiply Eq.(18) by \( \left( 1 + \lambda_1 \frac{\partial}{\partial y} \right)^{-1} \), we get
\[
\frac{\partial S_{xy}}{\partial y} = \frac{\lambda_1}{1 + \lambda_1 \frac{\partial}{\partial y}} \left( \frac{\partial^2 u}{\partial y^2} S_{xy} + \frac{\partial v}{\partial y} S_{yy} \right) + \mu \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} \)  
(19)

Substitute Eq.(19) into Eq.(16), we get
\[
\rho \frac{\partial u}{\partial t} = -\frac{\partial \rho}{\partial x} + \left( \frac{\lambda_1}{1 + \lambda_1 \frac{\partial}{\partial y}} \left( \frac{\partial^2 u}{\partial y^2} S_{xy} + \frac{\partial v}{\partial y} S_{yy} \right) + \mu \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} \right) + \rho g \sin \theta - \sigma B_0^2 u - \frac{\mu}{K} u
\]
(20)

Multiply Eq.(20) by \( 1 + \lambda_1 \frac{\partial}{\partial y} \), we obtain
\[
\rho \left( 1 + \lambda_1 \frac{\partial}{\partial y} \right) \frac{\partial u}{\partial t} = \mu \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} + \rho g \sin \theta - \left( 1 + \lambda_1 \frac{\partial}{\partial y} \right) \sigma B_0^2 u - \left( 1 + \lambda_1 \frac{\partial}{\partial y} \right) \frac{\mu}{K} u
\]
(21)

Since from Eqs.(13) and (9), then \( S_{yy} \) is reduced to zero, which demonstrates that \( C(y) = 0 \). Therefore from Eqs.(13) and (19) and in the presence of zero pressure gradient, we get
\[
\rho \left( 1 + \lambda_1 \frac{\partial}{\partial y} \right) \frac{\partial u}{\partial t} = \mu \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} + \rho g \sin \theta - \left( 1 + \lambda_1 \frac{\partial}{\partial y} \right) \sigma B_0^2 u
\]
(22)

Divide the above equation by \( \rho \), we get
\[
\left( 1 + \lambda_1 \frac{\partial}{\partial y} \right) \frac{\partial u}{\partial t} = \mu \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} + \rho g \sin \theta - \left( 1 + \lambda_1 \frac{\partial}{\partial y} \right) \sigma B_0^2 u
\]
(23)

Introducing non-dimensional variables
\[
u = \frac{\bar{u}}{V}, y = \frac{\bar{y}}{\delta}, \bar{t} = \frac{\bar{t}}{\rho \delta^2}, k_1 = \frac{\lambda_1 \mu}{\rho \delta^2}, k_2 = \frac{\lambda_2 \mu}{\rho \delta^2}, \omega = \frac{\rho \bar{\omega} \delta^2}{\mu}, m = \frac{\delta^2 \rho g \sin \theta}{\mu V}, \]
\[
M = \frac{\sigma B_0^2 \delta^2}{\mu}
\]
(24)

Where \( \omega \) is the oscillating parameter, \( k_1 \) is the relaxation parameter, \( k_2 \) is the retardation parameter, \( m \) is the gravitational parameter and \( M \) is the Magnetic parameter.

We find Eqs.(24), (14) and (15) in dimensionless forms (for simplicity the mark " * " neglected)
\[
\left( 1 + k_1 \frac{\partial}{\partial \bar{y}} \right) \frac{\partial \bar{u}}{\partial \bar{t}} = \left( 1 + k_2 \frac{\partial}{\partial \bar{t}} \right) \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + m - M \left( 1 + k_1 \frac{\partial}{\partial \bar{y}} \right) \bar{u} - \frac{\delta^2}{K} \left( 1 + k_1 \frac{\partial}{\partial \bar{y}} \right) \bar{u}
\]
(25)

and
\[
\bar{u}(0, \bar{t}) = \cos(\omega \bar{t}) \text{ and } \frac{\partial \bar{u}(1, \bar{t})}{\partial \bar{y}} = 0
\]
(26)

V. Basic ideas of Optimal Homotopy Asymptotic Method

To illustrate the basic ideas of optimal homotopy asymptotic method, we will consider the following general of partial differential equation of the form
\[
A(u(y, t)) + Q(y, t) = 0 \quad B(u(y, t), \frac{\partial u(y, t)}{\partial \bar{y}}) = 0 \quad y \in \Omega
\]
(27)

Where \( A \) is a differential operator, \( B \) is a boundary operator, \( u(y, t) \) is unknown function, \( \Omega \) is the problem domain and \( Q(y, t) \) is known analytic function. The operator \( A \) can be decomposed as

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\[ A = L + N \]  \hspace{1cm} (28)

Where \( L \) is linear operator and \( N \) is a nonlinear operator. According to OHAM, one can construct an optimal homotopy \( \varphi(y, t, p) : \Omega \times [0,1] \rightarrow R \) which satisfies

\[ H(\varphi(y, t, p), p) = (1 - p)[L(\varphi(y, t, p) + Q(y, t)] - H(p, c_j)[A(\varphi(y, t, p) + Q(y, t)] = 0 \]  \hspace{1cm} (29)

Where \( p \in [0,1] \) is an embedding parameter and \( H(p, c_j) \) is non-zero auxiliary function, which can be defined in the form

\[ H(p, c_j) = p c_1 + p^2 c_2 + p^3 c_3 + \cdots \]  \hspace{1cm} (30)

Where \( c_1, c_2, c_3, \ldots \) are called convergence control parameters and will be determined accordingly. To obtain an approximate solution, \( \varphi(y, t, p) \) is expanded in series about \( p \) as

\[ \varphi(y, t, pc_1, c_2, c_3, \ldots) = u_0(y, t) + \sum_{k \geq 1} u_k(y, t, p, c_1, c_2, c_3, \ldots)p^k \]  \hspace{1cm} (31)

If \( p = 0 \), then \( H(0, c_j) = 0 \). \( \varphi(y, t, 0) = u_0(y, t) \) and from Eq. (29), we get

\[ H(\varphi(y, t, p), p) = [L(u_0(y, t)) + Qu_0(y, t)] = 0 \]  \hspace{1cm} (32)

and

If \( p = 1 \), then \( H(1, c_j) = 0 \). \( \varphi(y, t, 1) = u(y, t) \) and from Eq. (29), we get

\[ H(\varphi(y, t, 1), 1) = H(1, c_j)[A(u(y, t)) + Qu(y, t)] \]  \hspace{1cm} (33)

Now by inserting Eq. (31) into Eq. (27) and equating the coefficient of like powers of \( p \), we obtain the zero, first and second order are given by

\[ L(u_0(y, t)) + Q(y, t) = 0 \]  \hspace{1cm} (34)

\[ L(u_1(y, t)) = c_1N_0(u_0(y, t)) \]  \hspace{1cm} (35)

\[ L(u_2(y, t)) - L(u_1(y, t)) = c_2N_0(u_0(y, t)) + c_1[L(u_1(y, t)) + N_1(u_0(y, t), u_1(y, t))] \]

\[ B \left( u_2(y, t), \frac{\partial u_2(y, t)}{\partial y} \right) = 0 \]  \hspace{1cm} (36)

And hence, the general governing equation for \( u_j(y, t) \) are given by

\[ L \left( u_j(y, t) \right) - L \left( u_{j-1}(y, t) \right) = c_jN_0(u_0(y, t)) + \sum_{i=1}^{j-1} c_i \left[ L \left( u_{j-1}(y, t) \right) + N_{j-1} \left( u_0(y, t), u_1(y, t), \ldots, u_{j-1}(y, t) \right) \right] \]

\[ B \left( u_j(y, t), \frac{\partial u_j(y, t)}{\partial y} \right) = 0 \]  \hspace{1cm} (37)

Where \( N_m(u_0(y, t), u_1(y, t), \ldots, u_m(y, t)) \) is the coefficient of \( p^m \), obtained by expanding \( N(\varphi(y, t, p, c_1, c_2, c_3, \ldots)) \) is series with respect to the embedding parameter \( p \).

\[ N \left( \varphi(y, t, p, c_1, c_2, c_3, \ldots) \right) = N_0(u_0(y, t)) + \sum_{j=1}^{m} N_j \left( u_0(y, t), u_1(y, t), \ldots, u_j(y, t) \right) p^j \]  \hspace{1cm} (38)

The convergence of the series in Eq. (39) depends upon the auxiliary constants \( c_1, c_2, c_3, \ldots \) if it converges at \( p = 1 \), then the \( m \)th order approximation \( u \) is

\[ u(y, c_1, c_2, c_3, \ldots) = u_0(y, t) + \sum_{j=1}^{m} u_j(y, c_i), i = 1, 2, \ldots, m \]  \hspace{1cm} (39)

Substituting Eq. (38) into Eq. (27), we get the following expression for the residual error

\[ R(y, t, c_i) = L \left( u(y, t, c_i) \right) + Q(y, t) + N(u(y, t, c_i)) \]  \hspace{1cm} (40)

If \( R(y, t, c_i) = 0 \), then \( u(y, c_1, c_2, c_3, \ldots) \) is the exact solution. To find the optimal value of

\[ f(c_i) = \int_{a}^{b} R^2(y, t, c_i)dy \]  \hspace{1cm} (41)

Where the value \( a \) and \( b \) depend on the given problem. The unknown convergence control parameters \( c_i (i = 1, 2, \ldots, m) \) can be optimally identified from the conditions

\[ \frac{\partial f}{\partial c_i} = 0 \]  \hspace{1cm} (42)

Where the constants \( c_1, c_2, c_3, \ldots \) can be determined by using Numerical methods (least square method, Ritz method, Galerkin’s method and collocation method etc.).
VI. Solution of the problem

Rewrite the Eq. (33) in the form
\[
\frac{\partial^2 u}{\partial y^2} + k_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial y^2} \right) - \left( M + \frac{\delta^2}{K} \right) u - \left( \left( M + \frac{\delta^2}{K} \right) k_1 + 1 \right) \frac{\partial u}{\partial t} - k_1 \frac{\partial^2 u}{\partial t^2} + m = 0
\]
and the boundary conditions are
\[
u(0, t) = \cos(\omega t) \quad \text{and} \quad \frac{\partial u(1, t)}{\partial y} = 0
\]
we choose linear operator
\[
L(u(y, t)) = \frac{\partial^2 u}{\partial y^2} + m
\]
\[
A(u(y, t)) = \frac{\partial^2 u}{\partial y^2} + k_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial y^2} \right) - \left( M + \frac{\delta^2}{K} \right) u - \left( \left( M + \frac{\delta^2}{K} \right) k_1 + 1 \right) \frac{\partial u}{\partial t} - k_1 \frac{\partial^2 u}{\partial t^2} + m
\]
and
\[
Q(y, t) = 0
\]
substitute Eqs. (45) and (46) into Eq. (29), we get
\[
(1 - p) \left[ \frac{\partial^2 u}{\partial y^2} + m \right] - H(p, c_1) \left[ \frac{\partial^2 u}{\partial y^2} + k_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial y^2} \right) - \left( M + \frac{\delta^2}{K} \right) u - \left( \left( M + \frac{\delta^2}{K} \right) k_1 + 1 \right) \frac{\partial u}{\partial t} - k_1 \frac{\partial^2 u}{\partial t^2} + m \right] = 0
\]
substitute Eq. (30) and (31) into Eq. (48), we obtain

Zero-order problem given by
\[
p^0: \frac{\partial^2 u_0}{\partial y^2} = -m
\]
and the boundary conditions are
\[
u_0(0, t) = \cos(\omega t) \quad \text{and} \quad \frac{\partial u_0(1, t)}{\partial y} = 0
\]
The solution of Eq. (49) given by
\[
u_0(y, t) = -\frac{m}{2} y^2 + my + \cos(\omega t)
\]
First-order problem given by
\[
p^1: \frac{\partial^2 u_1}{\partial y^2} = (1 + c_1) \frac{\partial^2 u_0}{\partial y^2} + (1 + c_1)m + c_1 k_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 u_0}{\partial y^2} \right) - c_1 \left( M + \frac{\delta^2}{K} \right) u_0
\]
\[
- c_1 \left( M + \frac{\delta^2}{K} \right) k_1 + 1 \right) \frac{\partial u_0}{\partial t} - c_1 k_1 \frac{\partial^2 u_0}{\partial t^2} + m
\]
and the boundary conditions are
\[
u_1(0, t) = \cos(\omega t) \quad \text{and} \quad \frac{\partial u_1(1, t)}{\partial y} = 0
\]
The solution of Eqs. (52) and (53) is given by
\[ u_1(y, t) = \cos(\omega t) - \frac{1}{2} M y^2 c_1 + \frac{m}{24} M y^4 c_1 - \frac{m y^3 \delta^2 c_1}{6K} + \frac{m y^4 \delta^2 c_1}{24K} - \frac{M y^2 \cos(\omega t) c_1}{2K} - \frac{1}{2} y^2 \delta^2 \cos(\omega t) c_1 - \frac{1}{2} y^2 \omega \sin(\omega t) c_1 + \frac{1}{3} m M c_1^2 + \frac{1}{3} M c_1^2 + M \cos(\omega t) c_1^2 + \frac{\delta^2 \cos(\omega t) c_1^2}{K} - \frac{1}{2} y^2 \omega \sin(\omega t) c_1 k_1 + \frac{1}{2} M y^2 \omega \sin(\omega t) c_1 k_1 + \frac{1}{2} y^2 \omega^2 \cos(\omega t) c_1 k_1 + \frac{1}{2} y^2 \omega^2 \cos(\omega t) c_1^2 k_1 - M \omega \sin(\omega t) c_1^2 k_1 - \frac{1}{2} y^2 \omega \sin(\omega t) c_1^2 k_1 - M \omega \sin(\omega t) c_1^2 k_1 + \frac{\delta^2 \omega \sin(\omega t) c_1^2 k_1}{K} \right) \]

(54)

Second –order problem given by

\[ p^2 \frac{\delta^2 u_2}{\delta y^2} = (1 + c_1) \frac{\delta^2 u_1}{\delta y^2} + c_1 k_2 \frac{\delta^2 u_1}{\delta t^2} \left( \frac{\delta^2 u_1}{\delta y^2} \right) - c_1 \left( M + \frac{\delta^2}{K} \right) u_1 - c_1 \left( \frac{M + \frac{\delta^2}{K}}{K} \right) + 1 \]

\[ \frac{\partial u_1}{\partial t} - c_1 k_1 \left( \frac{\delta^2 u_1}{\delta y^2} \right) + c_2 \frac{\delta^2 u_0}{\delta y^2} + c_2 k_2 \frac{\partial^2 u_0}{\partial t^2} \left( \frac{\partial^2 u_0}{\partial y^2} \right) - c_2 \left( M + \frac{\delta^2}{K} \right) u_0 - c_2 \left( \frac{M + \frac{\delta^2}{K}}{K} \right) \]

\[ -c_2 \left( M + \frac{\delta^2}{K} \right) k_1 \frac{\partial u_0}{\partial t} + c_2 m - c_2 k_2 \frac{\partial u_0}{\partial t} - c_2 k_1 \frac{\partial^2 u_0}{\partial t^2} \right) \]

(55)

and the boundary conditions are

\[ u_2(0, t) = \cos(\omega t) \quad \text{and} \quad \frac{\partial u_2(1, t)}{\partial y} = 0 \]

(56)

The solution of Eqs.(55) and (56) is given by

\[ u_2(y, t) = \cos(\omega t) - \frac{1}{2} y^2 (M + \frac{\delta^2}{K}) \cos(\omega t) c_1 + \left( \frac{m y^4}{18} \right) \frac{y^6}{720} + \frac{1}{6} y^3 \cos(\omega t) c_1^2 - \frac{1}{2} y^2 \left( \frac{M + \frac{\delta^2}{K}}{K} \right) \frac{m}{3} \]

\[ + \cos(\omega t) c_1^2 + (1 + c_1) \frac{\delta^2 u_1}{\delta y^2} + \frac{m y^3 \delta^2 c_1}{6K} + \frac{m y^4 \delta^2 c_1}{24K} + \frac{M y^2 \cos(\omega t) c_1}{2K} + \frac{1}{2} y^2 \delta^2 \cos(\omega t) c_1 - \frac{1}{2} y^2 \omega \sin(\omega t) c_1 + \frac{1}{2} y^2 \omega^2 \cos(\omega t) c_1 k_1 + \frac{1}{2} \left( \frac{M + \frac{\delta^2}{K}}{K} \right) \omega^2 \cos(\omega t) c_1^2 k_1 - \frac{1}{2} y^2 \left( M + \frac{\delta^2}{K} \right) \omega^2 \cos(\omega t) c_1^2 k_1 + \frac{1}{24} y^4 \left( M + \frac{\delta^2}{K} \right) \omega^2 \cos(\omega t) c_1^2 k_1 + y \cos(\omega t) c_1 (1 + c_1) k_1 + \frac{1}{2} y^2 \omega^2 \cos(\omega t) c_1 k_1 + \omega \sin(\omega t) c_1 \frac{\delta^2 u_2}{\delta y^2} + \left( \frac{M + \frac{\delta^2}{K}}{K} \right) k_1 + \frac{1}{2} y^2 \left( M + \frac{\delta^2}{K} \right) k_1 \]

\[ + \frac{1}{2} y^2 \left( M + \frac{\delta^2}{K} \right) \omega \sin(\omega t) c_1^2 \left( 1 + \frac{M + \frac{\delta^2}{K}}{K} \right) k_1 + \frac{1}{2} y^2 \omega \sin(\omega t)(1 + c_1)(1 + (M + \frac{\delta^2}{K}) k_1 + \frac{1}{2} y^2 \omega \sin(\omega t) c_1 \frac{\delta^2 u_2}{\delta y^2} + \frac{1}{2} y^2 \omega^2 \sin(\omega t) c_1^2 k_1 (1 + c_1) (1 + (M + \frac{\delta^2}{K}) k_1) - \frac{1}{2} y^2 \omega^2 \sin(\omega t) c_1^2 k_1 (1 + (M + \frac{\delta^2}{K}) k_1) + \frac{1}{24} y^4 \omega^2 \sin(\omega t) c_1^2 k_1 (1 + (M + \frac{\delta^2}{K}) k_1) + \frac{1}{24} y^4 \omega^3 \sin(\omega t) c_1^2 k_1 (1 + (M + \frac{\delta^2}{K}) k_1) + \frac{1}{24} y^4 \omega^3 \sin(\omega t) c_1^2 k_1 (1 + (M + \frac{\delta^2}{K}) k_1) - \frac{1}{24} y^4 \left( M + \frac{\delta^2}{K} \right) \omega \sin(\omega t) c_1^2 k_1 \left( 1 + \left( M + \frac{\delta^2}{K} \right) k_1 \right) - \frac{1}{24} y^4 \left( M + \frac{\delta^2}{K} \right) \omega \sin(\omega t) c_1^2 k_1 \left( 1 + \left( M + \frac{\delta^2}{K} \right) k_1 \right) - \frac{1}{2} y^2 \omega^3 \sin(\omega t) c_1^2 k_1 (1 + (M + \frac{\delta^2}{K}) k_1) + Q \]

(57)

Where Q is a function of \( \omega, t, \delta^2, M, m, y, k_1, c_1, c_2, K \) and \( k_2 \).

Substituting Eqs.(51),(54) and(57) into Eq.(39), we get

\[ u(y, t) = u_0(y, t) + u_1(y, t) + u_2(y, t) \]

(58)
\[ u(y, t) = -\frac{m}{2}y^2 + my + \cos(\omega t) + \cos(\omega t) - \frac{1}{6}m My^3 c_1 + \frac{1}{24}m M y^4 c_1 - \frac{m y^3 \delta^2}{6K} c_1 + \frac{m y^4 \delta^2}{2K} c_1 \\
+ M \cos(\omega t) c_1 + \frac{y^2 \delta^2 \cos(\omega t) c_1}{2K} + \frac{1}{2}y^2 \omega \sin(\omega t) c_1 + \frac{1}{3}m M c_1^2 + \frac{m \delta^2}{3K} c_1^2 + \omega \sin(\omega t) c_1 k_1 \\
+ \frac{1}{2}y^2 \omega \sin(\omega t) c_1 k_1 + \frac{y^2 \delta^2 \omega \sin(\omega t) c_1 k_1}{2K} - \frac{1}{2}y^2 \omega \cos(\omega t) c_1 k_1 - M \omega \sin(\omega t) c_1 k_1 \\
- \frac{y^2 \cos(\omega t) c_1}{K} + \cos(\omega t) - \frac{1}{2}y^2(M + \delta^2 \omega)(\cos(\omega t) c_1 + (\frac{m y^4}{18} - \frac{m y^6}{720} K) + \\
+ \frac{1}{6}y^3 \cos(\omega t) c_1^2 - \frac{1}{2}y^2(M + \delta^2) + \frac{m y^4(M + \delta^2) c_1^3}{3} + (M + \frac{\delta^2}{K})(\frac{m y^3}{6} - \frac{m y^4}{24}) \\
+ \frac{1}{4}y^2 \omega \sin(\omega t) c_1 + \frac{1}{24}y^4(M + \delta^2) c_1^2 + \frac{1}{2}y^2 \omega \cos(\omega t) c_1 k_1 + \frac{1}{2}y^2(M + \delta^2) c_1 k_1 \\
- \frac{1}{2}y^2 \omega \cos(\omega t) c_1 k_1 - \frac{1}{2}y^2(M + \delta^2) \omega \sin(\omega t) c_1 k_1 + \frac{1}{2}y^2(M + \delta^2) \omega \sin(\omega t) c_2 k_1 \\
+ \frac{1}{2}y^2 \omega \sin(\omega t) c_1 (1 + c_1) + \frac{1}{2}y^2 \omega \sin(\omega t)(1 + c_1)(1 + M + \frac{\delta^2}{K} k_1) + \frac{1}{2}y^2 \omega \sin(\omega t)(1 + c_1)(1 + (M + \frac{\delta^2}{K} k_1) \\
+ \frac{1}{2}y^2 \omega \sin(\omega t) c_1^2 (1 + c_1) (1 + (M + \frac{\delta^2}{K} k_1) - \frac{1}{2}y^2 \omega \sin(\omega t) c_1^2 k_1 (1 + (M + \frac{\delta^2}{K} k_1) \\
+ \frac{1}{24}y^4(M + \frac{\delta^2}{K} \omega \sin(\omega t) c_1^2 (1 + (M + \frac{\delta^2}{K} k_1) \\
- \frac{1}{24}y^4(M + \frac{\delta^2}{K} \omega \sin(\omega t) c_1^2 (1 + (M + \frac{\delta^2}{K} k_1) \\
- \frac{1}{2}y^2 \omega \sin(\omega t) c_1^2 k_1 (1 + (M + \frac{\delta^2}{K} k_1) \\
+ \frac{\delta^2}{K} k_1) + Q \) (59) \\
\]

substituting the approximate solution of Eq.(59) into Eq.(40) yields the residual and the functional J, respectively

\[ R(y, t, c_1, c_2) = \frac{\partial^2 u}{\partial y^2} + k_2 \frac{\partial}{\partial t} \left( \frac{\partial^2 u}{\partial y^2} \right) - \left( M + \frac{\delta^2}{K} \right) u - \left( M + \frac{\delta^2}{K} \right) k_1 + 1 \frac{\partial u}{\partial t} - k_1 \left( \frac{\partial^2 u}{\partial t^2} \right) + m \) (60)

\[ J(c_1, c_2) = \int_0^t R(y, t, c_1, c_2)dy \) (61)

Differentiating Eq.(61) with respect to \( c_1 \) and \( c_2 \), respectively and solving the result Equations to calculate the unknown auxiliary constants \( c_1 \) and \( c_2 \), in the particular cases \( m = 0.2, \omega = 0.2, k_1 = 0.5, k_2 = 0.3, M = 0.3, t = 5, \delta = 0.1 \) and \( K = 1 \), we obtain

The values of \( c_i \) for the velocity components

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<thead>
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<th>( c_2 )</th>
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<td>-1.688934901</td>
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<td>-0.164379301</td>
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</tr>
</tbody>
</table>

| Table 1: the values of c_i for the velocity component |

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VII. Numerical results and conclusions

This section presents the effects of controlling parameters on the velocity profile in the form of graphical and tabulated results. In order to validate the accuracy of our approximate solution via OHAM, we have used a semi-numerical technique optimal homotopy asymptotic method (OHAM), for solving unsteady (MHD) thin film flow of an incompressible Oldroyd – B fluid over an oscillating inclined belt making a certain angle with the horizontal in porous medium. The effects of pertinent parameters (define fluid behavior and flow geometries) such as gravitational parameter (m), magnetic parameter (M), relaxation time parameter ($k_1$), retardation time parameter ($k_2$), permeability parameter ($K$), the oscillating parameter $\omega$, time $t$, and the parameters ($c_1$, $c_2$, $\delta$). All the graphs are plotted by using MATHEMATICA Software.

In Table (1), we calculated the values of $c_1$ by using least square method.

Fig. (1) shows the physical configuration of the problem. Fig. (2) is depicted to show the changes of the velocity with gravitational parameter (m), it is observed that as (m) increasing then the velocity is decreasing. Fig. (3) reveals the velocity $u$ is obtained for different values of magnetic parameter $M$. The results indicated that the velocity of $u$ increases with increasing magnetic parameter $M$ when $0 \leq \omega < 0.37$ but it has the same value when $\omega = 0.37$ and the velocity is decreased when $0.37 < \omega < 0.8$ and it has the same value when $\omega = 0.8$ but it is increased when $\omega > 2$. Fig. (4) is sketch to show the profiles of the velocity field for different values of the relaxation time parameter $k_1$. From this figures, it is obvious to note that the velocity has the same value with increasing the relaxation time parameter when $0 \leq \omega < 0.42$ but it is increasing with increasing the relaxation time parameter when $0.42 < \omega < 0.99$ and it has the same value with increasing the relaxation time parameter when $\omega = 0.42$ but it is increased when $\omega > 1.0$. Fig. (5) shows the behavior of the velocity profiles for different values of retardation time parameter $k_2$. It is observed that the velocity is increased with increasing the value of retardation time parameter when $0 < \omega < 0.632$ and it has the same value with increasing the retardation time parameter when $0.632 < \omega < 1.253$ and it has the same value with increasing the retardation time parameter when $\omega = 1.253$. Fig. (6) illustrate the influence of the parameter ($c_2$) on the velocity field, we can see that the velocity is decreasing with increasing $c_2$ when $0 < \omega < 0.232$ then the velocity is increasing when $0.232 < \omega < 0.65$ but it has decreased when increasing the value of ($c_2$) when $0.65 < \omega < 1.29$ and then it has decreased with increasing the value of $c_2$ when $\omega > 1.29$. Fig. (7) is established to show the behavior of different time $t$, we can see the velocity $u$ is decreasing with increasing the time $t$ when $0 < \omega < 0.1$ but it is increasing with increasing the time $t$ when $0.7 < \omega < 1.387$. Fig. (8) displays the impact of the parameter $c_1$ on the velocity field. It can be seen that the velocity becomes similar with increasing the value of $c_1$. Fig. (9) sketches to show the profiles of the velocity field for different values of the permeability parameter ($K$). It is observed that the velocity is decreasing with increasing the value of the permeability parameter ($K$) when $0 < \omega < 0.39$ and the velocity is increasing with increasing the value of the permeability parameter ($K$) on the interval $0.39 < \omega < 0.82$ but it has decreased with increasing the parameter when $\omega > 0.82$. Fig. (10) shows the effect of the different values of the parameter $y$ on the velocity. It is seen that the velocity increases as the parameter $y$ increasing with $\omega < 0.323$ but it has decreased with increasing the parameter $y$ in the interval $0.323 < \omega < 0.6$ and then the velocity is increasing when $0.6 < \omega < 1.35$ but it is increased when $\omega > 1.35$. Figs. (11 and 12) show the behavior of the velocity for different values gravitational parameter (m) and magnetic parameter ($M$). It is seen that the velocity increase as the parameters increasing. Fig. (13) indicate that, by increasing the relaxation time parameter ($k_1$), the velocity decrease with increasing relaxation time parameter. Fig. (14) displays the impact of retardation time parameter ($k_2$) on the velocity. It is seen that the velocity increase with increasing retardation time. Figs. (15 and 16) indicate that the velocity reduces with increasing magnitude the parameter $c_2$ and time $t$. Fig. (17) show the behavior of $c_1$ on the velocity field, we can see the velocity has the same value with increasing $c_1$. Figs. (18 and 19) illustrate the influence of permeability parameter ($K$) and the oscillating parameter $\omega$. That implies that the velocity reduces with increasing the value of permeability parameter ($K$) and the oscillating parameter $\omega$.

Fig.2: The velocity $u$. Eq. (59) for different value of $m$ when keeping other parameters fixed $[y=0.5, \delta=0.1, c_1=-0.191837933, c_2=-1.688934901, k_1=0.2, k_2=0.3, M=0.3, t=5, k =1]$
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Fig.3: the velocity $u$, Eq. (59) for different value of $M$ when keeping other parameters fixed \{$y=0.5$, $\delta=0.1$, $c_1=-0.191837933$, $c_2=1.688934901$, $k_1=0.2$, $k_2=0.3$, $m=0.2$, $t=5$, $K=1$\}

Fig.4: the velocity $u$, Eq. (59) for different value of $k_1$ when keeping other parameters fixed \{$y=0.5$, $\delta=0.1$, $c_1=-0.191837933$, $c_2=1.688934901$, $M=0.3$, $k_2=0.3$, $m=0.2$, $t=5$, $K=1$\}

Fig.5: the velocity $u$, Eq. (59) for different value of $k_2$ when keeping other parameters fixed \{$y=0.5$, $\delta=0.1$, $c_1=-0.191837933$, $c_2=1.688934901$, $M=0.3$, $k_1=0.2$, $m=0.2$, $t=5$, $K=1$\}

Fig.6: the velocity $u$, Eq. (59) for different value of $c_2$ when keeping other parameters fixed \{$y=0.5$, $\delta=0.1$, $c_1=-0.191837933$, $M=0.3$, $k_1=0.2$, $m=0.2$, $k_2=0.3$, $t=5$, $K=1$\}
Effects of MHD on the Unsteady Thin Flow of Non-Newtonian Oldroyd–B Fluid over an

Fig. 7: The velocity $u$, Eq. (59) for different value of $t$ when keeping other parameters fixed \{$y=0.5$, $\delta =0.1$, $c_1=-0.191837933$, $c_2=-1.688934901$, $M=0.3$, $k_1=0.2$, $m=0.2$, $k_2=0.3$, $K=1$\}

Fig. 8: The velocity $u$, Eq. (59) for different value of $c_1$ when keeping other parameters fixed \{$y=0.5$, $\delta =0.1$, $c_2=-1.688934901$, $M=0.3$, $k_1=0.2$, $m=0.2$, $t=5$, $k_2=0.3$, $K=1$\}

Fig. 9: The velocity $u$, Eq. (59) for different value of $K$ when keeping other parameters fixed \{$y=0.5$, $\delta =0.1$, $c_1=-0.191837933$, $c_2=-1.688934901$, $M=0.3$, $k_1=0.2$, $m=0.2$, $t=5$, $k_2=0.3$\}

Fig. 10: The velocity $u$, Eq. (59) for different value of $\gamma$ when keeping other parameters fixed \{$\delta =0.1$, $c_1=-0.191837933$, $c_2=-1.688934901$, $M=0.3$, $k_1=0.2$, $m=0.2$, $t=5$, $k_2=0.3$, $K=1$\}

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**Fig. 11:** The velocity $u$, Eq. (59) for different values of $m$ when keeping other parameters fixed: $\omega=0.2$, $\delta=0.1$, $c_1=-0.191837933$, $c_2=-1.688934901$, $k_1=0.2$, $k_2=0.3$, $M=0.3$, $t=5$, $K=1$.

**Fig. 12:** The velocity $u$, Eq. (59) for different values of $M$ when keeping other parameters fixed: $\omega=0.2$, $\delta=0.1$, $c_1=-0.191837933$, $c_2=-1.688934901$, $k_1=0.2$, $k_2=0.3$, $m=0.2$, $t=5$, $K=1$.

**Fig. 13:** The velocity $u$, Eq. (59) for different values of $k_1$ when keeping other parameters fixed: $\omega=0.2$, $\delta=0.1$, $c_1=-0.191837933$, $c_2=-1.688934901$, $M=0.3$, $k_2=0.3$, $m=0.2$, $t=5$, $K=1$.

**Fig. 14:** The velocity $u$, Eq. (59) for different values of $k_2$ when keeping other parameters fixed: $\omega=0.2$, $\delta=0.1$, $c_1=-0.191837933$, $c_2=-1.688934901$, $k_1=0.2$, $m=0.2$, $t=5$, $K=1$.
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**Fig.15:** The velocity $u$, Eq. (59) for different value of $c_2$ when keeping other parameters fixed $\{ \omega=0.2, \delta=0.1, c_1=-0.191837933, M=0.3, k_1=0.2, m=0.2, k_2=0.3, t=5, K=1 \}$

**Fig.16:** The velocity $u$, Eq. (59) for different value of $t$ when keeping other parameters fixed $\{ \omega=0.2, \delta=0.1, c_1=-0.191837933, c_2=-1.688934901, M=0.3, k_1=0.2, m=0.2, k_2=0.3, K=1 \}$

**Fig.17:** The velocity $u$, Eq. (59) for different value of $c_1$ when keeping other parameters fixed $\{ \omega=0.2, \delta=0.1, c_2=-1.688934901, M=0.3, k_1=0.2, m=0.2, t=5, k_2=0.3, K=1 \}$

**Fig.18:** The velocity $u$, Eq. (59) for different value of $K$ when keeping other parameters fixed $\{ \omega=0.2, \delta=0.1, c_1=-0.191837933, c_2=-1.688934901, M=0.3, k_1=0.2, m=0.2, t=5, k_2=0.3 \}$

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![Fig.19: the velocity $u$, Eq.(59) for different value of $\omega$ when keeping other parameters fixed { $\delta =0.1$ , $c_1=-0.191837933$ , $c_2=1.688934901$ , $M=0.3$ , $k_1=0.2$ , $m=0.2$ , $t=5$ , $k_2=0.3$ , $K=1$}]

References