Sums Of Squares Of Fractal Sequences: Farey Sequence & Negative Jacobsthal Sequence

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Abstract: Negative Jacobsthal numbers are fractal sequence of numbers. We find sums of squares of negative Jacobsthal sequence of numbers. We also find sum of squares of Farey sequence of numbers.

Keywords: Sums of squares, Jacobsthal sequence, Jacobsthal Lucas Sequence, Farey sequence

I. Introduction

The Jacobsthal and Jacobsthal–Lucas sequences \( I_n \) and \( j_n \) are defined by the recurrence relations
\[ f_0 = 0, f_1 = 1, \quad f_n = f_{n-1} + 2f_{n-2} \text{ for } n \geq 2, \]
\[ j_0 = 2, j_1 = 1, \quad j_n = j_{n-1} + 2j_{n-2} \text{ for } n \geq 2. \]

The first ten terms of the sequence \( f_n \) are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171 and 341. It is given by the formula \( f_n = (2^n - (-1)^n) \). The first ten terms of the sequence \( j_n \) are 2, 1, 5, 7, 17, 31, 65, 127, 257, 511 and 1025. It is given by the formula \( j_n = (2^n + (-1)^n) \). Negative Jacobsthal numbers are fractal sequence of numbers. Here we find the sums of squares of Negative Jacobsthal and Negative Jacobsthal Lucas sequence of numbers.

A Farey sequence of order \( N \) is a set of irreducible fractions between 0 and 1 arranged in increasing order, the denominators of which do not exceed \( N \). In this paper we establish a formula to find the sum of squares of Farey sequence of numbers.

Sums of Squares of Jacobsthal and Jacobsthal-Lucas Sequences

Theorem
\[ f_n^2 + j_n^2 = \frac{10}{9} f_{2n} + \frac{1}{3} 2^{-n+2} (-1)^n \]

Proof

By the formula \( f_n = \frac{2^{-n} - (-1)^n}{3} \) : \( j_n = 2^{-n} + (-1)^n \)
we have
\[ f_n^2 + j_n^2 = \left[ \frac{2^{-n} - (-1)^n}{3} \right]^2 + \left[ 2^{-n} + (-1)^n \right]^2 \]

Squaring and adding the above formula
\[ = \frac{1}{9} 2^{-2n} + 2^{-2n} + \frac{1}{9} \left[ (-1)^{-2n} + (-1)^{-2n} - 2 \cdot 2^{-n} (-1)^{-n} \right] \]
\[ = \frac{10}{9} 2^{-2n} + \frac{10}{9} (-1)^{-2n} + 2.2^{-n} (-1)^{-n} \left( 1 - \frac{1}{3} \right) \]
\[ = \frac{10}{9} (2^{-2n} + (-1)^{-2n}) + \frac{2^2}{3} 2^{-n} (-1)^{-n} \]
\[ = \frac{10}{9} f_{2n} + \frac{1}{3} 2^{-n+2} (-1)^n \]

Remark
When \( n \) is odd
\[ f_n^2 + j_n^2 = \frac{10}{9} f_{2n} + \frac{1}{3} 2^{-n+2} \]
When \( n \) is even
\[ f_n^2 + j_n^2 = \frac{10}{9} f_{2n} + \frac{1}{3} 2^{-n+2} \]
Summation of the Jacobsthal-Lucas sequence

Theorem

\[ \sum_{n=1}^{k} j^2_n = \frac{1}{3} + k + 2 \sum_{n=1}^{k} \frac{(-1)^{-n}}{\binom{n}{i}} \]

Proof

By the formula \( j_n = 2^{-n} + (-1)^{-n} \)

We have

\[ \sum_{n=1}^{k} j^2_n = \sum_{n=1}^{k} (2^{-n} + (-1)^{-n})^2 \]
\[ = \sum_{n=1}^{k} 2^{-2n} + \sum_{n=1}^{k} (-1)^{-2n} + 2 \sum_{n=1}^{k} 2^{-n} (-1)^{-n} \]
\[ = \left\{ \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \cdots \right\} + \left\{ \frac{1}{3} + \frac{1}{3^3} + \cdots \right\} + 2 \left\{ -\frac{1}{2} + \frac{1}{2^3} - \frac{1}{2^5} + \frac{1}{2^7} - \cdots \right\} \]

Using the geometric series

\[ = \left( \frac{1}{1 - \frac{1}{2^2}} - 1 \right) + k + 2 \sum_{n=1}^{k} \frac{(-1)^{-n}}{2^n} \]
\[ = \left( \frac{4}{3} - 1 \right) + k + 2 \sum_{n=1}^{k} \frac{(-1)^{-n}}{2^n} \]
\[ \sum_{n=1}^{k} j^2_n = \frac{1}{3} + k + 2 \sum_{n=1}^{k} \frac{(-1)^{-n}}{\binom{n}{i}} \]

Theorem

Sums of squares of Farey fractions of order \( N \) is given by

\[ S_N^2 = \sum_{n=2}^{N} \left( \frac{X_n}{K} \right)^2 \]

Proof

In [1] it is proved that

\[ S_N = S_{N-1} + \frac{\varphi(N)}{2} \]

Where \( S_N \) denotes the sum of Farey fractions of order \( N \).

The corresponding formula for summation of squares of Farey sequence can be taken as

\[ S_N^2 = S_{N-1}^2 + \sum_{x=1}^{N} \left( \frac{x}{N} \right)^2 \]

Where \( S_N^2 \) denotes the sum of squares of Farey fractions of order \( N \).

Now

\[ S_N^2 = S_{N-2}^2 + \sum_{x=1}^{N-1} \left( \frac{x}{N-1} \right)^2 + \sum_{x=1}^{N} \left( \frac{x}{N} \right)^2 \]
\[ S_N^2 = S_{N-3}^2 + \sum_{x=1}^{N-2} \left( \frac{x}{N-2} \right)^2 + \sum_{x=1}^{N-1} \left( \frac{x}{N-1} \right)^2 + \sum_{x=1}^{N} \left( \frac{x}{N} \right)^2 \]
\[ S_N^2 = S_{N-3}^2 + \sum_{x=1}^{N-2} \left( \frac{x}{N-2} \right)^2 + \sum_{x=1}^{N-1} \left( \frac{x}{N-1} \right)^2 + \sum_{x=1}^{N} \left( \frac{x}{N} \right)^2 + \cdots \]

Continuing we get

\[ S_N^2 = \sum_{x=1}^{N} \left( \frac{x}{K} \right)^2 \]
Illustration

\[ S_5^2 = S_4^2 + \left( \frac{1}{5} \right)^2 + \left( \frac{2}{5} \right)^2 + \left( \frac{3}{5} \right)^2 + \left( \frac{4}{5} \right)^2 \]

Proof

\[ S_5^2 = \sum_{x,k=1}^{5} \left( \frac{x}{k} \right)^2 \]

\[ S_5^2 = \left( \frac{1}{4} \right)^2 + \left( \frac{3}{4} \right)^2 + \left( \frac{1}{3} \right)^2 + \left( \frac{2}{3} \right)^2 + 1^2 \]

\[ S_5^2 = S_4^2 + \left( \frac{1}{5} \right)^2 + \left( \frac{2}{5} \right)^2 + \left( \frac{3}{5} \right)^2 + \left( \frac{4}{5} \right)^2 \]

Reference


