Invariant Submanifolds in an Indefinite Trans-Sasakian Manifold

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Abstract: In this paper, invariant submanifolds in an indefinite trans-Sasakian manifold are studied. Necessary and sufficient conditions are given on a submanifold of an indefinite trans-Sasakian manifold to be an invariant submanifold. Here we show that an invariant submanifold of an indefinite trans-Sasakian manifold is totally geodesic.

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I. Introduction


II. Preliminaries

Let \( \overline{M} \) be an \((2n+1)\)-dimensional indefinite almost contact metric manifold with indefinite almost contact metric structure \((\phi, \xi, \eta, g)\) then they satisfy:

\[
\begin{align*}
\phi^2 &= -I + \eta \otimes \xi, \\
\eta(\xi) &= 1, \quad \phi \xi = 0, \\
g(\phi X, \phi Y) &= g(X, Y) - \alpha \eta(X) \eta(Y), \\
g(X, \xi) &= \eta(X),
\end{align*}
\]

where \( X, Y \) are vector fields on \( \overline{M} \) and where \( \epsilon = g(\xi, \xi) = \pm 1 \).

An indefinite almost contact metric structure \((\phi, \xi, \eta, g)\) on \( M \) is called indefinite trans-Sasakian if

\[
(\overline{\nabla}_X \phi)(Y) = \alpha \{g(X, Y) \xi - \epsilon \eta(Y) X\} + \beta \{ \eta(\phi X, Y) \xi - \epsilon \eta(Y) \phi X \}
\]

where \( \alpha \) and \( \beta \) are non zero scalar functions on \( \overline{M} \) of type \((\alpha, \beta)\). \( \overline{\nabla} \) is a Riemannian connection on \( \overline{M} \). In particular, an indefinite trans-Sasakian manifold is normal.

From above formula, one easily obtains

\[
\overline{\nabla}_X \xi = - \alpha \epsilon \phi X + \beta \{ \epsilon X - \epsilon \eta(X) \xi \},
\]

Let \( M \) be an \((2m+1)\) dimensional \((n > m)\) manifold imbedded in \( \overline{M} \). The induced metric \( g \) of \( M \) is given by

\[
g(X, Y) = \overline{g}(\overline{X}, \overline{Y})
\]

for any vector fields \( X, Y \) on \( M \).

Let \( T_x(M) \) and \( T_x(M)^\perp \) denote that tangent and normal bundles of \( M \) and \( x \in M \). Let \( \nabla_X \) denote the Riemannian connection on \( M \) determined by the induced metric \( g \) and \( R \) denote the Riemannian curvature tensor of \( M \). Then Gauss-Weingarten formula is given by:

\[
\nabla_X Y = \nabla_X g(Y, Z) + g(\nabla_X Z, Y) - R(X, Y) Z + R(Z, X) Y - R(Y, Z) X - R(Z, X) Y
\]
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(2.7) \[ \nabla_X Y = \nabla_X Y + B(X, Y). \]

(2.8) \[ \nabla_X N = -A_N(X) + D_N N \]
for any vector fields \( X, Y \) tangent to \( M \) and any vector field \( N \) normal to \( M \), where \( D \) is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle \( T_x(M) \). Both \( A \) and \( B \) are called the second fundamental forms of \( M \) satisfy

\[ g(B(X, Y), N) = g(A_N(X, Y)). \]

A submanifold \( M \) of \( \overline{M} \) is said to be invariant if \( \xi \) is tangent to \( M \) everywhere on \( M \) and \( \overline{\phi} X \) is tangent to \( M \) for any tangent vector \( X \) to \( M \). An invariant submanifold \( M \) has the induced structure tensor \( (\phi, \xi, \eta, g) \).

III. Invariant Submanifolds in Indefinite Trans-Sasakian Manifold

Let \( \overline{M} \) be a \((2n+1)\) dimensional indefinite trans-Sasakian manifold and \( M \) a \((2m+1)\) dimensional \((n > m)\) manifold imbedded in \( \overline{M} \). For the second fundamental form \( B \) of an invariant submanifold \( M \) of a indefinite trans-Sasakian manifold. We define its covariant derivative \( (\nabla_X B) \) by

\begin{equation}
(3.1) \quad \nabla_X B(Y, Z) = D_X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z),
\end{equation}

where \( X, Y, Z \in \chi(M) \) - the set of all differential vector field on \( M \).

Then by (2.7). We obtain

\begin{equation}
\end{equation}

Lemma 3.1. If \( M \) is an invariant submanifold of a indefinite trans-Sasakian manifold \( \overline{M} \), then its second fundamental form \( B \) satisfies \( B(X, \xi) = 0 \), for any \( X \in \chi(M) \).

**Proof:** Since \( \xi \) is tangent to \( M \) everywhere on \( M \), we have

\[ \nabla_X \xi = \nabla_X \xi = \nabla_X \xi + B(X, \xi). \]

Since by equation,

\[ \nabla_X \xi = \nabla_X \xi = -\alpha \varepsilon \phi X + \beta (\varepsilon X - \epsilon \eta(X) \xi) \]

\[ \nabla_X \xi = \nabla_X \xi + B(X, \xi). \]

then by taking the normal parts of (3.3) we get \( B(X, \xi) = 0 \).

Lemma 3.2. Any invariant submanifolds \( M \) with induced structure tensors of a indefinite trans-Sasakian manifold \( \overline{M} \) is also indefinite trans-Sasakian manifold.

**Proof:** From (3.2) and lemma (3.1), we have

\begin{equation}
(3.4) \quad \overline{R}(X, \xi) \xi = \overline{R}(X, \xi)\xi + (\nabla_X B)(\xi, \xi) - (\nabla_{\xi} B)(X, \xi).
\end{equation}

Again from equation From (3.1) and lemma (3.1), we get

\begin{equation}
(3.5) \quad (\nabla_X B)(\xi, \xi) = 0, \quad (\nabla_X B)(X, \xi) = 0.
\end{equation}

Finally using From (3.5) in (3.4), we obtain

\[ \overline{R}(X, \xi) \xi = R(X, \xi) \xi + 0 + 0, \]
\[ R(X, \xi, \xi) = -\alpha (\epsilon \eta(X) \xi - X) + \beta (\phi X). \]

Hence the lemma.
Lemma 3.3. Let $M$ be an invariant submanifold of a indefinite trans-Sasakian manifold $\overline{M}$, then $\overline{R}(X, \xi)Y$ is tangent to $M$ iff $\phi B(X, \phi Y) = B(X, \phi Y)$ for any $X, Y \in \chi(M)$.

Proof: 

\[
(\nabla_X \phi)Y = \nabla_X \phi Y - \phi(\nabla_X Y) = \nabla_X \phi Y + B(X, \phi Y) - \phi(\nabla_X Y) - \phi(B(X, Y)) = (\nabla_X \phi)Y + B(X, \phi Y) - \phi(B(X, Y))
\]

Then we have 

\[
\alpha(g(Y, Y)\xi - \epsilon\eta(Y))X + \beta(g(\phi X, Y)\xi - \epsilon\eta(Y))\phi X
\]

Thus we get

\[
B(X, \phi Y) = \phi(B(X, Y))
\]

Lemma 3.4. Let $M$ be invariant submanifold of the indefinite trans-Sasakian manifold $M$ then, 

\[
\nabla_X B(Y, \xi) = -B(Y, \nabla_X \xi)
\]

for any $X, Y \in \chi(M)$ 

Proof: By using Lemma 3.1 we get 

\[
\nabla_X B(Y, \xi) = \nabla_X B(Y, \xi) - B(\nabla_X Y, \xi) - B(Y, \nabla_X \xi)
\]

Then, we have 

\[
\nabla_X B(Y, \xi) = -B(Y, \nabla_X \xi)
\]

Theorem 3.1. Let $M$ be an invariant submanifold of an indefinite trans-Sasakian manifold $\overline{M}$. Then $B$ is parallel if and only if $M$ is totally geodesic. 

Proof: Suppose that $B$ is parallel. For each $X, Y \in \chi(M)$ and using lemma 3.4 we get, 

\[
\nabla_X B(Y, \xi) = 0
\]

and 

\[
B(Y, \nabla_X \xi) = 0
\]

BY equation (2.1), we have 

\[
\nabla_X \xi = -\alpha \epsilon \phi X + \beta \{\epsilon X - \epsilon\eta(X)\xi\}
\]

Hence 

\[
B(Y, -\alpha \epsilon \phi X - \beta \epsilon \phi^2 X) = 0
\]

\[
-\alpha B(Y, \epsilon \phi X) - \beta B(Y, \epsilon \phi^2 X) = 0
\]

Since $M$ is an invariant submanifold of $\overline{M}$, we have $\phi(B(X, Y)) = 0$. 

From Lemma 3.3 it follows that 

\[
\phi(B(X, Y)) = B(X, \phi Y) = 0
\]

Then we get 

\[
\beta B(Y, \epsilon \phi^2 X) = 0
\]

hence it follows that 

\[
B(Y, -\epsilon X + \epsilon\eta(X)\xi) = 0
\]

so 

\[
B(Y, X) = 0
\]

viceversa let $M$ is totally geodesic, Then $B=0$, for all $X, Y, Z \epsilon TM$.

\[
(\overline{\nabla}_X B)(Y, Z) = D_X (B(Y, Z)) = B(\nabla_X Y, Z) - B(Y, \nabla_X Z) = 0
\]

thus we have $\nabla B = 0$
References

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