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Bivariate Vieta-Fibonacci and Bivariate Vieta-Lucas Polynomials

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Abstract: *In this paper, we consider the bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials which are generalized of Vieta-Fibonacci, Vieta-Lucas, Vieta-Pell, Vieta-Pell-Lucas polynomials. Also, we give the some properties. Afterwards, we obtain the some identities for the bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials by using the known properties of bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials.*

Keywords: *Binet's Formula, Fibonacci Polynomials, Lucas Polynomials, Vieata Polynomials*.

I. Introduction

In [1], Horadam consider the Vieta-Fibonacci and Vieta-Lucas polynomials which are defined by the following recurrence relations

$$V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$$

with

$$V_0(x) = 0,\ V_1(x) = 1$$

and

$$v_n(x) = xv_{n-1}(x) - v_{n-2}(x)$$

with

$$v_0(x) = 2,\ v_1(x) = x\ .$$

Also, the author give the relationships among Vieta, Jacobsthal and Morgan-Voyce polynomials by using the known connections with Fibonacci, Lucas and Chebyshev polynomials.

In [2, 3], the author defined the bivariate Fibonacci and bivariate Lucas polynomials and give the some properties of these polynomials. Also, Catalini obtain the some identities for Bivariate Fibonacci and Bivariate Lucas polynomials derived from a book of Gould. In [4], Catalini defined the generalized bivariate Fibonacci polynomial and give the summation and inversion formulas.

Swammy give the generalized Fibonacci and Lucas polynomials and their associated diagonal polynomials [5].

In [6], Andre-Jeannin define a general class of polynomials by the following recurrence relations

$$U_n(p,q;x) = (x+p)U_{n-1}(p,q;x) - qU_{n-2}(p,q;x),\ \ n \geq 2$$

with $U_0(p,q;x) = 0,\ U_1(p,q;x) = 1.$ Particular cases of $U_n(p,q;x)$ are Fibonacci polynomials, Pell polynomials, the first Fermat polynomials and the Morgan-Voyce polynomials of the second kind. Also, the author give the combinatorial properties of the polynomials $U_n(p,q;x)$.

In [7], Djordjevic define a generalization of the polynomials $U_n(p,q;x)$ which is given by Andre-Jeannin. Also, the author obtain the some properties of the generalized polynomials.

In [8], the author define the new polynomials by using the polynomial $U_n(p,q;x)$ and investigate the properties of a new polynomials.

In [9], Robbins developed some properties of a special infinite triangular array which was discovered by Vieta. Also, the author prove some irreducibility properties of Vieta polynomials.

In [10], the authors define the Vieta-Pell and Vieta-Pell-Lucas polynomials and give the properties of these polynomials.

In [11], the authors introduce the generalized Vieta-Jacobsthal and Vieta-Jacobsthal-Lucas polynomials. Also, various families of multilinear and multilateral generating functions for these polynomials are derived.

In light of the foregoing, we can consider the bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials. Clearly, these polynomials are a generalization of the Vieta-Fibonacci, Vieta-Lucas, Vieta-Pell and Vieta-Pell-Lucas polynomials.

In the rest sections of this paper, we give the definition of bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials. Afterwards, we obtain some properties of these polynomials and give the Pascal arrays generating these polynomials.

II. Bivariate Vieta-Fibonacci and Bivariate Vieta-Lucas Polynomials

Definition 2.1. Let $n \geq 2$ be integer. The recurrence relations of the bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials are

$$V_n(x,y) = xV_{n-1}(x,y) - yV_{n-2}(x,y) \tag{2.1}$$

with the initial conditions

$$V_0(x,y) = 0,\ V_1(x,y) = 1$$

and

$$v_n(x,y) = xv_{n-1}(x,y) - yv_{n-2}(x,y) \tag{2.2}$$

where $v_0(x,y) = 2,\ v_1(x,y) = x$.

The first few terms of $V_n(x,y)$ and $v_n(x,y)$ polynomials are as following table.

n	$V_n(x,y)$	$v_n(x,y)$
0	0	2
1	1	x
2	x	$x^2 - 2y$
3	$x^2 - y$	$x^3 - 3xy$
4	$x^3 - 2xy$	$x^4 - 4x^2y + 2y^2$
5	$x^4 - 3x^2y + y^2$	$x^5 - 5x^3y + 5xy^2$
6	$x^5 - 4x^3y + 3xy^2$	$x^6 - 6x^4y + 9x^2y^2 - 2y^3$
$\vdots$	$\vdots$	$\vdots$

The characteristic equation of the bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials is

$$\lambda^2 - x\lambda + y = 0 \tag{2.3}$$

Let α and β be the roots of the characteristic equation (2.3). α and β satisfy the following equations

$$\alpha + \beta = x,\ \alpha\beta = y,\ \alpha - \beta = \sqrt{x^2 - 4y}\,.$$

Using the standart tecniques, we have the Binet's formulas of $V_n(x,y)$ and $v_n(x,y)$ polynomials as

$$V_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{2.4}$$

and

$$v_n(x,y) = \alpha^n + \beta^n. \tag{2.5}$$

The generating functions for the infinite sets of polynomials $\{V_n(x,y)\}$ and $\{v_n(x,y)\}$ are found in the usual way to be

$$\sum_{n=0}^{\infty} V_n(x,y)t^n = t\left(1 - xt + yt^2\right)^{-1} \tag{2.6}$$

and

$$\sum_{n=0}^{\infty} v_n(x,y)t^n = (2 - xt)\left(1 - xt + yt^2\right)^{-1}. \tag{2.7}$$

We can also extend the definition of $V_n(x,y)$ and $v_n(x,y)$ to the negative index

$$V_{-n}(x,y)=\frac{-1}{y^n}V_n(x,y) \tag{2.8}$$

and

$$v_{-n}(x,y)=\frac{1}{y^n}v_n(x,y). \tag{2.9}$$

III. Elementary Properties

Some Interrelationship

The relationships between $V_n(x,y)$ and $v_n(x,y)$ can be given as follows

$$V_{n+1}(x,y)-yV_{n-1}(x,y)=v_n(x,y)$$
$$v_{n+1}(x,y)-yv_{n-1}(x,y)=(x^2-4y)V_n(x,y)$$
$$V_{2n}(x,y)=V_n(x,y)v_n(x,y)$$
$$V_{2n+1}(x,y)=V_{n+1}^2(x,y)-yV_n^2(x,y)$$
$$v_{n+1}^2(x,y)-yv_n^2(x,y)=(x^2-4y)V_{2n+1}(x,y)$$
$$v_{n+1}^2(x,y)+yv_n^2(x,y)=xv_{2n+1}(x,y)+4y^{n+1}$$
$$v_n^2(x,y)+(x^2-4y)V_n^2(x,y)=2v_{2n}(x,y)$$
$$v_n^2(x,y)-(x^2-4y)V_n^2(x,y)=4y^n$$
$$v_n(x,y)v_{n+1}(x,y)-(x^2-4y)V_n(x,y)V_{n+1}(x,y)=2xy^n$$
$$v_n(x,y)v_{n+1}(x,y)+(x^2-4y)V_n(x,y)V_{n+1}(x,y)=2v_{2n+1}(x,y)$$

Summation Formulas

Using the Binet's formulas (2.4)-(2.5), we can give the summation of the $V_n(x,y)$ and $v_n(x,y)$ polynomials as

$$\sum_{k=0}^{n}V_k(x,y)=\frac{V_{n+1}(x,y)-yV_n(x,y)-1}{x-y-1} \tag{3.1}$$

and

$$\sum_{k=0}^{n}v_k(x,y)=\frac{v_{n+1}(x,y)-yv_n(x,y)+x-2}{x-y-1} \tag{3.2}$$

where $x-y\neq 1$.

Explicit Formulas

Induction can be used, with a little effort, to establish the explicit formulas of bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials as

$$V_n(x,y)=\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}(-1)^i\binom{n-i-1}{i}x^{n-2i-1}y^i \tag{3.3}$$

and

$$v_n(x,y)=\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^i\frac{n}{n-i}\binom{n-i}{i}x^{n-2i}y^i\quad . \tag{3.4}$$

Differentiation Formulas

If we use the expilicit formula (3.4) of bivariate Vieta-Lucas Polynomial $v_n(x,y)$, we have

$$\frac{\partial v_n(x,y)}{\partial x} = nV_n(x,y), \ \frac{\partial v_n(x,y)}{\partial y} = -nV_{n-1}(x,y). \tag{3.5}$$

Using (3.5) and recurrences (2.1)-(2.2), we can give the following equations

$$x\frac{\partial v_n(x,y)}{\partial x} + 2y\frac{\partial v_n(x,y)}{\partial y} = nv_n(x,y)$$

$$x\frac{\partial V_n(x,y)}{\partial x} + 2y\frac{\partial V_n(x,y)}{\partial y} = (n-1)V_n(x,y)$$

$$n\frac{\partial v_{n+1}(x,y)}{\partial y} = -(n+1)\frac{\partial v_n(x,y)}{\partial x}$$

$$\frac{\partial v_{n+1}(x,y)}{\partial y} + \frac{\partial v_n(x,y)}{\partial x} + V_n(x,y) = 0$$

Some Identities

If we use the Binet's formulas (2.4) - (2.5) and the expilicit formulas (3.3)-(3.4), we have the following identities for the bivariate Vieta-Fibonacci and bivariate Vieta-Lucas polynomials.

$$V_n(x,y) = 2^{1-n}\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}\binom{n}{2i+1}x^{n-2i-1}\left(x^2-4y\right)^i \tag{3.6}$$

$$V_{2n}(x,y) = x\sum_{i=0}^{n-1}\binom{2n-i-1}{i}\left(x^2-4y\right)^{n-i-1}y^i \tag{3.7}$$

$$v_{n-1}(x+2, x+y+1) = \frac{1}{n}\sum_{i=1}^{n}\binom{n}{i}iv_{i-1}(x,y) \tag{3.8}$$

$$v_{2n-1}(x,y) = \frac{1}{n}\sum_{i=1}^{n}\binom{n}{i}ix^{i-1}(-y)^{n-i}v_i(x,y) \tag{3.9}$$

Cassini and Honsberger Formulas

Bivariate Vieta-Fibonacci polynomials are generated by the matrix Q,

$$Q = \begin{pmatrix} x & 1 \\ -y & 0 \end{pmatrix}.$$

It can be proved by mathematical induction on *n*

$$Q^n = \begin{pmatrix} V_{n+1}(x,y) & V_n(x,y) \\ -yV_n(x,y) & -yV_{n-1}(x,y) \end{pmatrix}$$

where $V_n(x,y)$ is $n-th$ bivariate Vieta-Fibonacci polynomial. Using the determinants of the matrices Q and Q^n, we can give the Cassini identity for the bivariate Vieta-Fibonacci polynomials as

$$V_{n+1}(x,y)V_{n-1}(x,y) - V_n^2(x,y) = -y^{n-1}. \tag{3.10}$$

Similarly, the Cassini identity for the bivariate Vieta-Lucas polynomials is

$$v_{n+1}(x,y)v_{n-1}(x,y) - v_n^2(x,y) = \left(x^2-4y\right)y^{n-1}. \tag{3.11}$$

Also, From the matrix Q, we have the other identity which is called Honsberger Formula as

$$V_{m+n}(x,y) = V_{m+1}(x,y)V_n(x,y) - yV_m(x,y)V_{n-1}(x,y). \tag{3.12}$$

Taking $m=n$ in the formula (3.12), we have the following identitiy

$$V_{2n}\left(x,y\right)=xV_n^2\left(x,y\right)-2yV_n\left(x,y\right)V_{n-1}\left(x,y\right) \tag{3.13}$$

Using $n+1$ instead of m in the formula (3.12), we have the other identitiy as follows

$$V_{2n+1}\left(x,y\right)=V_{n+1}^2\left(x,y\right)-yV_n^2\left(x,y\right) \tag{3.14}$$

IV. Pascal Arrays Generating Bivariate Vieta-Fibonacci Polynomials and Bivariate Vieta-Lucas Polynomials

We consider the following table. Denote the coefficient of the power of x and y in the $n-th$ row and $i-th$ column by $F\left(n,i\right)$.

n/i	1	2	3	4	5	6	$\cdots$
1	1						
2	x	$-y$					
3	x^2	$-2xy$	y^2				
4	x^3	$-3x^2y$	$3xy^2$	$-y^3$			
5	x^4	$-4x^3y$	$6x^2y^2$	$-4xy^3$	y^4		
6	x^5	$-5x^4y$	$10x^3y^2$	$-10x^2y^3$	$5xy^4$	$-y^5$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table1: Bivariate Vieata-Fibonacci Polynomials from Rising Diagonals

Define the entries in the row n as the terms in the expansion $\left(x-y\right)^{n-1}$, that is

$$\sum_{i=1}^{n}F\left(n,i\right)x^{n-i}\left(-y\right)^{i-1}=\left(x-y\right)^{n-1},\quad n\geq i. \tag{4.1}$$

From (4.1), we obtain

$$F\left(n,i\right)=\binom{n-1}{i-1}. \tag{4.2}$$

Now, using the rising diagonal lines in Table 1, we have

$$\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}F\left(n-i+1,i\right)x^{n-2i+1}\left(-y\right)^{i-1}.$$

Using (4.2), we obtain

$$\begin{aligned}\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}F\left(n-i+1,i\right)x^{n-2i+1}\left(-y\right)^{i-1} &= \sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\binom{n-i}{i-1}x^{n-2i+1}\left(-y\right)^{i-1}\\ &= \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-i-1}{i}x^{n-2i-1}\left(-y\right)^{i}\\ &= \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\left(-1\right)^{i}\binom{n-i-1}{i}x^{n-2i-1}y^{i}.\end{aligned}$$

From the expilicit formula (3.3), we have

$$\sum_{i=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}F\left(n-i+1,i\right)x^{n-2i+1}\left(-y\right)^{i-1}=V_n\left(x,y\right).$$

Namely, the sum of the elements on the rising diagonal lines in the Table 1 is the bivariate Vieta-Fibonacci polynomial $V_n\left(x,y\right)$.

Now, we consider the Table 2. Denote the coefficient of the power of x and y in the $n-th$ row and $i-th$ column by $L(n,i)$.

n/i	1	2	3	4	5	6	$\cdots$
1	x	$-2y$					
2	x^2	$-3xy$	$2y^2$				
3	x^3	$-4x^2y$	$5xy^2$	$-2y^3$			
4	x^4	$-5x^3y$	$9x^2y^2$	$-7xy^3$	$2y^4$		
5	x^5	$-6x^4y$	$14x^3y^2$	$-16x^2y^3$	$9xy^4$	$-2y^5$	
6	x^6	$-7x^5y$	$20x^4y^2$	$-30x^3y^3$	$25x^2y^4$	$-11xy^5$	$2y^6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 2: Bivariate Vieata-Lucas Polynomials from Rising Diagonals

We may define the entries in the row n as the terms in the expansion of

$$(x-y)^n+(x-y)^{n-1}=(x-y)^{n-1}(x-2y)$$

that is

$$\sum_{i=1}^{n+1} L(n,i)\,x^{n-i+1}(-y)^{i-1}=(x-y)^{n-1}(x-2y). \tag{4.3}$$

Using (4.3) and Pascal's formula , we obtain

$$L(n,i)=\binom{n-1}{i-2}+\binom{n}{i-1}. \tag{4.4}$$

Now, using the rising diagonal lines in Table 2, we have

$$\sum_{i=1}^{\left\lfloor\frac{n+2}{2}\right\rfloor} L(n-i+1,i)\,x^{n-2i+2}(-y)^{i-1}.$$

Using (4.4), we obtain

$$\begin{aligned}\sum_{i=1}^{\left\lfloor\frac{n+2}{2}\right\rfloor} L(n-i+1,i)\,x^{n-2i+1}(-y)^{i-1} &= \sum_{i=1}^{\left\lfloor\frac{n+2}{2}\right\rfloor}\left[\binom{n-i}{i-2}+\binom{n-i+1}{i-1}\right]x^{n-2i+2}(-y)^{i-1}\\ &=\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\left[\binom{n-i-1}{i-1}+\binom{n-i}{i}\right]x^{n-2i}(-y)^{i}\\ &=\sum_{i=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^i\frac{n}{n-i}\binom{n-i}{i}x^{n-2i}y^i.\end{aligned}$$

From the expilicit formula (3.4), we have

$$\sum_{i=1}^{\left\lfloor\frac{n+2}{2}\right\rfloor} L(n-i+1,i)\,x^{n-2i+2}(-y)^{i-1}=v_n(x,y).$$

It' clearly, the sum of the elements on the rising diagonal lines in the Table 2 is the bivariate Vieta-Lucas polynomial $v_n(x,y)$.

V. Conclusion

In a future paper, we shall investigate the sequences $\{V_{p,n}(x,y)\}$ and $\{v_{p,n}(x,y)\}$ of polynomials, defined by the recurrences relations

$$V_{p,n}\left(x,y\right)=xV_{p,n-1}\left(x,y\right)-yV_{p,n-p-1}\left(x,y\right) \text{ for } n>p$$

with the initial conditions

$$V_{p,0}\left(x,y\right)=0,\ V_{p,n}\left(x,y\right)=x^{n-1},\ n=1,2,\ldots,p$$

and

$$v_{p,n}\left(x,y\right)=xv_{p,n-1}\left(x,y\right)-yv_{p,n-p-1}\left(x,y\right) \text{ for } n>p$$

with the initial conditions

$$v_{p,0}\left(x,y\right)=p+1,\ v_{p,n}\left(x,y\right)=x^{n},\ n=1,2,\ldots,p.$$

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