Rate of Equi-Summability of Some Series Associated with Fourier series and Certain Integrals

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Abstract: Salem and Zygmund have studied the equi-summability problem of factored Fourier series and conjugate series the factor being n^2, 0 < γ < 1. In the present work we study the rate of equi-summability of some series of functions belonging to H_p^{(w)} space, p≥1.


Keywords: Equi-summability, Abel mean, Generalised Hölder metric, H_p^{(w)} space, Poisson’s kernel.

I. Definition and Notation

Let f be 2π-periodic function and f ∈ L_p[0,2π], p ≥ 1. Let the Fourier series of f at x be given by

\[ f(x) = \sum_{n=-\infty}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \]  

(1.1)

The series conjugate to (1.1) is given by

\[ \sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} A_n(x) \]  

(1.2)

We write

\[ \psi_n(t) = f(x + t) - f(x - t) \]

\[ \psi_n(t) = \frac{1}{\pi} \int_0^\pi \frac{\psi_n(t)}{t} \cot \frac{1}{2} dt \]

\[ f(x; \epsilon) = \lim_{t \to 0^+} f(x; \epsilon) \]

f is said to belong to (see Zygmund [8])lip α if \[ |f(x + t) - f(x)| = o(t^\alpha) \] and to Lip α if |f(x + t) - f(x)| = O(t^\alpha), 0 < α ≤ 1.

It was Prossdorf [5], who first studied the degree of approximation problems of Fourier series inH_q(0 < α ≤ 1) space in the Hölder metric. Generalizing the Hölder metric, Leindler [4] introduced the space H^w space replacing t^α by an arbitrary function w which is given by

\[ H^w = \{ f \in C_2; |f(\delta, \alpha) - f(\delta, \alpha)| = O(w(\alpha)) \} \]

where w is a modulus of continuity. The norm \|f\|_w in H^w is defined by

\[ \|f\|_w = \sup_{x \neq y} \frac{|f(x) - f(y)|}{x - y} \]

In the case w(\delta) = \delta^a, 0 < \alpha ≤ 1 the space H^w reduces to H_\alpha space (the norm)||,\|_w being replaced by \|f\|_\alpha, introduced earlier by Prossdorf[5]. It is known that [4]

\[ H_\alpha \subseteq H_p \subseteq C_2, \quad 0 \leq \beta < \alpha \leq 1. \]

Das, Nath and Ray [3] have studied a further generalization of H^w space which is defined as follows. For f ∈ L_p[0,2π], p ≥ 1, we write

\[ A(f, w) = \sup_{t \neq 0} \frac{|f(x + t) - f(x)|}{|t|} \]

where w is a modulus of continuity. We say that f ∈ Lip (w, p) if

\[ \|f\|_{\alpha, p} = O(\omega(t)) \]

Define

\[ H_p^{(w)} = \{ f \in [0,2\pi], p ≥ 1 | A(f, \omega) < \infty \}. \]

and \[ \|f\|_{p, w} = \|f\|_p + A(f, \omega). \]

It can be easily verified that \[ \|f\|_{p, w} \] is a norm in H_p^{(w)}. If we put \[ \omega(t) = t^\alpha, \quad 0 < \alpha ≤ 1 \] then H_p^{(w)} reduces to H(\alpha, p) space with the norm \[ \|f\|_{p, w} \] replaced by \[ \|f\|_{(\alpha, p)} \] introduced earlier by

Das, Ghosh and Ray [2]. If further p = \infty then H(\alpha, p) reduces to H_\alpha space introduced earlier

By Prossdorf, if \[ \frac{\omega(t)}{t} \to 0 \] as \[ t \to 0 \] then f(x) exists and zero everywhere, and f is constant. Given

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the space $H_p^{(\omega)}$ and $H_p^{(\nu)}$ if $\frac{\omega(t)}{v(t)}$ is non-decreasing then

$$H_p^{(\omega)} \subseteq H_p^{(\nu)} \subseteq L_p, \ p \geq 1 \quad (1.5)$$

since $\|f\|_p^{(\nu)} = \max\{1, \frac{\omega(2\pi)}{v(2\pi)}\} \|f\|_p^{(\omega)}$. If $\omega(t) = t^\alpha$ and $\omega(t) = t^\beta$ then (1.5) reduces to the following

Inclusion relation

$$H_{(\alpha,\nu)} \subseteq H_{(\beta,\nu)} \subseteq L_p, \ p \geq 1$$

II. Introduction

For $0 < \gamma < 1$ and $0 < \epsilon < \infty$ we write

$$I_f(x; \epsilon) = -\frac{1}{\pi} f'(y + 1) \cos \frac{\pi}{2} \int_\epsilon^\infty \frac{w(x)}{s+1} \, ds \quad (2.1)$$

$$J_f(x; \epsilon) = -\frac{1}{\pi} f'(y + 1) \sin \frac{\pi}{2} \int_\epsilon^\infty \frac{w(x)}{s+1} \, ds \quad (2.2)$$

We define

$$I_f(x) = \lim_{\epsilon \to 0^+} I_f(x; \epsilon) \text{ and } J_f(x) = \lim_{\epsilon \to 0^+} J_f(x; \epsilon) \quad (2.3)$$

when ever the limits exist.

Salem and Zygmund ([6],p.30) have studied the equi-summability of

$$\sum_{n=1}^\infty n^n B_n(x) \quad (2.4)$$

and

$$\sum_{n=1}^\infty n^n A_n(x), 0 < a < 1$$

respectively with the integrals $I_f(x)$ and $J_f(x)$.

Let $U_f(r, x)$ and $V_f(r, x)$ respectively denote the Abel transforms of the series (2.4) and (2.5) that is

$$U_f(r, x) = \sum_{n=1}^\infty n^n B_n(x)r^n, 0 < r < 1, \quad 0 < \gamma < 1$$

$$V_f(r, x) = \sum_{n=1}^\infty n^n A_n(x)r^n, 0 < r < 1, \quad 0 < \gamma < 1$$

Salem and Zygmund proved the following.

Theorem A([6],p.30)

Let the function $f(x)$ be $2\pi$ - periodic and belong to $lip \gamma$, where $0 < \gamma < 1$. Then the difference

$$I_f(x; 1 - r) - U_f(r, x) \quad (2.6)$$

tends to zero uniformly in $x$ as $r \to 1^-$. If $f \in Lip \gamma, 0 < \gamma < 1$ the above expression in (2.6) is uniformly bounded.

Theorem B([6],p.31)

Let the function $f(x)$ be defined as in Theorem A. Then

$$I_f(x; 1 - r) - V_f(r, x) \quad (2.7)$$

tends to zero uniformly in $x$ as $r \to 1^-$. If $f \in Lip \gamma, 0 < \gamma < 1$ the above expression in (2.7) is uniformly bounded.

The object of the present paper is to obtain the degree at which the expression (2.6) and (2.7) converges for functions in $H_p^{(\omega)}$ space.

III. Main Result

We prove the following theorems.

Theorems 1: Let $v$ and $\omega$ be moduli of continuity such that $\omega/v$ is non-decreasing. If $f \in H_p^{(\omega)}, p \geq 1$ and, $0 < \gamma < 1$ then

$$\|I_f(\cdot; 1 - r) - U_f(\cdot, r)\|_p = O(1)(1 - r)^{1-\gamma} \int_{1-r}^\infty \frac{\omega(u)}{v(u)^{2\gamma}} \, du$$

Theorems 2: Let $v$ and $\omega$ be moduli of continuity such that $\omega/v$ is non-decreasing. If $f \in H_p^{(\omega)}, p \geq 1$ and, $0 < \gamma < 1$ then

$$\|I_f(\cdot; 1 - r) - V_f(\cdot, r)\|_p = O(1)(1 - r)^{1-\gamma} \int_{1-r}^\infty \frac{\omega(u)}{v(u)^{2\gamma}} \, du$$

Remark: See ([3], p.47) and ([1], p.25) where the authors have obtained the degree of approximation in $H_p^{(\omega)}, (p \geq 1)$ space respectively for the partial sum and Rogosinski mean.
IV. We need following notations and lemmas for the proof of Theorem 1.

For $0 < \gamma < 1$, we write

$$B(r, x, t) = -\sum_{n=1}^{\infty} 2 \sin nt R_n(x)r^n, \quad (4.1)$$

$$h(x, t) = B(r, x, t) - \Psi(x, t), \quad (4.2)$$

$$i(x, t, u) = \Psi(x, t + u) - \Psi(x, t - u) - 2\Psi(x, t) \quad (4.3)$$

$$j(x, t, u) = \Psi(x, t + u) - \Psi(x, t - u) \quad (4.4)$$

$$P(r, u) = \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos nu = \frac{1}{2} \frac{1 - r^2}{1 - 2\cos \omega + r^2} \quad (4.5)$$

$$f(r, x) = \frac{1}{2} + \sum_{n=1}^{\infty} A_n(x)r^n \quad (4.6)$$

We need the following lemmas for the proof of Theorem 1.

**Lemma 1:** Let $\omega(t)$ and $\nu(t)$ be defined as in Theorem 1

If $f \in H^p_{(\omega)}$, $p \geq 1$, then for $0 < u \leq \pi$

(i) $||\Psi(.t + u) - \Psi(.t)||_p = O(\omega(u))$

(ii) $||\Psi(.t) - \Psi(.t + y, t)||_p = O(\omega(|y|))$

(iii) $||i(.t, t, u)||_p = O(\omega(u))$

(iv) $||i(.t, t, u) - i(.t + y, t, u)||_p = O(1) \frac{\omega(u)}{\omega(|y|)}$

(v) $||i(.t, t) - i(.t + y, t)||_p = O(1) \frac{\omega(u)}{\nu(u)}$

(vi) $||i(.t, t, u)||_p = O(\omega(u))$

(vii) $||f(.t, t, u) - f(.t + y, t, u)||_p = O(1) \frac{\omega(u)}{\nu(u)}$

(viii) $||f(.t, t, u) - f(.t + y, t, u)||_p = O(1) \frac{\omega(u)}{\nu(u)}$

**Proof:** We have

$$\Psi(x, t + u) - \Psi(x, t) = f(x + t + u) - f(x - t - u) - f(x + t) + f(x - t)$$

$$= \{f(x + t + u) - f(x + t)\} - \{f(x + t - u) - f(x - t)\}$$

It follows from the Minkowski’s inequality for $0 < u \leq \pi$

$$||\Psi(.t + u) - \Psi(.t)||_p \leq ||\Psi(.t + u) - \Psi(.t + y)||_p + ||\Psi(.t - u) - \Psi(.t - y)||_p$$

$$= O(\omega(u))$$

By using the fact that $f \in L^p(\omega, p)$. It is easy to see that the result holds when $\Psi(x, t + u)$ is replaced by $\Psi(x, t - u)$.

For the proof of (ii), writing

$$\Psi(x, t) - \Psi(x + y, t) = \{f(x + t) - f(x + y + t)\} - \{f(x - t) - f(x - t - y)\}$$

and proceeding as above, we have

$$||\Psi(.t) - \Psi(.t + y, t)||_p = O(\omega(|y|)).$$

Writing

$$i(x, t, u) = \{\Psi(x, t + u) - \Psi(x, t)\} + \{\Psi(x, t - u) - \Psi(x, t)\}$$

and applying Minkowski’s inequality and part(i) we obtain (iii). First part of (iv) follows from (iii). We write

$$i(x, t, u) - i(x + y, t, u) = \{\Psi(x, t + u) - \Psi(x + y, t + u)\}$$

$$+\{\Psi(x, t - u) - \Psi(x + y, t - u)\}$$

$$-2\{\Psi(x, t) - \Psi(x + y, t)\}$$

from which the second part of (iv) follows by applying Minkowski’s inequality and (ii).

Now, by first part of (iv)

$$||i(.t, t, u) - i(.t + y, t, u)||_p = O(\omega(u))$$

$$= O\left(\frac{\omega(u)}{\nu(u)}\right)$$

for $u \leq |y|$ as $v$ is non-decreasing.

If $u \geq |y|$, then

$$\frac{\omega(u)}{\nu(u)} \geq \frac{\omega(|y|)}{\nu(|y|)}$$

as $\frac{\omega(u)}{\nu(u)}$ is non-decreasing

(4.7)

so that

$$||i(.t, t, u) - i(.t + y, t, u)||_p = O(\omega(|y|))$$

(by second part of (iv))
Lemma 2: Let $B(r, x, t)$ and $h(x, t)$ be defined as in (4.1) and (4.2). Then under the hypothesis of Theorem 1 as $r \to 1^-$

(i) $\| h(\cdot, t) - h(\cdot, +y, t) \|_p = O(1) \nu(|y|)(1 - r) \int_{1-r}^{1} \frac{\omega(u)}{u^2} du$

(ii) $\| h(\cdot, t) \|_p = O(1)(1 - r) \int_{1-r}^{1} \frac{\omega(u)}{u^2} du$

(iii) $\| h(\cdot, t) - h(\cdot, +y, t) \|_p = O(1)(1 - r) \int_{1-r}^{1} \frac{\omega(u)}{v(u)u^2} du$

Proof: Now from (4.1) $B(r, x, t) = - \sum_{n=1}^\infty 2 \sin nt B_n(x)r^n, 0 < r < 1$

It can easily verify that $B(r, x, t) = \sum_{m=1}^\infty A_n(x + t) - \sum_{n=1}^\infty A_n(x - t)$

which by ((7)p.96) equal to

$$f(r, x + t) - f(r, x - t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \{ f(x + t + u) - f(x - t + u) \} P(r, u) du$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \{ \psi(x, t + u) - \psi(x, t - u) \} P(r, u) du$$

(4.8)

where the Poisson's kernel $P$ is defined in (4.5). As

$$\int_{0}^{\pi} P(r, u) du = \frac{\pi}{2}$$

from (4.2), we get

$$h(x, t) = B(r, x, t) = - \sum_{n=1}^\infty A_n(x + t) - \sum_{n=1}^\infty A_n(x - t)$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \{ \psi(x, t + u) + \psi(x, t - u) - 2\psi(x, t) \} P(r, u) du$$

$$= \frac{1}{\pi} \int_{0}^{\pi} i(x, t, u) P(r, u) du$$

(4.9)

from which it follows that

$$h(x, t) - h(x + y, t) = \frac{1}{\pi} \int_{0}^{\pi} \{ i(x, t, u) - i(x + y, t, u) \} P(r, u) du$$

Applying generalised Minkowski’s inequality, we get for $p \geq 1$

$$\pi \| h(\cdot, t) - h(\cdot, +y, t) \|_p \leq \int_{0}^{\pi} \| i(\cdot, t, u) - i(\cdot, +y, t, u) \|_p^P(r, u) du$$

$$= \left( \int_{0}^{1-r} + \int_{1-r}^{\pi} \right) \| i(\cdot, t, u) - i(\cdot, +y, t, u) \|_p P(r, u) du$$

(4.10)

We know ([7], Vol.1,p.96) that for $0 < u \leq \pi$ and $0 < r < 1$

$$P(r, u) = O\left( \frac{1}{u^2} \right)$$

(4.11)

$$P(r, u) = O\left( \frac{1-r}{u^2} \right)$$

(4.12)

Now applying Lemma 1(v) and the estimates (4.11) and (4.12) respectively for the integrals $\int_{0}^{1-r}$ and $\int_{1-r}^{\pi}$ in (4.10), we obtain

$$\pi \| h(\cdot, t) - h(\cdot, +y, t) \|_p = O(1) \nu(|y|) \left[ \frac{1}{1-r} \int_{0}^{1-r} \frac{\omega(u)}{v(u)u^2} du + (1 - r) \int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du \right]$$

(4.13)

As $\omega(u) v(u)$ is monotonic non-decreasing, we have

$$\int_{1-r}^{\pi} \frac{\omega(u)}{v(u)u^2} du \geq \frac{\omega(1-r)}{v(1-r)} \int_{1-r}^{\pi} \frac{du}{u^2} \geq k \frac{\omega(1-r)}{1-r} \frac{1}{v(1-r)}$$

(4.14)

As $f \in Lip(\omega, p), p \geq 1$, proceeding as above Lemma 2(ii) can be proved. Lemma 2(iii) is an immediate consequence of (i) and (ii) of Lemma 2 as by definition.

$$\| h(\cdot, t) - h(\cdot, +y, t) \|_p \leq \| h(\cdot, t) \|_p$$

(4.9)

Lemma 3: Let $B(r, x, t)$ be defined as in (4.1). Then under the hypothesis of Theorem 1, we have as

$$r \to 1^- \text{ for } p \geq 1$$

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(i) \( \left\| \frac{\partial}{\partial t}(B(r,+,t) - B(r,+y,t)) \right\|_p = O(1)(|y|)(1-r) \int_1^\pi \frac{u}{v(u)}mu^3 du \)

(ii) \( \left\| \frac{\partial}{\partial t}(B(r,,t)) \right\|_p = O(1)(1-r) \int_1^\pi \frac{u}{v(u)}mu^3 du \)

(iii) \( \left\| \frac{\partial}{\partial t}(B(r,+,t) - B(r,+y,t)) \right\|_p = O(1)(1-r) \int_1^\pi \frac{u}{v(u)}mu^3 du \)

Proof: Proceeding as in the proof of Lemma 2 and using the integral given in (4.6) for \( f(r,x) \), we get from (4.8) that

\[ B(r,x,t) = \frac{1}{\pi} \int_{-\pi}^\pi f(u)(P(r,u - x - t) - P(r,u - x + t))du \]

which on partial differentiation with respect to \( t \) gives, after simplification,

\[ \frac{\partial}{\partial t}B(r,x,t) = \frac{1}{\pi} \int_{-\pi}^\pi f(u)(-P'(r,u - x - t) - P'(r,u - x + t))du \]

where \( f(x,t,u) \) has been defined in (4.4)

Therefore

\[ \frac{\partial}{\partial t}B(r,x,t) = \frac{1}{\pi} \int_{-\pi}^\pi f(u)(j(x,t,u) - j(x,y,t,u))P'(r,u)du \]

Applying generalized Minkowski’s inequality, we have for \( p \geq 1 \)

\[ \left\| \frac{\partial}{\partial t}(B(r,+,t) - B(r,+y,t)) \right\|_p \leq \frac{1}{\pi} \int_{-\pi}^\pi \left\| j(x,t,u) - j(x,y,t,u) \right\|_p |P'(r,u)|du \]

We know ([6], p.31) that for \( 0 < u \leq \pi \) and \( 0 < r < 1 \)

\[ P'(r,u) = \frac{(1-r^2)\sin u}{(1-2\cos u+r^2)^2} \]

(4.16)

(4.17)

(4.18)

Applying Lemma 1(viii) and the estimates (4.17) and (4.18) respectively for the integrals \( \int_0^1r^2 \) and \( \int_1^r \) in (4.15) we obtain

\[ \left\| \frac{\partial}{\partial t}(B(r,+,t) - B(r,+y,t)) \right\|_p = O(1)u(|y|) \left\{ \frac{1}{(1-r)^2} \int_0^1 \frac{u\omega(u)}{v(u)}du + (1-r) \int_1^\pi \frac{\omega(u)}{v(u)}mu^3 du \right\} \]

\[ = O(1)u(|y|) \left\{ \frac{\omega(1-r)}{(1-r)\pi} + (1-r) \int_1^\pi \frac{\omega(u)}{v(u)}mu^3 du \right\} \]

\[ = O(1)u(|y|) \left\{ (1-r) \int_1^\pi \frac{\omega(u)}{v(u)}mu^3 \right\} \]

as by the monotonicity of \( \frac{\omega(u)}{v(u)} \)

\[ \int_1^\pi \frac{\omega(u)du}{v(u)\pi} \geq \frac{K}{(1-r)^2} \]

where \( K \) is some positive constant and this completes the proof of (i). As by the hypothesis \( f \in Lip(\omega,p) \), \( p \geq 1 \) proceeding as above part (ii) can be proved.

By definition

\[ \left\| \frac{\partial}{\partial t}(B(r,+,t)) \right\|_p = \sup_{y \neq 0} \frac{\left\| \frac{\partial}{\partial t}(B(r,+,t)) - \frac{\partial}{\partial t}(B(r,+,t)) \right\|_p}{v(|y|)} \]

and hence part (iii) follows from (i) and (ii).

Remark: In what follows, in the proof of Theorem 1 we do not need part (iii) of Lemma 2. However, part (iii) has been included for the sake of completeness of Lemma 2. Similar remarks also applies to part (iii) of Lemma 3.

V. Proof of Theorem

From (4.1), we have for \( 0 < y < 1 \)

\[ B(r,x,t) = \frac{1}{t^{1+\gamma}} \left\{ -2 \sum_{n=1}^{\infty} \sin nt \frac{B_n(x)r^n}{t^{1+\gamma}} \right\} , t \neq 0 \]

The above series the right is uniformly convergent for \( t > \epsilon > 0 \). Hence integrating term by term over the interval \((\epsilon,T)\) and observing that (see[6],p.31)

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\[
\left| \int_0^\infty \sin nt \, \frac{dt}{1 + ty} \right| < C \eta \frac{e^{1 + \nu}}{t^{1 + \nu}} \quad \text{for some } C > 1, 0 < \nu < 1
\]
(5.3)

Using (5.3) in (5.2), we get after simple computation,
\[
\frac{-\Gamma(y + 1) \cos \frac{n \pi}{\gamma}}{\pi} \int_0^\infty \frac{t^n}{1 + ty} \, dt = \sum_{n=1}^\infty n^r B_n(x) r^n = U_r(x, y)
\]
which ensures that
\[
\frac{-\Gamma(y + 1) \cos \frac{n \pi}{\gamma}}{\pi} \int_0^\infty \frac{t^n}{1 + ty} \, dt = U_r(x, y)
\]
By the notations of (2.1), (4.2) the above result takes the following form.
\[
U_r(x, y) = \frac{\Gamma(y + 1) \cos \frac{n \pi}{\gamma}}{\pi} \frac{1}{D(r, x)}
\]
(5.4)

where
\[
D(r, x) = \int_0^1 h(x, t) \, dt + \int_{-1}^0 \frac{B(r, x, t)}{1 + ty} \, dt, 0 < \nu < 1
\]
(5.5)

For the proof of Theorem 1, it is sufficient to show that as \( r \to 1^- \)
\[
\|D(r, \cdot) - D(r, \cdot + y)\|_p = O(1)(1 - r)^{1 - \nu} \int_0^\pi \frac{\omega(u)}{\nu(u)u^2} \, du
\]
(5.6)

Now
\[
D(r, x) - D(r, x + y) = \int_0^1 \frac{B(r, x, t) - B(r, x + y, t)}{1 + ty} \, dt + \int_{-1}^0 \frac{B(r, x, t)}{1 + ty} \, dt
\]
(5.7)

By Lemma2(i)
\[
I_2 = O(1) v(y) (1 - r) \int_0^\pi \frac{\omega(u)}{\nu(u)u^2} \, du \int_{-1}^0 \frac{dt}{1 + ty}
\]
\[
= O(1) v(y) (1 - r)^{1 - \nu} \int_0^\pi \frac{\omega(u)}{\nu(u)u^2} \, du
\]
(5.8)

As \( B(r, x, 0) = 0 = B(r, x + y, 0) \), by Mean value Theorem we have for some \( \theta \) with \( 0 < \theta < 1 \).
\[
B(r, x, t) - B(r, x + y, t) = \left[ \frac{\partial}{\partial t} (B(r, x, t) - B(r, x + y, t)) \right]_{t=\theta}
\]
Hence by Lemma3
\[
\| \{ (B(r, \cdot, t) - B(r, \cdot + y, t)) \} \|_p = O(1) v(y) (1 - r) \int_0^\pi \frac{\omega(u)}{\nu(u)u^2} \, du
\]
(5.9)

Using (5.9), we get
\[
I_1 = \int_0^1 \frac{\|B(r, \cdot, t) - B(r, \cdot + y, t)\|_p}{1 + ty} \, dt
\]
\[
= O(1) v(y) (1 - r) \int_0^\pi \frac{\omega(u)}{\nu(u)u^2} \, du \int_{-1}^0 \frac{dt}{1 + ty}
\]
\[
= O(1) v(y) (1 - r)^{1 - \nu} \int_0^\pi \frac{\omega(u)}{\nu(u)u^2} \, du
\]
(5.10)

Collecting the results from (5.7), (5.8) and (5.10), we obtain
\[
\|D(r, \cdot) - D(r, \cdot + y)\|_p = O(1) v(y) (1 - r)^{1 - \nu} \left[ \int_0^\pi \frac{\omega(u)}{\nu(u)u^2} \, du + (1 - r) \int_{-1}^0 \frac{\omega(u)}{\nu(u)u^2} \, du \right]
\]
\[
= O(1) v(y) (1 - r)^{1 - \nu} \int_0^\pi \frac{\omega(u)}{\nu(u)u^2} \, du
\]
(5.11)

from which it follows that

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\[
\sup_{y \neq 0} \frac{\|D(r, \cdot) - D(r, \cdot + y)\|_p}{v(y)} = O(1(1 - r)^{1 - \gamma}) \int_{1 - r}^{1} \frac{\omega(u)}{v(u)u^2} \, du
\]
(5.12)
As \( f \in \text{Lip}(\omega, p) \), \( p \geq 1 \) by the hypothesis, proceeding as above it can be proved that
\[
\|D(r, \cdot)\|_p = O(1)(1 - r)^{1 - \gamma} \int_{1 - r}^{1} \frac{\omega(u)}{u^2} \, du
\]
(5.13)
From (5.12) and (5.13), it follows that
\[
\|D(r, \cdot) - D(r, \cdot + y)\|_p = O(1)(1 - r)^{1 - \gamma} \int_{1 - r}^{1} \frac{\omega(u)}{v(u)u^2} \, du
\]
and this completes the proof of Theorem 1.

VI. Additional notations and Lemmas for the proof of Theorem 2
We know ([7], Vol.1,p.50) that
\[
\emptyset(x, t) = 2\sum_{n=1}^{\infty} \cos nt A_n(t)
\]
(6.1)
Let \( A(r, x, t) \) denote the Abel mean of the series
\[
-2\sum_{n=1}^{\infty}(1 - \cos nt) A_n(x)
\]
(6.2)
that is:
\[
A(r, x, t) = -2\sum_{n=1}^{\infty}(1 - \cos nt) A_n(x) r^n, 0 < r < 1
\]
(6.3)
We write
\[
H(x, t) = A(r, x, t) - \emptyset(x, t)
\]
(6.4)
\[
I(x, t, u) = \emptyset(x, t + u) + \emptyset(x, t - u) - 2\emptyset(x, u) - 2\emptyset(x, t)
\]
(6.5)
\[
f(x, t, u) = \emptyset(x, t + u) - \emptyset(x, t - u)
\]
(6.6)
\[
\Delta(r, x) = \int_{1-r}^{1} \frac{H(x,t)}{x^{1+y}} \, dt + \int_{0}^{1-r} \frac{\Delta(r, x)}{t^{1+y}} \, dt, 0 < y < 1
\]
(6.7)
We need the following Lemmas for the proof of Theorem 2.

Lemma 4 Let \( \omega(t) \) and \( v(t) \) be defined as in Theorem 2. If \( f \in H_p^{(\omega)}, p \geq 1 \) then for \( 0 < u \leq \pi \n\)
(i) \( \|\emptyset, y\|_p = O(\omega(u)) \)
(ii) \( \|\emptyset, t + u - \emptyset, t\|_p = O(\omega(u)) \)
(iii) \( \|\emptyset, t - \emptyset, +y, t\|_p = O(\omega(\omega)) \)
(iv) \( \|I, t, u\|_p = O(\omega(u)) \)
(v) \( \|I, t, u\| - I, +y, t, u\|_p = O\left\{ \frac{\omega(u)}{\omega(\omega)} \right\} \)
(vi) \( \|I, t, u\| - I, +y, t, u\|_p = O(1)v(\omega) \)
(vii) \( \|I, t, u\|_p = O(\omega(u)) \)
(viii) \( \|I, t, u\| - I, +y, t, u\|_p = O(1) \left\{ \frac{\omega(u)}{\omega(\omega)} \right\} \)
We omit the proof of Lemma 4 as its proof is similar to Lemma 1.

Lemma 5 Let \( A(r, t) \) and \( H(x, t) \) be defined as in section 3. Let \( \omega(t) \) and \( v(t) \) be defined as in Theorem 2. If \( f \in H_p^{(\omega)}, p \geq 1 \) then \( r \to 1^- \n\)
(i) \( \|H, (\cdot, t) - H, (\cdot + y, t)\|_p = O(1)v(\omega)(1 - r) \int_{1-r}^{1} \frac{\omega(u)}{v(u)u^2} \, du \)
(ii) \( \|H, (\cdot, t)\|_p = O(1)(1 - r) \int_{1-r}^{1} \frac{\omega(u)}{u^2} \, du \)
(iii) \( \|H, (\cdot, t) - H, (\cdot + y, t)\|^{(v)}_p = O(1)(1 - r) \int_{1-r}^{1-r} \frac{\omega(u)}{v(u)u^2} \, du \)

Proof: We have
\[
A(r, x, t) = -2\sum_{n=1}^{\infty}(1 - \cos nt) A_n(x) r^n
\]
\[
= \sum_{n=1}^{\infty}[A_n(x + t) + A_n(x - t) - A_n(x)] r^n
\]
\[
= f(r, x + t) + f(r, x - t) - f(r, x)
\]
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\[
\frac{1}{\pi} \int_{-\pi}^{\pi} \left( f(x + t + u) + f(x - t + u) - 2f(x + u) \right) P(r, u) du
\]

(6.8)

using (4.6), adopting the argument used in deriving the integral representation of \( B(r, x, t) \) in section 3.

It can be shown that

\[
A(r, x, t) = \frac{1}{\pi} \int_{0}^{\pi} (\emptyset(x, t + u) + \emptyset(x, t - u) - 2\emptyset(x, u)) P(r, u) du
\]

(6.9)

\[
A(r, x, t) - \emptyset(x, t) = \frac{1}{\pi} \int_{0}^{\pi} I(x, t, u) P(r, u) du
\]

(6.10)

The remaining part of the proof of the Lemma is similar to the proof of Lemma 2.

**Lemma 6** Let \( A(r, x, t) \) be defined as in section 5. Let \( \omega(t) \) and \( v(t) \) be defined as in Theorem 2.

If \( f \in H_p^\omega, p \geq 1 \) then \( r \to 1^- \)

\[
\begin{align*}
(i) \quad & \left\| \frac{\partial}{\partial t} \{ A(r, ., t) - A(r, ., +y, t) \} \right\|_p^r = O(1) \nu(b^y) (1 - r) \int_{1-r}^{r} \frac{\omega(u)}{u^2} \, du \\
(ii) \quad & \left\| \frac{\partial}{\partial t} A(r, ., t) \right\|_p^r = O(1) (1 - r) \int_{1-r}^{r} \frac{\omega(u)}{u^2} \, du \\
(iii) \quad & \left\| \frac{\partial}{\partial t} \{ A(r, ., t) - A(r, ., +y, t) \} \right\|_p^r = O(1) (1 - r) \int_{1-r}^{r} \frac{\omega(u)}{u^2} \, du
\end{align*}
\]

**Proof:** We know that

\[A(r, x, t) = f(r, x, t) + f(r, x - t) - 2f(r, x).\]

Using (4.6) we get after simple computation

\[
\frac{\partial}{\partial t} A(r, x, t) = \frac{1}{\pi} \int_{0}^{\pi} f(u) \{ P(r, u - x + t) + P(r, u + x - t) - 2P(r, u - x) \} du
\]

which has similarity with \( j(x, t, u) \) replaced by \( j(x, t, u) \) the integral representation of \( \frac{\partial}{\partial t} B(r, x, t) \) in the proof of Lemma 3 and hence proceeding as in the proof of Lemma 3 the results of Lemma 6 can be established.

**Proof of Theorem 2**

Proceeding as in Theorem 1 and making use of the fact that \([6], p.32\)

\[
\int_{0}^{\infty} \frac{\sin \frac{\pi x}{r^{1+y}}}{r^{1+y}} \, dx = \frac{\pi}{4^y(\gamma+1)^{\gamma+1}} 0 < \gamma < 1.
\]

We have

\[
-\frac{\gamma + 1}{\pi} \int_{0}^{\infty} \frac{A(r, x, t)}{1+y} \, dx = \sum_{n=1}^{\infty} n^y A_n(x) r^n = V_f(r, x)
\]

which further ensures that (using the notation given in (2.2), (6.4) and (6.7))

\[
I_f(x, 1-r) - V_f(r, x) = \frac{r^{\gamma+1}}{\pi} \int_{0}^{\infty} \frac{A(r, x, t)}{1+y} \, dx (r, x)
\]

(7.1)

For the proof of Theorem 2, it is enough to show that as \( r \to 1^- \)

\[
\left\| \Delta(r, .) - \Delta(r, +y) \right\|_p^{(r, y)} = O(1) (1 - r)^{1-y} \int_{1-r}^{r} \frac{\omega(u)}{u^2} \, du
\]

(7.2)

Using Lemma 5 and Lemma 6 (in place of Lemma 2 and Lemma 3) and adopting the technique used in proving (5.6) (see proof of Theorem 1) the validity of (7.2) can be established and this completes the proof of Theorem 2.

**VII. Corollaries**

By taking \( \omega(t) = t^\alpha \) and \( v(t) = t^\beta, 0 \leq \beta < \alpha \leq 1 \), in Theorem 1, we obtain the following corollaries:

**Corollary 1**

If \( f \in H(\alpha, p), p \geq 1, 0 \leq \beta < \alpha \leq 1, \) and \( 0 < \gamma < 1 \) then

\[
\left\| I_f (.; 1-r) - U_n (r ; .) \right\|_{(\beta, p)} = O(1) \left\{ \begin{array}{ll}
(1 - r)^{\alpha - (\beta + y)}, & 0 \leq \beta < \beta + y \leq \alpha < 1 \\
(1 - r)^{1-(\beta + y)}, & 0 \leq \beta < \beta + y \leq \alpha = 1 \\
(1 - r)^{1-y} \log \frac{1}{1-r}, & \beta = 0, \alpha = 1, \gamma < 1
\end{array} \right.
\]

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Putting $p = \infty$, in Corollary 1, we get

**Corollary 2**

If $f \in H_{\alpha}, 0 \leq \beta < \alpha \leq 1$, and $0 < \gamma < 1$ then

$$|I_{\gamma}(\cdot;1-r) - U_n(r;\cdot)|_\beta = O(1) \left\{ \begin{array}{ll} (1-r)^{\alpha-(\beta+\gamma)}, & 0 \leq \beta < \beta + \gamma \leq \alpha < 1 \\ (1-r)^{1-\beta+\gamma}, & 0 < \beta < \beta + \gamma \leq \alpha = 1 \\ (1-r)^{1-\gamma \log \frac{1}{1-r}}, & \beta = 0, \alpha = 1, 0 < \gamma < 1 \end{array} \right.$$ 

**Corollary 3**

If $f \in L_{\text{lip}}$, $0 < \alpha \leq 1$ and $0 < \gamma < 1$ then

$$\|I_{\gamma}(\cdot;1-r) - U_n(r;\cdot)\|_c = O(1) \left\{ \begin{array}{ll} (1-r)^{\alpha-\gamma}, & 0 < \gamma \leq \alpha < 1 \\ (1-r)^{1-\gamma \log \frac{1}{1-r}}, & \beta = 0, \alpha = 1, 0 < \gamma < 1 \end{array} \right.$$ 

In the case $\alpha = \gamma, 0 < \gamma < 1$, Corollary 3 reduces to the second part of Theorem A due to Salem and Zygmund.

Analogous results can be obtained for Theorem 2.

**References**


[4]. 43-60.


