Diophantine Equation’s Integer Solutions

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Abstract: In this study; where the x, y, z numbers are unknowns, a and n positive integers, existence of x, y, z integer solutions’ for each positive n integers of \( x^n + ay^2 = z^2 \) Diophantine equation was shown by induction. Also change of x was analyzed according to the values which a will take and each situation was exemplified.

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I. Introduction

The conjecture which is stated as like, where the Number theory’s famous “\( p, q, r \)’s are natural numbers \( x, y, z \) integer solutions’ for each positive n integers of \( x^n + ay^2 = z^2 \) Diophantine equation was shown by induction. Also change of x was analyzed according to the values which a will take.

First let’s give the lemma that guarantees the existence of all integer solutions of equation (1) in the special situation \( n = 2 \).

Lemma 1.1 Where the \( a, u, v \) positive integers, \( a \) doesn’t contain square factor (multiplier) and \( u+av \equiv 1 (mod \ 2) \) and \( (u, av) = 1 \). Also the let the operation \([5]\) of \((2,1),(2,0),(0,2),(0,1)\) respectively.

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Now let’s give our theorem regarding integer solutions of equation (2).

Theorem 2.1. Let the integer z be given as \( x = u^2 - av^2 \) where \( a, u, v \) positive integers, \( a \) doesn’t contain square factor (multiplier), \( u+av = 1(mod \ 2) \) and \( (u, av) = 1 \). Also the let the operation \([5]\) of

\[
(u_1,v_1,x_1) \bigotimes (u_2,v_2,x_2) = (u_1u_2 + av_1v_2, u_1v_2 + v_1u_2, x_1x_2)
\]
be defined for the solution triples of \((u_1, v_1, x_1), (u_2, v_2, x_2)\) in order to check \(z = u^2 + av^2\) equation
by \(u_1, u_2, v_1, v_2, z_1, z_2\) integers. Then, under the operation (5), all integer solutions of \(x^n + ay^2 = z^2\)
Diophantine equation shall be given as:

\[
z = \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} a^k u^{n-2k} v^{2k}, \quad y = \sum_{k=0}^{\left\lfloor \frac{n-1}{2k+1} \right\rfloor} a^k u^{n-2k-1} v^{2k+1}, \quad x = u^2 - av^2
\]

(6)

**Proof.** Let us make the proof with induction on \(n\). According to induction hypothesis for \(n = 1\), we see;
\(z = u, y = v, x = u^2 - av^2\)
In \(n = 2\) special condition we could be able to draw attention to \(z = u^2 + av^2, y = 2uv, x = u^2 - av^2\)
because of Lemma 1.1.
Now, consider claim is correct for \(n\). So let’s accept it is

\[
z = \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} a^k u^{n-2k} v^{2k}, \quad y = \sum_{k=0}^{\left\lfloor \frac{n-1}{2k+1} \right\rfloor} a^k u^{n-2k-1} v^{2k+1}, \quad x = u^2 - av^2
\]
and show the truth of claim for \(n + 1\). According to operation (5) it is;

\[
(u, v, x)^n \circledast (u, v, x) = \left( \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} a^k u^{n-2k} v^{2k}, \sum_{k=0}^{\left\lfloor \frac{n-1}{2k+1} \right\rfloor} a^k u^{n-2k-1} v^{2k+1}, x^n \right) \circledast (u, v, x)
\]

\[
= \left( \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} a^k u^{n-2k} v^{2k} + \sum_{k=0}^{\left\lfloor \frac{n-1}{2k+1} \right\rfloor} a^k u^{n-2k-1} v^{2k+1}, \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} a^k u^{n-2k} v^{2k} + \sum_{k=0}^{\left\lfloor \frac{n-1}{2k+1} \right\rfloor} a^k u^{n-2k-1} v^{2k+1}, x^n + x \right)
\]

\[
= \left( \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} a^k u^{n-2k} v^{2k} + \sum_{k=0}^{\left\lfloor \frac{n-1}{2k+1} \right\rfloor} a^k u^{n-2k-1} v^{2k+1}, \sum_{k=0}^{\left\lfloor \frac{n}{2k} \right\rfloor} a^k u^{n-2k} v^{2k} + \sum_{k=0}^{\left\lfloor \frac{n-1}{2k+1} \right\rfloor} a^k u^{n-2k-1} v^{2k+1}, x^n + x \right)
\]

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\[
\begin{align*}
\left( n+1 \right) z^{n+1} & + \left( \sum_{k=1}^{n} \left( \frac{n}{2k} \right) u^n \cdot v^{2k} - 1 \right) k^{a^2} u^n \cdot v^{2k} - 1 = \left( \sum_{k=1}^{n} \left( \frac{n}{2k} \right) u^n \cdot v^{2k} - 1 \right) k^{a^2} u^n \cdot v^{2k} - 1, \\
\left( n - 1 \right) z^{n-1} & + \left( \sum_{k=0}^{n-1} \left( \frac{n}{2k} \right) u^n \cdot v^{2k} - 1 \right) k^{a^2} u^n \cdot v^{2k} - 1 = \left( \sum_{k=0}^{n-1} \left( \frac{n}{2k} \right) u^n \cdot v^{2k} - 1 \right) k^{a^2} u^n \cdot v^{2k} - 1, \\
\left( n + 1 \right) z^{n+1} & + \left( \sum_{k=0}^{n+1} \left( \frac{n}{2k} \right) u^n \cdot v^{2k} - 1 \right) k^{a^2} u^n \cdot v^{2k} - 1 = \left( \sum_{k=0}^{n+1} \left( \frac{n}{2k} \right) u^n \cdot v^{2k} - 1 \right) k^{a^2} u^n \cdot v^{2k} - 1, \\
\left( n - 1 \right) z^{n-1} & + \left( \sum_{k=0}^{n-1} \left( \frac{n}{2k} \right) u^n \cdot v^{2k} - 1 \right) k^{a^2} u^n \cdot v^{2k} - 1 = \left( \sum_{k=0}^{n-1} \left( \frac{n}{2k} \right) u^n \cdot v^{2k} - 1 \right) k^{a^2} u^n \cdot v^{2k} - 1
\end{align*}
\]

so it means that this claim is valid for \(n + 1\).

**Corollary 2.1.1** i) If \(a = 1\) is taken at equation (2), the solution triples which was given by (6) transforms as following:

\[
z = \sum_{k=0}^{n} \left( \frac{n}{2k} \right) u^n \cdot v^{2k}, \quad y = \sum_{k=0}^{n-1} \left( \frac{n-1}{2k+1} \right) u^n \cdot v^{2k+1} + 1, \quad x = u^2 - v^2
\]

ii) If it is accepted that \(a = t^2\), \(n = 2n_1\) and \((z, ty) = 1\) at equation (2) for a \(t\) integer which is different than zero, then the triple of \((x^{n_1}, ty, z)\) becomes a primitive Pythagorean triple [6].

iii) If \(a = t^2\), \(x = x_1^2\) and \((z, ty) = 1\) where \(t\) and \(x_1\) are integers which are different that zero at equation (2), then the \((x_1^{n_1}, ty, z)\) triple is a primitive Pythagorean triple [6].

**Proof.** i) It is from the formulas of (6) [4].

ii) If it is accepted that \(a = t^2\), \(n = 2n_1\) and \((z, ty) = 1\) at equation (2) for a \(t\) integer which is different than zero, then

\[
z^2 - ay^2 = x^n \Rightarrow z^2 - t^2y^2 = x^{2n_1} \Rightarrow z^2 - (ty)^2 = (x^{n_1})^2 \Rightarrow (x^{n_1})^2 + (ty)^2 = z^2
\]
the triple of \((x_{n1}, ty, z)\) becomes a primitive Pythagorean triple [6].

iii) If \(a = t^2\), \(z = z_1^2\) and \((x, ty) = 1\) where \(t\) and \(z_1\) are integers which are different that zero at equation (2), then
\[
z^2 - ay^2 = x^n \Rightarrow z^2 - t^2y^2 = (x_1^2)^n \Rightarrow z^2 - (ty)^2 = (x_1^n)^2 \Rightarrow (x_1^n)^2 + (ty)^2 = z^2
\]
the \((x_1^n, ty, z)\) triple is a primitive Pythagorean triple [6].

Now let's give the examples below which are the application of the theorem and results above.

**Example 2.1.1.1** If we accept \(a = 1\) at (2) equation then this equation transforms into \(z^2 - y^2 = x^n\). Solution formulas corresponding to \(n = 1, 2, 3, 4, 5, 6, 7\) values at solution formulas of (6) are given by the table below.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(u)</th>
<th>(v)</th>
<th>(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(u)</td>
<td>(v)</td>
<td>(u^2 - v^2)</td>
</tr>
<tr>
<td>2</td>
<td>(u^2 + v^2)</td>
<td>(v)</td>
<td>(u^2 - v^2)</td>
</tr>
<tr>
<td>3</td>
<td>(u^2 + 3uv^2)</td>
<td>(3uv + v^3)</td>
<td>(u^2 - v^2)</td>
</tr>
<tr>
<td>4</td>
<td>(u^2 + 6uv^2 + v^4)</td>
<td>(4uv + 4v^3)</td>
<td>(u^2 - v^2)</td>
</tr>
<tr>
<td>5</td>
<td>(u^2 + 10uv^2 + 5v^4)</td>
<td>(5uv + 10v^3 + v^5)</td>
<td>(u^2 - v^2)</td>
</tr>
<tr>
<td>6</td>
<td>(u^2 + 15uv^2 + 15v^4 + v^6)</td>
<td>(6uv + 20v^3 + 6v^5)</td>
<td>(u^2 - v^2)</td>
</tr>
<tr>
<td>7</td>
<td>(u^2 + 21uv^2 + 35v^4 + 7uv^3 + 7v^5)</td>
<td>(7uv + 35v^3 + 21uv^2 + 2v^5 + v^7)</td>
<td>(u^2 - v^2)</td>
</tr>
</tbody>
</table>

In this situation, the table values below are obtained if \(x, y, z\) integers are that are shown by \(X = X_n\) and \(Z = Z_n\) and that are corresponding to \(n\) values at formulas of (6) by giving some values in accordant with the hypothesis of Theorem 2.1 in place of \(u\) and \(v\).

<table>
<thead>
<tr>
<th>(u)</th>
<th>(v)</th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
<th>(z_1)</th>
<th>(z_2)</th>
<th>(z_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
</tr>
</tbody>
</table>

From above it is calculated as below:
\[
(37)^2 - (12)^2 = (35)^2 \Rightarrow (12)^2 + (35)^2 = (37)^2,
\]
\[
(365)^2 - (364)^2 = (9)^3 \Rightarrow (9)^3 + (364)^2 = (365)^2 \Rightarrow (3)^6 + (364)^2 = (365)^2,
\]
\[
(1241)^2 - (1160)^2 = (21)^4 \Rightarrow (21)^4 + (1160)^2 = (1241)^2,
\]
\[
(8404)^2 - (8403)^2 = (7)^5 \Rightarrow (7)^5 + (8403)^2 = (8404)^2,
\]
\[
(7813)^2 - (7812)^2 = (5)^6 \Rightarrow (5)^6 + (7812)^2 = (7813)^2,
\]
\[
(40156)^2 - (37969)^2 = (15)^7 \Rightarrow (15)^7 + (37969)^2 = (40156)^2.
\]

Again from the table above it is;
\[
(29525)^2 - (29524)^2 = (9)^5 \Rightarrow (3)^5 + (29524)^2 = (29525)^2
\]
\[
\Rightarrow (3)^5 + (29524)^2 = (29525)^2,
\]
\[
(66637)^2 - (51012)^2 = (35)^6 \Rightarrow [(35)^3]^2 + (51012)^2 = (66637)^2
\]
and from the expressions at the right side of the equations, the ordered triples of
\[
(3^5, 29524, 29525), (35^3, 51012, 66637)
\]
are obtained, and these are Pythagorean triples.

**Example 2.1.1.2** If we take \(a = 2\) at equation (2) then this equation transforms into \(z^2 - 2y^2 = x^n\). If values of \(n = 1, 2, 3, 4, 5, 6, 7\) are taken at solution formulas of (6), the solution formulas that are corresponding to these shall be given by the table below.

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Here, since the $a$ is even, $u$ can’t be even. In this situation, by giving some values in accord with the hypothesis of Theorem 2.1 in place of $u$ and $v$, if the $x, y, z$ integers that are corresponding to $n$ values at formulas of (6) are shown by $x = x_n, y = y_n$ and $z = z_n$, then the table below is generated.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$z$</th>
<th>$v$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$u$</td>
<td>$v$</td>
<td>$u^2 - 2v^2$</td>
</tr>
<tr>
<td>2</td>
<td>$u^2 + 2v^2$</td>
<td>$2uv$</td>
<td>$u^2 - 2v^2$</td>
</tr>
<tr>
<td>3</td>
<td>$u^2 + 6uv^2$</td>
<td>$3u^2 + 2v^2$</td>
<td>$u^2 - 2v^2$</td>
</tr>
<tr>
<td>4</td>
<td>$u^2 + 12u^2v^2 + 4v^2$</td>
<td>$4u^2 + 8uv^2$</td>
<td>$u^2 - 2v^2$</td>
</tr>
<tr>
<td>5</td>
<td>$u^2 + 20u^2v^2 + 20uv^2$</td>
<td>$5u^2 + 20u^2v^2 + 4v^2$</td>
<td>$u^2 - 2v^2$</td>
</tr>
<tr>
<td>6</td>
<td>$u^2 + 30u^2v^2 + 60u^2v^2 + 8v^2$</td>
<td>$6u^2 + 40u^2v^2 + 24uv^2$</td>
<td>$u^2 - 2v^2$</td>
</tr>
<tr>
<td>7</td>
<td>$u^2 + 42u^2v^2 + 140u^2v^2 + 56uv^2$</td>
<td>$7u^2 + 70u^2v^2 + 84uv^2 + 8v^2$</td>
<td>$u^2 - 2v^2$</td>
</tr>
</tbody>
</table>

Example 2.1.1.3 If we take $a = 3$ at equation (2), then the equation (2) transforms into $z^2 - 3y^2 = x^2$. When $n = 1, 2, 3, 4, 5, 6, 7$ is considered at solution triples of (6), the corresponding solution formulas to these are can be seen at the table below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$z$</th>
<th>$y$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$u$</td>
<td>$v$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>2</td>
<td>$u^2 + 3v^2$</td>
<td>$2uv$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>3</td>
<td>$u^2 + 9uv^2$</td>
<td>$3u^2 + 3v^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>4</td>
<td>$u^2 + 18u^2v^2 + 9v^2$</td>
<td>$4u^2 + 12uv^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>5</td>
<td>$u^2 + 30u^2v^2 + 45uv^2 + 9v^2$</td>
<td>$5u^2 + 30u^2v^2 + 9v^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>6</td>
<td>$u^2 + 45u^2v^2 + 135u^2v^2 + 27v^2$</td>
<td>$6u^2 + 60u^2v^2 + 54uv^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>7</td>
<td>$u^2 + 63u^2v^2 + 315u^2v^2 + 56uv^2$</td>
<td>$7u^2 + 105u^2v^2 + 189uv^2 + 27v^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
</tbody>
</table>

By giving some values in accord with the hypothesis of Theorem 2.1 in place of $u$ and $v$ at formula table, if the $x, y, z$ integers that are corresponding to $n$ values at formulas of (6) are shown by $x = x_n, y = y_n$ and $z = z_n$, then the table below is generated.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$z$</th>
<th>$y$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$u$</td>
<td>$v$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>2</td>
<td>$u^2 + 3v^2$</td>
<td>$2uv$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>3</td>
<td>$u^2 + 9uv^2$</td>
<td>$3u^2 + 3v^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>4</td>
<td>$u^2 + 18u^2v^2 + 9v^2$</td>
<td>$4u^2 + 12uv^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>5</td>
<td>$u^2 + 30u^2v^2 + 45uv^2 + 9v^2$</td>
<td>$5u^2 + 30u^2v^2 + 9v^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>6</td>
<td>$u^2 + 45u^2v^2 + 135u^2v^2 + 27v^2$</td>
<td>$6u^2 + 60u^2v^2 + 54uv^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
<tr>
<td>7</td>
<td>$u^2 + 63u^2v^2 + 315u^2v^2 + 56uv^2$</td>
<td>$7u^2 + 105u^2v^2 + 189uv^2 + 27v^2$</td>
<td>$u^2 - 3v^2$</td>
</tr>
</tbody>
</table>

From the above table,

\[
(26)^2 - 3(15)^2 = (1)^3 \Rightarrow (26)^2 - 3(15)^2 = 1,
\]
\[
(97)^2 - 3(56)^2 = (1)^4 \Rightarrow (97)^2 - 3(56)^2 = 1, \]
\[
(362)^2 - 3(209)^2 = (1)^5 \Rightarrow (362)^2 - 3(209)^2 = 1
\]

are obtained and these are solutions of $x^2 - 3y^2 = 1$ Pell equation [6]. Again from the table above

\[
(553)^2 - 3(304)^2 = (13)^4 \Rightarrow (13)^4 + 3(304)^2 = (553)^2.
\]
\[
(2569)^2 - 3(1480)^2 = (13)^4 \Rightarrow (13)^4 + 3(1480)^2 = (2569)^2
\]
\[
(845)^2 - 3(492)^2 = (-23)^3 \Rightarrow (-23)^3 + 3(492)^2 = (845)^2
\]
\[
(120725)^2 - 3(69716)^2 = (-23)^5 \Rightarrow (-23)^5 + 3(69716)^2 = (120725)^2
\]

is obtained.

Example 2.1.1.4 If we accept $a = 5$ at equation (2) then the equation of (2) shall transforms into form of $z^2 - 5y^2 = x^2$. Solution formulas that are corresponding to $n = 1, 2, 3, 4, 5, 6, 7$ values at solution formulas of (6) could be seen at the table below.

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In this situation if the $x, y, z$ integers that are corresponding to the $n$ values at formulas of (6) by giving some values in accordant with the Theorem 2.1 hypothesis in place of $u$ and $v$ are shown by $x = x_n$, $y = y_n$ and $z = z_n$ the table values below are obtained.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>$u^2 + 5v^2$</td>
<td>$2uv$</td>
<td>$u^2 - 5v^2$</td>
</tr>
<tr>
<td>3</td>
<td>$u^2 + 15uv^2$</td>
<td>$3uv + 5v^2$</td>
<td>$u^2 - 5v^2$</td>
</tr>
<tr>
<td>4</td>
<td>$u^2 + 30u^2v^2 + 25v^4$</td>
<td>$4uv + 20v^3$</td>
<td>$u^2 - 5v^2$</td>
</tr>
<tr>
<td>5</td>
<td>$u^2 + 50u^3v^2 + 125v^6$</td>
<td>$5uv + 50uv^2 + 25v^4$</td>
<td>$u^2 - 5v^2$</td>
</tr>
<tr>
<td>6</td>
<td>$u^2 + 75uv^2 + 375uv^4 + 125v^6$</td>
<td>$6uv + 100uv^2 + 150uv^3$</td>
<td>$u^2 - 5v^2$</td>
</tr>
<tr>
<td>7</td>
<td>$u^2 + 105u^3v^2 + 875uv^4 + 875v^6$</td>
<td>$7uv + 175uv^2 + 525uv^3 + 125v^4$</td>
<td>$u^2 - 5v^2$</td>
</tr>
</tbody>
</table>

From table we obtain;

$$(38)^2 - 5(17)^2 = (1)^3 \Rightarrow (38)^2 - 5(17)^2 = -1,$$

$$(161)^2 - 5(72)^2 = (1)^4 \Rightarrow (161)^2 - 5(72)^2 = 1$$

are obtained and these are solutions of $x^2 - 5y^2 = \pm 1$ Pell equations [6]. From table we obtain;

$$(604)^2 - 5(279)^2 = (-29)^3 \Rightarrow (-29)^3 + 5(279)^2 = (604)^2,$$

$$(8681)^2 - 5(3864)^2 = (29)^4 \Rightarrow (29)^4 + 5(3864)^2 = (8681)^2,$$

$$(19326)^2 - 5(8305)^2 = (31)^5 \Rightarrow (31)^5 + 5(8305)^2 = (19326)^2.$$
\( x^n + ay^2 = z^2 \) \text{ Diophantine Equation’s Integer Solutions}

\(-2 = 2 (\text{mod } 4)\) and then \(x = 4k + 2\).

iv) If it is \(a = 4a_1 + 3\),
\[ x = u^2 - av^2 = (2u_1)^2 - (4a_1 + 3)(2v_1 + 1)^2 = 4u_1^2 - (4a_1 + 3)(4v_1^2 + 4v_1 + 1) \]
\[ = 4u_1^2 - (4a_1 + 3)[4(v_1^2 + v_1) + 1] = 4[u_1^2 - 4a_1v_1^2 - 4a_1v_1 - 4 - 3v_1^2 - 3v_1] = 3 \]
is found. Here if we say it is \([u_1^2 - 4a_1v_1^2 - 4a_1v_1 - 4 - 3v_1^2 - 3v_1] = k\) then this means \(x = 4k + 3\) and that is \(3 = 1 (\text{mod } 4)\) and then \(x = 4k + 1\).

b) If \(u\) is odd and \(v\) is even so for \(u_1, v_1 \in \mathbb{Z}\) it is \(u = 2u_1 + 1\) and \(v = 2v_1\) then
\[ x = u^2 - av^2 = (2u_1 + 1)^2 - a(2v_1)^2 = 4u_1^2 + 4u_1 + 1 - a(4v_1)^2 = 4[u_1^2 + u_1 - av_1^2] + 1 \]
is found. Here if we say it is \([u_1^2 + u_1 - av_1^2] = k\) then this means \(x = 4k + 1\).

Let’s give examples as the application of this theorem below.

**Example 2.2.1** Let \(u, v\) and \(a_1\) are integers and \(a = 4a_1 + 1\) and \(a\) are different than square. Then it is \(x = u^2 - av^2 = u^2 - (4a_1 + 1)v^2\) and \(x\) takes the values at the table according to values which \(u, v\) and \(a\) will have.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline u & v & u^2 - v^2 & u^2 - 5v^2 & u^2 - 13v^2 & u^2 - 17v^2 & u^2 - 21v^2 \\
\hline 2 & 1 & -3 & -7 & -11 & -15 & -19 \\
\hline 4 & 1 & 15 & 31 & 47 & 63 & 79 \\
\hline 6 & 3 & 7 & -29 & -101 & -137 & -173 \\
\hline 8 & 5 & 11 & -89 & -289 & -389 & -489 \\
\hline 10 & 7 & 15 & -181 & -573 & -769 & -965 \\
\hline
\end{array}
\]

**Example 2.2.2** Let it is given in the form of \(a = 4a_1 + 3\) where \(u\) is even, \(v\) is odd and \(a_1\) are integers. Then \(x = u^2 - av^2 = (4a_1 + 3)v^2\) takes the values at table according to some values which \(u, v\) and \(a\) will have.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline u & v & u^2 - 3v^2 & u^2 - 7v^2 & u^2 - 11v^2 & u^2 - 15v^2 & u^2 - 19v^2 & u^2 - 23v^2 & u^2 - 27v^2 & u^2 - 31v^2 & u^2 - 35v^2 \\
\hline 6 & 13 & -9 & 5 & 1 & -3 & -7 & -11 & -15 & -19 & -23 \\
\hline 10 & 21 & 5 & -11 & -27 & -43 & -59 & -75 & -91 & -107 & -123 \\
\hline
\end{array}
\]

**Example 2.2.3** When it is given in the form of \(a = 4a_1 + 1\) where the \(u\) is odd, \(v\) is even and \(a_1\) are integers, \(x = u^2 - (4a_1 + 1)v^2\) will take the values at table according to some values which \(u, v\) and \(a\) will have.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline u & v & u^2 - v^2 & u^2 - 5v^2 & u^2 - 13v^2 & u^2 - 17v^2 & u^2 - 21v^2 & u^2 - 25v^2 \\
\hline 3 & 2 & -11 & -43 & -59 & -75 & -91 & -107 \\
\hline 5 & 21 & 5 & -27 & -43 & -59 & -75 & -91 \\
\hline 4 & 9 & -55 & -183 & -247 & -311 & -385 & -461 \\
\hline 7 & 245 & -31 & -39 & -59 & -75 & -91 & -107 \\
\hline 6 & -31 & -71 & -159 & -223 & -351 & -419 & -487 \\
\hline 9 & -131 & -275 & -563 & -707 & -851 & -995 & -1143 \\
\hline 8 & 17 & -239 & -571 & -1007 & -1543 & -2081 & -2625 \\
\hline
\end{array}
\]

**Example 2.2.4** Let it is given in the form of \(a = 2a_1\) where \(u\) is odd, \(v\) is even, \(a\) is square free and \(a_1\) are integers. Then \(x = u^2 - (2a_1)v^2\) takes the values at table according to some values which \(u, v\) and \(a\) will have.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline u & v & u^2 - v^2 & u^2 - 5v^2 & u^2 - 8v^2 & u^2 - 10v^2 & u^2 - 12v^2 & u^2 - 14v^2 & u^2 - 18v^2 & u^2 - 20v^2 & u^2 - 22v^2 & u^2 - 24v^2 & u^2 - 26v^2 \\
\hline 5 & 2 & 17 & -1 & -23 & -31 & -39 & -47 & -63 & -71 & -87 & -95 & -111 \\
\hline
\end{array}
\]
Example 2.2.5 Let it is given in the form of \(a = 2a_1\) where \(a\) is odd, \(v\) is odd, \(a\) is square free and \(a_1\) are even integers. Then \(x = u^2 - (2a_1)v^2\) takes the values at table according to some values which \(u\), \(v\) and \(a\) will have.

<table>
<thead>
<tr>
<th>(u)</th>
<th>(v)</th>
<th>(u^2 - 8v^2)</th>
<th>(u^2 - 12v^2)</th>
<th>(u^2 - 20v^2)</th>
<th>(u^2 - 24v^2)</th>
<th>(u^2 - 28v^2)</th>
<th>(u^2 - 32v^2)</th>
<th>(u^2 - 40v^2)</th>
<th>(u^2 - 44v^2)</th>
<th>(u^2 - 48v^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>-3</td>
<td>-11</td>
<td>-15</td>
<td>-19</td>
<td>-23</td>
<td>-31</td>
<td>-35</td>
<td>-39</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>-47</td>
<td>-83</td>
<td>-155</td>
<td>-191</td>
<td>-227</td>
<td>-263</td>
<td>-335</td>
<td>-371</td>
<td>-407</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>41</td>
<td>37</td>
<td>29</td>
<td>25</td>
<td>21</td>
<td>17</td>
<td>9</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>49</td>
<td>13</td>
<td>-59</td>
<td>-95</td>
<td>-131</td>
<td>-167</td>
<td>-239</td>
<td>-275</td>
<td>-311</td>
</tr>
<tr>
<td>29</td>
<td>5</td>
<td>641</td>
<td>541</td>
<td>341</td>
<td>241</td>
<td>141</td>
<td>41</td>
<td>-159</td>
<td>-259</td>
<td>-359</td>
</tr>
</tbody>
</table>

Example 2.2.5 Let it is given in the form of \(a = 2a_1\) where \(u\), \(v\), \(a_1\) are odd positive integers and \(a\) is square free. Then \(x = u^2 - (2a_1)v^2\) takes the values at table according to some values which \(u\), \(v\) and \(a\) will have.

<table>
<thead>
<tr>
<th>(u)</th>
<th>(v)</th>
<th>(u^2 - 2v^2)</th>
<th>(u^2 - 6v^2)</th>
<th>(u^2 - 10v^2)</th>
<th>(u^2 - 14v^2)</th>
<th>(u^2 - 18v^2)</th>
<th>(u^2 - 22v^2)</th>
<th>(u^2 - 26v^2)</th>
<th>(u^2 - 30v^2)</th>
<th>(u^2 - 34v^2)</th>
<th>(u^2 - 38v^2)</th>
<th>(u^2 - 42v^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-5</td>
<td>-9</td>
<td>-13</td>
<td>-17</td>
<td>-21</td>
<td>-25</td>
<td>-29</td>
<td>-33</td>
<td>-37</td>
<td>-41</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>7</td>
<td>3</td>
<td>-1</td>
<td>-5</td>
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<td>-13</td>
<td>-17</td>
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<td>5</td>
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<td>7</td>
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<td>-101</td>
<td>-137</td>
<td>-173</td>
<td>-209</td>
<td>-245</td>
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<td>-317</td>
<td>-353</td>
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<td>13</td>
<td>3</td>
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<td>-29</td>
<td>-65</td>
<td>-101</td>
<td>-137</td>
<td>-173</td>
<td>-209</td>
<td></td>
</tr>
</tbody>
</table>

References


