

## Ricci Solitons on Para-Sasakian Manifolds

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**Abstract:** The present paper aims at studying a Ricci solitons on para-Sasakian manifolds. We give the results on Ricci solitons in para-Sasakian manifold satisfying the conditions  $S(\xi, X) \cdot R = 0$ ,  $R(\xi, X) \cdot \bar{P} = 0$  and  $R(\xi, X) \cdot \bar{H} = 0$ , where  $\bar{P}$ ,  $\bar{H}$  are pseudo-projective and quasi-conharmonic curvature tensors, respectively.

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### I. Introduction

A Ricci soliton is a natural generalization of Einstein metric. A Ricci soliton  $(g, V, \lambda)$  is defined on a pseudo-Riemannian manifold  $(M, g)$  by

$$(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad (1.1)$$

where  $\mathcal{L}_V g$  denotes the Lie derivative of Riemannian metric  $g$  along a vector field  $V$ ,  $\lambda$  is a constant, and  $X, Y$  are arbitrary vector fields on  $M$ . A Ricci soliton is said to be *shrinking*, *steady*, and *expanding* according as  $\lambda$  is *negative*, *zero*, and *positive*, respectively. Theoretical physicists have also been taking interest in the equation of Ricci soliton in relation with string theory, and the fact that equation (1.1) is a special case of Einstein field equations. The Ricci soliton in Riemannian Geometry was introduced [1] as self-similar solutions of the Ricci flow. Recent progress on Riemannian Ricci solitons may be found in [2]. Also, Ricci solitons have been studied extensively in the context of pseudo-Riemannian Geometry; we may refer to [3, 4, 5, 6, 7, 8] and references therein.

In 1976, Sato [9] introduced the notion of almost paracontact structure  $(\phi, \xi, \eta)$  on a differentiable manifolds. This structure is an analogue of the almost contact structure. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be of even dimension as well. Takahashi [10] defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated Pseudo-Riemannian metric.

In 1977, Adati and Matsumoto defined Para-Sasakian and special Para-Sasakian manifolds [11], which are special classes of an almost paracontact manifold. Also, The geometry of these manifolds is extensively studied by [12, 13, 14, 15, 16, 17] and many others. In 1985, Kaneyuki and Williams [18] defined the notion of almost para contact structure on pseudo-Riemannian manifold of dimension  $(2n+1)$ . Later, Zamkovoy [19] showed that any almost para contact structure admits a pseudo-Riemannian metric with signature  $(n+1, n)$ .

In the present paper, we study Ricci solitons on para-Sasakian manifolds. The paper is organised as follows: section 2 is devoted to preliminaries on para-Sasakian manifolds. In section 3, 4 and 5 we study para-Sasakian Ricci solitons satisfying  $S(\xi, X) \cdot R = 0$ ,  $R(\xi, X) \cdot \bar{P} = 0$  and  $R(\xi, X) \cdot \bar{H} = 0$ , where  $\bar{P}$  and  $\bar{H}$  are pseudo-projective and quasi-conharmonic curvature tensors, respectively.

### II. Preliminaries

An *almost paracontact structure* on a manifold  $M$  of dimension  $n$  is a triplet  $(\phi, \xi, \eta)$  consisting of a  $(1,1)$ -tensor field  $\phi$ , a vector field  $\xi$ , a 1-form  $\eta$  satisfying:

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad (2.1)$$

$$\eta \cdot \phi = 0, \quad \text{rank}(\phi) = n-1, \quad (2.2)$$

where  $I$  denotes the identity transformation. A pseudo-Riemannian metric  $g$  on  $M$  is compatible with the almost paracontact structure  $(\phi, \xi, \eta)$  if

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y). \quad (2.3)$$

In such case,  $(\phi, \xi, \eta, g)$  is called an *almost paracontact metric structure*. By (2.1)-(2.3), it is clear that,  $g(X, \xi) = \eta(X)$  for any compatible metric. Any almost paracontact structure admits compatible metrics. The fundamental 2-form  $\Phi$  of an almost paracontact structure  $(\phi, \xi, \eta, g)$  is defined by  $\Phi = g(X, \phi Y)$ , for all tangent vector fields  $X, Y$ . If  $\Phi = d\eta$ , then the manifold  $(M, \phi, \xi, \eta, g)$  is called a *paracontact metric manifold* associated to the metric  $g$ .

In case the paracontact metric structure is normal. The structure is called para-Sasakian. Equivalently, a paracontact metric structure  $(\phi, \xi, \eta, g)$  is para-Sasakian if An almost paracontact metric structure becomes a paracontact metric structure if

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \tag{2.4}$$

for any vector fields  $X, Y \in \Gamma(TM)$ , where  $\nabla$  is Levi-Civita connection of  $g$ .

From (2.4), it follows that

$$\nabla_X \xi = -\phi X. \tag{2.5}$$

Also, in an  $n$ -dimensional para-Sasakian manifold, the following relations hold:

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.6}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X, \tag{2.8}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.9}$$

$\forall X, Y, Z \in \Gamma(TM)$ . Here  $R$  is Riemannian curvature tensor and  $S$  is Ricci tensor defined by  $S(X, Y) = g(QX, Y)$ , where  $Q$  is Ricci operator.

Let  $M(\phi, \xi, \eta, g)$  be an  $n$ -dimension para-Sasakian manifold and let  $(g, \xi, \lambda)$  be a Ricci soliton on  $M$ . Then the relation (1.1) implies

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0$$

or

$$2S(X, Y) = -(\mathcal{L}_\xi g)(X, Y) - 2\lambda g(X, Y) \tag{2.10}$$

for any  $X, Y \in \Gamma(M)$ .

On a para-Sasakian manifold  $M$ , from (2.5) and the skew-symmetric property of  $\phi$ , we obtain

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \tag{2.11}$$

By plugging (2.11) in (2.10), we have

$$S(X, Y) = -\lambda g(X, Y). \tag{2.12}$$

Also, By Putting  $Y = \xi$  in (2.12) we have

$$S(X, \xi) = -\lambda \eta(X). \tag{2.13}$$

By virtue of (2.9), we obtain from (2.13) that

$$\lambda = (n-1). \tag{2.14}$$

Thus, we can state the following:

**Theorem 1.** *A para-Sasakian Ricci soliton in an  $n$ -dimensional ( $n > 1$ ) para-Sasakian manifold is expanding.*

### III. Ricci Soliton in a Para-Sasakian Manifold Satisfying $S(\xi, X) \cdot R = 0$ .

Consider a para-Sasakian manifold  $(M^n, g)$ , satisfying the condition

$$S(\xi, X) \cdot R = 0. \tag{3.1}$$

By definition we have

$$\begin{aligned} S((X, \xi) \cdot R)(U, V)W &= ((X \wedge_S \xi)R)(U, V)W \\ &= (X \wedge_\xi)R(U, V)W + R((X \wedge_S)U, V)W \\ &\quad + R(U, (X \wedge_S \xi)V)W + R(U, V)(X \wedge_S \xi)W, \end{aligned} \tag{3.2}$$

where the endomorphism  $X \wedge_S Y$  is defined by

$$(X \wedge_s Y)Z = S(Y, Z)X - S(X, Z)Y. \tag{3.3}$$

In the view of (3.3) in (3.2), we get

$$\begin{aligned} & S((X, \xi) \cdot R)(U, V)W \\ = & S(\xi, R(U, V)W)X - S(X, R(U, V)W)\xi + S(\xi, U)R(X, V)W - S(X, U)R(\xi, V)W \\ & + S(\xi, V)R(U, X)W - S(X, V)R(U, \xi)W + S(\xi, W)R(U, V)X - S(X, W)R(U, V)\xi. \end{aligned} \tag{3.4}$$

Using (2.12) and (2.13), we obtain

$$\begin{aligned} & -\lambda[\eta(R(U, V)W)X - g(X, R(U, V)W)\xi + \eta(U)R(X, V)W - g(X, U)R(\xi, V)W \\ & + \eta(V)R(U, X)W - g(X, V)R(U, \xi)W + \eta(W)R(U, V)X - g(X, W)R(U, V)\xi] = 0. \end{aligned} \tag{3.5}$$

By taking an inner product with  $\xi$  and by Using (2.6), (2.7) and (2.8) in (3.5), we get

$$\begin{aligned} & \lambda[g(X, R(U, V)W) - g(X, U)g(V, W) + g(X, V)g(U, W) \\ & + 2\eta(X)\{g(V, W)\eta(U) - g(U, W)\eta(V)\}] = 0. \end{aligned} \tag{3.6}$$

Putting  $X = U = e_i$  in (3.6) and summing over  $i = 1, 2, \dots, n$ , we get

$$\lambda[S(V, W) - (n-3)g(V, W) - 2\eta(V)\eta(W)] = 0. \tag{3.7}$$

Putting  $V = W = \xi$  in (3.7) and by virtue of (2.12) and (2.13), we have

$$\lambda\{\lambda + (n-1)\} = 0. \tag{3.8}$$

Implies either  $\lambda = 0$  or  $\lambda = -(n-1)$ .

Therefore, we can state the following theorem.

**Theorem 2.** A Ricci soliton in a para-Sasakian manifold ( $n > 1$ ) satisfying  $S(\xi, X) \cdot R = 0$  is either steady or shrinking.

#### IV. Ricci Soliton in a Para-Sasakian Manifolds Satisfying $R(\xi, X) \cdot \bar{P} = 0$ .

Pseudo-projective curvature tensor  $\bar{P}$  is defined by B. Prasad [20].

$$\bar{P}(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] - \frac{r}{n} \left( \frac{a}{n-1} + b \right) [g(Y, Z)X - g(X, Z)Y] \tag{4.1}$$

where  $a, b \neq 0$  are constants.

Taking  $Z = \xi$  in (4.1) and by using (2.7) and (2.12) in (4.1), we have

$$\bar{P}(X, Y)\xi = \left[ a + b\lambda + \frac{r}{n} \left( \frac{a}{n-1} + b \right) \right] [\eta(X)Y - \eta(Y)X]. \tag{4.2}$$

Similarly using (2.6), (2.12) in (4.1), we get

$$\eta(\bar{P}(X, Y)Z) = \left[ a + b\lambda + \frac{r}{n} \left( \frac{a}{n-1} + b \right) \right] [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]. \tag{4.3}$$

By using the condition  $R(\xi, X) \cdot \bar{P} = 0$ , we have

$$R(\xi, X)\bar{P}(U, V)W - \bar{P}(R(\xi, X)U, V)W - \bar{P}(U, R(\xi, X)V)W - \bar{P}(U, V)R(\xi, X)W = 0. \tag{4.4}$$

In the view of (2.8) in (4.4), we obtain

$$\begin{aligned} & \eta(\bar{P}(U, V)W)X - g(X, \bar{P}(U, V)W)\xi - \eta(U)\bar{P}(X, V)W + g(X, U)\bar{P}(\xi, V)W \\ & - \eta(V)\bar{P}(U, X)W + g(X, V)\bar{P}(U, \xi)W - \eta(W)\bar{P}(U, V)X + g(X, W)\bar{P}(U, V)\xi = 0. \end{aligned} \tag{4.5}$$

Taking inner product with  $\xi$  in (4.5), we get

$$\begin{aligned} & \eta(\bar{P}(U, V)W)\eta(X) - g(X, \bar{P}(U, V)W) - \eta(U)\eta(\bar{P}(X, V)W) + g(X, U)\eta(\bar{P}(\xi, V)W) \\ & - \eta(V)\eta(\bar{P}(U, X)W) + g(X, V)\eta(\bar{P}(U, \xi)W) - \eta(W)\eta(\bar{P}(U, V)X) + g(X, W)\eta(\bar{P}(U, V)\xi) = 0. \end{aligned} \tag{4.6}$$

By virtue of (4.2), (4.3) in (4.6), we have

$$-g(X, \bar{P}(U, V)W) + \left[ a + b\lambda + \frac{r}{n} \left( \frac{a}{n-1} + b \right) \right] [g(U, W)g(V, X) - g(X, U)g(V, W)] = 0. \quad (4.7)$$

Using (4.1) in (4.7), we obtain

$$-ag(X, R(U, V)W) + a[g(U, W)g(X, V) - g(X, U)g(V, W)] = 0. \quad (4.8)$$

Putting  $X = U = e_i$  in (4.8) and summing over  $i = 1, 2, \dots, n$ , we get

$$S(V, W) + (n-1)g(V, W) = 0. \quad (4.9)$$

Putting  $V = W = \xi$  in (4.9) and by virtue of (2.12), we obtain

$$\lambda = (n-1). \quad (4.10)$$

It implies  $\lambda$  is positive for every  $(n > 1)$ . Hence we state the following theorem.

**Theorem 3.** A Ricci soliton in a para-Sasakian manifold  $(n > 1)$  satisfying  $R(\xi, X) \cdot \bar{P} = 0$  is expanding.

**V. Ricci Soliton in a Para-Sasakian Manifold Satisfying  $R(\xi, X) \cdot \bar{H} = 0$ .**

Quasi-conharmonic curvature tensor  $\bar{H}$  ( $n > 3$ ) of the type (1,3) is defined [21].

$$\begin{aligned} \bar{H}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y] + c[g(Y, Z)QX - g(X, Z)QY] \\ &\quad - \frac{r}{n} \left( \frac{2a}{n-2} + b + c \right) [g(Y, Z)X - g(X, Z)Y], \end{aligned} \quad (5.1)$$

where  $a, b$  and  $c$  are constants such that  $a, b, c \neq 0$ .

Taking  $Z = \xi$  in (5.1) and by using (2.7) and (2.12) in (5.1), we have

$$\bar{H}(X, Y)\xi = \left[ a + (b+c)\lambda + \frac{r}{n} \left( \frac{2a}{n-2} + b + c \right) \right] [\eta(X)Y - \eta(Y)X]. \quad (5.2)$$

Similarly using (2.6), (2.12) in (5.1), we get

$$\eta(\bar{H}(X, Y)Z) = \left[ a + (b+c)\lambda + \frac{r}{n} \left( \frac{2a}{n-2} + b + c \right) \right] [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]. \quad (5.3)$$

We assume the condition  $R(\xi, X) \cdot \bar{H} = 0$ , then we have

$$R(\xi, X)\bar{H}(U, V)W - \bar{H}(R(\xi, X)U, V)W - \bar{H}(U, R(\xi, X)V)W - \bar{H}(U, V)R(\xi, X)W = 0. \quad (5.4)$$

In the view of (2.8) in (5.4), we obtain

$$\begin{aligned} \eta(\bar{H}(U, V)W)X - g(X, \bar{H}(U, V)W)\xi - \eta(U)\bar{H}(X, V)W + g(X, U)\bar{H}(\xi, V)W \\ - \eta(V)\bar{H}(U, X)W + g(X, V)\bar{H}(U, \xi)W - \eta(W)\bar{H}(U, V)X + g(X, W)\bar{H}(U, V)\xi = 0. \end{aligned} \quad (5.5)$$

Taking inner product with  $\xi$  in (5.5), we get

$$\begin{aligned} \eta(\bar{H}(U, V)W)\eta(X) - g(X, \bar{H}(U, V)W) - \eta(U)\eta(\bar{H}(X, V)W) + g(X, U)\eta(\bar{H}(\xi, V)W) \\ - \eta(V)\eta(\bar{H}(U, X)W) + g(X, V)\eta(\bar{H}(U, \xi)W) - \eta(W)\eta(\bar{H}(U, V)X) + g(X, W)\eta(\bar{H}(U, V)\xi) = 0. \end{aligned} \quad (5.6)$$

By virtue of (5.2), (5.3) in (5.6), we have

$$-g(X, \bar{H}(U, V)W) + \left[ a + (b+c)\lambda + \frac{r}{n} \left( \frac{2a}{n-2} + b + c \right) \right] [g(U, W)g(X, V) - g(V, W)g(X, U)] = 0. \quad (5.7)$$

Using (5.1) in (5.7), we obtain

$$-ag(X, R(U, V)W) + a[g(U, W)g(X, V) - g(V, W)g(X, U)] = 0. \quad (5.8)$$

Putting  $X = U = e_i$  in (5.8) and summing over  $i = 1, 2, \dots, n$ , we get

$$S(V, W) + (n - 1)g(V, W) = 0. \quad (5.9)$$

Putting  $V = W = \xi$  in (4.9) and by virtue of (2.12), we obtain

$$\lambda = (n - 1). \quad (5.10)$$

It implies  $\lambda$  is positive for every  $(n > 1)$ . Hence we state the following theorem.

**Theorem 4 .** A Ricci soliton in a para-Sasakian manifold  $(n > 1)$  satisfying  $R(\xi, X) \cdot \bar{H} = 0$  is expanding.

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