The First Triangular Representation of The Symmetric Groups with \( p \) divides (n-1)

Eyad M.A.Al-Aamaly1, Ali Abdulsahib M. Al-Butaihi2,*,Abid Ali H Al-Taraf3
1(Ph.D. student in Department of Mathematics, College of Sciences/ Al-Mustansiriyah University, Baghdad, Iraq)
2(M. Sc. student in Department of Mathematics, College of Sciences/ Al-Mustansiriyah University, Baghdad, Iraq)
3(Ph.D. student in Department of Mathematics, College of Sciences/ Al-Mustansiriyah University, Baghdad, Iraq)

Abstract: In this paper we will introduce new type of representations of the symmetric groups we call them the triangular representations of the symmetric groups and we will study the first of them which we call it the first triangular representation of the symmetric groups when \( p \) divides (n-1).

Keywords: symmetric group, group algebra \( KS_n \), \( KS_n \) module, Specht module, exact sequence.

I. Introduction

When Prof. W. Specht was a student under the supervision of Prof. I. Schur, he began investigating representation theory of the symmetric group. During that time it was well known that standard Young tableaux of a given partition \( \lambda \) of a positive integer \( n \) form a basis of an ordinary irreducible representationspace of \( S_n \). The problem that W. Specht was facing in his investigating in that time is that the symmetric group acts in a natural way on tableaux, but the result of the application of a permutation to a standard tableau can be a nonstandard tableau, and it is by no means clear how a nonstandard tableau can be written as a linear combination of standard ones. For this reason W. Specht introduced in 1935 polynomials corresponding to the tableaux (known nowadays as Specht polynomials), and it is obvious how a given polynomial can be written as a linear combination of other polynomials (see [1]).

In 1962 H. K. Farahat studied the representation which deals with the partition \( \lambda = (n-1,1) \) of the positive integer \( n \) and called it the natural representation of the symmetric groups [2].

In 1969 M. H. Peel renamed the natural representation of the symmetric groups by the first natural representation of the symmetric groups and studied the second representation of the symmetric group which deal with the partition \( \lambda = (n-2,2,2) \) of the positive integer \( n \) [3].

In 1971 Peel introduced the \( r \)-Hook representations which deal with the partitions \( \lambda = (n-r,1^r) \); \( r \geq 1 \) [4].

Now in this paper we will introduce new representations of the symmetric groups we call them the triangular representations of the symmetric groups and we will study the first of them which we call it the first triangular representation of the symmetric groups.

Throughout this paper let \( K \) be a field of characteristic \( p \) (which may be zero or a prime number not equal 2), and \( x_1, x_2, ..., x_n \) be linearly independent commuting variables over \( K \).

II. Preliminaries

Definition 1: Let \( S_n \) be the set of all permutations \( \sigma \) on the set \( \{x_1, x_2, ..., x_n\} \) and \( K[x_1, x_2, ..., x_n] \) be the ring of polynomials in \( x_1, x_2, ..., x_n \) with coefficients in \( K \). Then each permutation \( \tau \in S_n \) can be regarded as a bijective function from \( K[x_1, x_2, ..., x_n] \) onto \( K[x_1, x_2, ..., x_n] \) defined by \( f(x_1, x_2, ..., x_n) = f(\tau(x_1), \tau(x_2), ..., \tau(x_n)) \) for all \( f(x_1, x_2, ..., x_n) \in K[x_1, x_2, ..., x_n] \). Then \( K[S_n] \) forms a group algebra with respect to addition of functions, product of functions by scalars and composition of functions which is called the group algebra of the symmetric group \( S_n \) [3].

Definition 2: Let \( n \) be a positive integer then the sequence \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_l) \) is called a partition of \( n \) if \( \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_l > 0 \) and \( \lambda_1 + \lambda_2 + ... + \lambda_l = n \). Then the set \( D_\lambda = \{(i,j)\mid 1 \leq i, j \leq n \} \) is called \( \lambda \)-diagram. And any bijective function \( \lambda : D_\lambda \to \{(i,j)\mid 1 \leq i, j \leq n \} \) is called \( \lambda \)-face. And any bijective function \( \lambda : D_\lambda \to \{(i,j)\mid 1 \leq i, j \leq n \} \) is called \( \lambda \)-tableau. \( \lambda \)-tableau may be thought as an array consisting of rows and \( \lambda \)-columns of distinct variables \( (i,j) \) where the variables occur in the first \( \lambda_i \) positions of the \( i^{th} \) row and each variable \( t((i,j)) \) occurs in the \( r^{th} \) row and the \( f^{th} \) column \( ((i,j)) \) position of the array.

The set of all \( \lambda \)-tableaux will be denoted by \( T_\lambda \), i.e. \( T_\lambda = \{t \mid t \text{ is a } \lambda \text{-tableau}\} \).

DOI: 10.9790/5728-12020272734 www.iosrjournals.org 27 | Page
Then the function \( g: T_i \to K[x_1, x_2, \ldots, x_n] \) which is defined by \( g(t) = \prod_{i=1}^{n} \prod_{j=1}^{t(i,j)} (t(i,j))^{i-1}, \forall t \in T_i \) is called the row position monomial function of \( T_i \), and for each \( \lambda \)-tableau \( t \), \( g(t) \) is called the row position monomial of \( t \). So \( M(\lambda) \) is the cyclic \( KS_n \) -module generated by \( g(t) \) over \( KS_n \).\[5\]

### III The First Triangular Representation of \( S_n \)

In the beginning we define some denotations which we need them in this paper.

1) Let \( \sigma_1(n) = \sum_{i=1}^{n} x_i \).

2) Let \( \sigma_2(n) = \sum_{l=1}^{n} x_i x_j \).

3) Let \( C_2(n) = x_i^2 (\sigma_2(n) - \sum_{j=1}^{n} x_j), l = 1, 2, \ldots, n \).

We denote \( \bar{N} \) to be the \( KS_n \) module generated by \( C_1(n) \) over \( KS_n \). The set \( B = \{ C_1(n) \mid i = 1, 2, \ldots, n \} \) is a \( K \)-basis for \( \bar{N} = KS_n \cdot C_1(n) \), and \( \dim_K \bar{N} = n \).

4) Let \( \sigma_2(n) = C_1(n) - C_2(n) \), \( i, j = 1, 2, \ldots, n \).

The set \( N_0 \) is the \( KS_n \) submodule of \( \bar{N} \) generated by \( u_{12}(n) \).

5) Let \( \sigma_3(n) = \sum_{i=1}^{n} \sum_{l=1}^{n} x_i x_j x_k \).

Then \( \sum_{i=1}^{n} C_2(n) = \sigma_3(n) \) and \( \dim_K (\sigma_3(n)) = \dim_K (\sigma_3(n)) = \dim_K (\sigma_3(n)) = 1 \). \( K \sigma_1(n), K \sigma_2(n) \) and \( K \sigma_3(n) \) are all \( KS_n \) modules, since \( \sigma_0(n) = \sigma_0(n) \forall k = 1, 2, 3 \).

**Definition 3:** The \( KS_n \) -module \( M \left( n - \frac{(r+2)(r+1)}{2}, r+1, r, \ldots, 1 \right) \) defined by

\[
M \left( n - \frac{(r+2)(r+1)}{2}, r+1, r, \ldots, 1 \right) = KS_n x_1 x_2 \cdots x_{r+1} x_{r+2} \cdots x_n = x_1 x_2 \cdots x_{r+1} x_{r+2} \cdots x_n^{r+1}
\]

is called the \( r \)-th triangular representation module of \( S_n \) over \( K \), where \( n \geq \frac{(r+3)(r+2)}{2} \).

**Remark:** The first triangular representation module of \( S_n \) over \( K \) is the \( KS_n \) -module \( M(n \geq 3, 2, 1) \), the second triangular representation module of \( S_n \) over \( K \) is the \( KS_n \) -module \( M(n \geq 6, 3, 2, 1) \), and the third triangular representation module of \( S_n \) over \( K \) is the \( KS_n \) -module \( M(n \geq 10, 4, 3, 2, 1) \), and so on.

**Lemma 1:** The set \( B(n \geq 3, 2, 1) = \{ x_i x_j x_k \mid i \leq j < k \leq n, 1 \leq i \leq l \leq n, l \neq i, j \} \) is an \( K \)-basis of \( M_0(n \geq 3, 2, 1) \), and \( \dim_K M_0(n \geq 3, 2, 1) = \binom{n}{3}(n-2) - 1 ; n \geq 6 \).

**Theorem:** The set \( B_0(n \geq 3, 2, 1) = \{ x_i x_j x_k \mid i \leq l \leq k \leq n, i \leq l \leq n, l \neq i, j, \} \) is an \( K \)-basis of \( M_0(n \geq 3, 2, 1) \), and \( \dim_K M_0(n \geq 3, 2, 1) = \binom{n}{3}(n-2) - 1 ; n \geq 6 \).

**Proof:** Since the \( KS_n \) module \( M_0(n \geq 3, 2, 1) \) consists of all polynomials of the form

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j x_k \text{with} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} = 0 \quad \text{and} \quad k_{ij}, k_{ji} \in K \text{i.e.}
\]

\[
M_0(n \geq 3, 2, 1) = \{ \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j x_k \mid \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} = 0, k_{ij}, k_{ji} \in K \}
\]

it is clear that \( B_0(n \geq 3, 2, 1) \subseteq M_0(n \geq 3, 2, 1) \). To prove \( B_0(n \geq 3, 2, 1) \) generates \( M_0(n \geq 3, 2, 1) \) over \( K \). Let \( x \in M_0(n \geq 3, 2, 1) \).

\[
\Rightarrow x = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j x_k \quad \text{with} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} = 0
\]

\[
\Rightarrow x = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j x_k \quad - \quad \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j x_k = 0
\]

\[
\Rightarrow x = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j x_k \quad - \quad \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j x_k
\]

\[
\Rightarrow x = \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} x_i x_j x_k
\]

**DOI:** 10.9790/5728-1202022734 | www.iosrjournals.org 28 | Page
\[
\Rightarrow x = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} \left( x_i x_j x_1^2 - x_1 x_2 x_3^2 \right).
\]

\[
\Rightarrow x = \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} (x_i x_j x_1^2 - x_1 x_2 x_3^2) \right) \text{with the term 123 excluded from the double summation since}
\]

\[
k_{123} (x_1 x_2 x_3^2 - x_1 x_2 x_3^2) = 0. \text{Thus } B_0(n-3, 2, 1) \text{ generates } M_\alpha(n-3, 2, 1) \text{ over } K. B_0(n-3, 2, 1) \text{ is linearly}
\]

independent if

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} (x_1 x_2 x_3^2) = 0
\]

\[
\Rightarrow \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} x_i x_j x_1^2 - \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} x_i x_2 x_3^2 \right) = 0
\]

\[
\Rightarrow \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} x_i x_j x_1^2 = \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} x_i x_2 x_3^2 \right) = 0, \text{ where } k_{123} = - \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} \text{ with } \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} k_{ij} = 0
\]

Hence \(k_{ij} = 0\) \(\forall i, j, l; 1 \leq i < j \leq n, 1 \leq l \leq n, l \neq i, j, \text{ since } B(n-3, 2, 1) \) is linearly independent by

lemma 1.

Thus \(B_0(n-3, 2, 1)\) is a \(K\)-basis of \(M_\alpha(n-3, 2, 1)\), and

\[
dim_k M_\alpha(n-3, 2, 1) = \left( \frac{n+1}{2} \right) \left( n - 2 \right) - 1 = \frac{n^3 - 3n^2 + 2n - 2}{2}
\]

**Theorem 2:** \(\overline{N} = KS_nC_i(n)\) and \(M(n-1,1)\) are isomorphic over \(KS_n\).

**Proof:** Let \(\varphi : M(n-1,1) \rightarrow \overline{N}\) be defined as follows:

\[
\varphi(x_i) = C_i(n); i = 2, \ldots, n. \text{ Then for each } \tau = (x_i, x_j) \in S_n \text{ such that } \tau(x_i) = x_j \text{ we get that } \varphi(\tau x_i) = \varphi(x_i) = C_i(n) = \tau C_i(n) = \tau \varphi(x_i).
\]

Hence \(\varphi\) is a \(KS_n\)-homomorphism. Also \(\gamma = \sum\limits_{i=1}^{n} k_i C_i(n)\) for any \(\gamma \in \overline{N}\).

Thus \(\forall \gamma \in \overline{N}, \exists w = \sum\limits_{i=1}^{n} k_i x_i \in M(n-1,1)\) s.t. \(\varphi(\omega) = \varphi(\sum\limits_{i=1}^{n} k_i x_i) = \sum\limits_{i=1}^{n} \varphi(k_i x_i) = \sum\limits_{i=1}^{n} k_i \varphi(x_i) = \gamma\). Hence \(\varphi\) is an epimorphism. Thus \(\text{dim}_k \text{ker}\varphi = \text{dim}_k M(n-1,1) - \text{dim}_k \overline{N} = n - n = 0\).

\(\Rightarrow \text{ker}\varphi = 0\). Then \(\varphi\) is monomorphism. Thus \(\varphi\) is a \(KS_n\)-isomorphism. Hence \(M(n-1,1)\) and \(\overline{N}\) are isomorphic over \(KS_n\).

**Theorem 3:** \(\overline{N}_0 = KS_n u_{1,2}(n)\) and \(M_0(n-1,1)\) are isomorphic over \(KS_n\).

**Proof:** From theorem (2) we have a \(KS_n\)-homomorphism \(\varphi : M(n-1,1) \rightarrow \overline{N}\) s.t.

\[
\varphi(x_i) = C_i(n); i = 1, 2, \ldots, n. \text{ And since } M_0(n-1,1) = KS_n(x_2 - x_1) \in M(n-1,1), \text{ then}
\]

\[
\varphi(x_i - x_j) = \varphi(x_i) - \varphi(x_j) = C_i(n) - C_j(n) = u_{1,2}(n) \in \overline{N}_0. \text{ Let } \psi = \varphi|_{M_0(n-3,2,1)}.
\]

Thus \(\psi : M_0(n-1,1) \rightarrow \overline{N}_0\) s.t. \(\psi(x_i - x_j) = u_{1,2}(n); i = 1, 2, \ldots, n. \text{ Which is } KS_n\)-homomorphism.

Also have \(\forall u_{ij} \in \overline{N}_0, \exists x_i - x_j \in M_0(n-1,1)\) s.t.

\[
\psi(x_i - x_j) = \psi(x_i - x_2 + x_1 - x_j) = \psi(x_i - x_1) - \psi(x_j - x_1) = u_{ij}(n) - u_{ij}(n) = C_i(n) - C_j(n) = C_i(n) - C_j(n) = u_{ij}(n).
\]

Thus \(\psi\) is an epimorphism. Since \(\text{dim}\, M_0(n-1,1) = n -1\) and \(\text{dim}\, \overline{N}_0 = n - 1\).

Then \(\text{dim}_k \text{ker}\psi = \text{dim}_k M_0(n-1,1) - \text{dim}_k \overline{N}_0 = 0\). Hence \(\text{ker}\psi = 0\). Thus \(\psi\) is a monomorphism. Therefore \(\psi : KS_n\)-isomorphism. Thus \(M_0(n-1,1)\) and \(\overline{N}_0\) are isomorphic over \(KS_n\).

**Corollary 1:** The \(KS_n\)-module \(\overline{N}_0 = KS_n u_{1,2}(n)\) is irreducible over \(KS_n\) if \(p\) does not divide \(n\).

**Proof:** From theorem (3) we have \(\overline{N}_0 \cong M_0(n-1,1)\). Since \(M_0(n-1,1)\) is irreducible over \(KS_n\) if \(p\) does not divide \(n\) by [4]. Hence \(\overline{N}_0\) is irreducible over \(KS_n\) if \(p\) does not divide \(n\) since the sum of the coefficients

DOI: 10.9790/5728-1202022734
Thus $M$ is a composition series.

2. The $K_{S_n}$ is an irreducible submodule over $KS_n$.

Hence $0 \leq 0 \subset N_0 \subset ... \subset N_0 \subset N$.

The following sequence of $K_{S_n}$-modules is exact

$0 \to Ker d \to M(n - 3, 2, 1) \to M(n - 2, 2) \to 0 \cdots (1)$

Proof: It is clear that the map $d$ is onto since $\forall \sum_{i \neq i = 0} k_{i,j} x_i x_j \in M(n - 2, 2)$,

$\exists 2 \sum_{i \neq i = 0} k_{i,j} x_i x_j x_i x_j \in M(n - 3, 2, 1)$

for some $l (\neq i, j)$ s.t. $d(\sum_{i \neq i = 0} k_{i,j} x_i x_j) = \sum_{i \neq i = 0} k_{i,j} x_i x_j$.

Since the inclusion map $\pi$ is 1-1 and $Im \pi = Ker d$. Hence the sequence (1) is exact.

Theorem 5: The sequence (1) is split iff $p$ does not divide $(n-2)$.

Proof: Assume $p$ does not divide $(n-2)$. We can define a function

$\varphi : M(n - 2, 2) \to M(n - 3, 2, 1)$ by $\varphi (x_i x_j) = \frac{1}{2(n-2)} \sum_{i \neq i = 0} x_i x_j t_i$. This is a $K_{S_n}$-homomorphism.

Since for any $\tau \in S_n$ then $\varphi(\tau(x_i x_j)) = \varphi(\tau(x_i x_j)) = \frac{1}{2(n-2)} \sum_{i \neq i = 0} \tau(x_i) \tau(x_j) x_i x_j$.

Where $\tau(x_i) = x_{i',} \tau(x_j) = x_{j'}$.

$\Rightarrow \varphi(\tau(x_i x_j)) = \frac{1}{2(n-2)} \tau(x_i x_j) = \tau(\frac{1}{2(n-2)} \sum_{i \neq i = 0} x_i x_j x_i x_j) = \tau \varphi(x_i x_j)$

And $\varphi(x_i x_j) = d(\frac{1}{2(n-2)} \sum_{i \neq i = 0} x_i x_j x_i x_j) = \frac{1}{2(n-2)} (2(n-2) x_i x_j) = x_i x_j$. Then $d \varphi = I$ on $M(n - 2, 2)$. Hence the sequence (1) is split.

Thus $M(n - 3, 2, 1) = L \oplus Ker d$, where $L = \varphi(M(n - 2, 2))$.

Now assume that the sequence (1) is split. Then there exist a $K_{S_n}$-homomorphism $\psi : M(n - 2, 2) \to M(n - 3, 2, 1)$ s.t. $d \psi = I$ on $(n - 2, 2)$.

i.e. $d \psi(x_i x_j) = x_i x_j$.

Then $\psi$ has the form $\psi(x_i x_j) = \sum_{i \neq i = 0} k_{i,j} x_i x_j$ for $1 \leq i < j \leq n$. 

DOI: 10.9790/5728-1202022734 www.iosrjournals.org
Thus we get 
\[ d \psi(x_i x_j) = d \left( \sum_{l=1}^n \sum_{j=1}^l k_{ij} x_i x_j x_l^2 \right) = \sum_{l=1}^n \left( 2 \sum_{j=1}^l k_{ij} x_i x_j \right) = x_i x_j \]

Which implies that 
\[ 2 \left( \sum_{l=1}^n k_{ij} \right) = 0 \text{ if } (i, j) \neq (i, j) \text{ and } 2 \left( \sum_{l=1}^n k_{ij} \right) = 1 \text{ if } (i, j) = (i, j). \]

Moreover if \( \tau = (x_r x_s) \in S_n; 1 \leq r < s \leq n \ s. t. \ \tau (x_i x_j) = x_{i_1} x_{j_1}. \)

Then \( \psi(\tau (x_i x_j)) = \psi(x_i x_j) = \tau \psi(x_i x_j) \Rightarrow \psi(x_i x_j) - \tau \psi(x_i x_j) = 0 \)
\[ \Rightarrow \sum_{i=1}^n \sum_{j=1}^l k_{ij} x_i x_j x_l^2 - \tau (\sum_{i=1}^n \sum_{j=1}^l k_{ij} x_i x_j x_l^2) = 0 \Rightarrow \sum_{i=1}^n \sum_{j=1}^l (k_{ij} x_i x_j x_l^2 - k_{ij} \tau(x_i x_j x_l^2)) = 0 \Rightarrow \]
\[ \sum_{i=1}^n \sum_{j=1}^l (k_{ij} x_i x_j x_l^2 - k_{ij} \tau(x_i x_j x_l^2)) = 0 \Rightarrow \sum_{i=1}^n \sum_{j=1}^l \sum_{r=1}^l (k_{ir} x_i x_j x_r x_s^2) + \sum_{i=1}^n \sum_{j=1}^l \sum_{r=1}^l (k_{rj} x_i x_j x_r x_s^2) + \sum_{i=1}^n \sum_{j=1}^l \sum_{r=1}^l (k_{rj} x_i x_j x_r x_s^2) = 0 \]

Then by equating the coefficient of the above equation we get
\[ k_{rj} = k_{sj} \forall r < j < n \ \exists s \text{ and } \forall 1 \leq l \leq n \ \exists l \neq r, s, j \]
\[ k_{i r l} = k_{i s l} \forall 1 \leq i < r \text{ and } \forall 1 \leq l \leq n \ \exists l \neq r, s, i \]
\[ k_{i j r} = k_{i j s} \forall 1 \leq l < j \leq n \ \exists i, j \neq r, s \]
\[ k_{i r s} = k_{i s r} \forall 1 \leq l < r \]
\[ k_{r j s} \forall r < j \leq n, j \neq s \]

So for any \( r, s; 1 \leq r < s \leq n \) we get \( k_{r j} = k_{s j} = k_{i r l} = k_{i s l} = k_{i r s} = k_{i s r} = k_{r j s} = k_{s j r} = k_{f} \) for any \( i, j, l \)

But we have \( \sum_{i=1}^n \sum_{j=1}^l k_{ij} = 0 \text{ when } (i, j) \neq (i, j) \) which implies that \( \sum_{i=1}^n \sum_{j=1}^l k_{ij} = \sum_{i=1}^n \sum_{j=1}^l k = 0. \)

i.e. \( (n - 2)k = 0 \Rightarrow p | (n - 2) \text{ or } k = 0. \)

From other side we get for any \( r, s; 1 \leq r < s \leq n \) that \( k_{i j r} = k_{i j s} = k_1. \)

But we have \( \sum_{i=1}^n \sum_{j=1}^l k_{ij} = 1 \text{ when } \)

\[ (i, j) = (i, j). \]Which implies that \( \sum_{i=1}^n \sum_{j=1}^l k_{ij} = \sum_{i=1}^n \sum_{j=1}^l k = (n - 2)k_1 = 1 \Rightarrow p | (n - 2) \text{ and } k_1 \neq 0. \)

Hence we get that \( p | (n - 2), k_1 \neq 0 \text{ and } k = 0. \)

i.e. if the sequence \( (1) \) is split, then \( p | (n - 2). \)

**Corollary 2:** The dimension of \( ker d \) over \( K \) of the \( KS_n \) -homomorphism
\[ d: M(n - 3, 1) \rightarrow M(n - 2, 2) \]

\[ \text{Proof: } \text{Since } d: M(n - 3, 1) \rightarrow M(n - 2, 2) \text{ is onto map. Then we have } \]
\[ M(n - 3, 1) \]
\[ \text{ker } d \approx M(n - 2, 2). \text{ Thus } \text{dim}_K \text{ ker } d = \text{dim}_K M(n - 3, 1) - \text{dim}_K M(n - 2, 2) = \]
\[ \frac{n(n-1)(n-2)}{2} - \frac{n(n-1)}{2} = \frac{n(n-1)(n-3)}{2}. \]

**Lemma 2:** \( \text{dim}_K S(n - 3, 2, 1) = \frac{n(n-2)(n-4)}{4}. \)

**Proposition 3:** \( S(n - 3, 2, 1) \) is a proper submodule of \( ker d. \)
Proof: Recall that \( S(n-3,2,1) = K_{S_n} \Delta (x_1, x_2, x_3) \Delta (x_4, x_5). \)
Let \( y = \Delta (x_1, x_2, x_3) \Delta (x_4, x_5). \) \( = (x_2 - x_1)(x_3 - x_1)(x_5 - x_4) \)
\( = (x_2 - x_1)(x_3 - x_1)(x_5 - x_4) \)
\( \Rightarrow y = x_2 x_3 x_4 x_5 - x_2 x_1 x_4 x_5 - x_2 x_3 x_5 x_4 + x_2 x_4 x_5 x_3 - x_3 x_4 x_5 x_2 + x_4 x_3 x_2 x_5 - x_4 x_2 x_5 x_3 + x_5 x_2 x_3 x_4 - x_5 x_2 x_4 x_3 \)
Then \( d(y) = 2x_2 x_3 x_4 - 2x_2 x_3 - 2x_2 x_4 + 2x_1 x_3 - 2x_1 x_4 + 2x_1 x_2 + 2x_3 x_4 - 2x_3 x_2 \) \( + 2x_2 x_4 = 0. \)
Therefore, \( e \in ker d. \) Hence \( S(n-3,2,1) \subset ker d. \)
But \( dim_{K} S(n-3,2,1) = \frac{n(n-2)(n-4)}{3} \leq \frac{n(n-1)(n-3)}{2} = dim_{K} ker d \)
Therefore \( S(n-3,2,1) \) is a proper submodule of \( ker d. \)

**Corollary 3:** The following sequence of \( KS_n - modules \) is exact
\[ 0 \rightarrow Ker \rightarrow M_0(n-3,2,1) \rightarrow M_0(n-2,2) \rightarrow 0 \] \( \quad \ldots (2) \)

**Proof:** Since \( M_0(n-3,2,1) \subset M(n-3,2,1) \) and the \( K- \) basis of
\( M_0(n-3,2,1) \) is \( \{ x_1 x_2 x_3 x_4 x_5 | 1 \leq i < j \leq n, 1 \leq l \leq n, l \neq i, j, (i, j, l) \neq (1,2,3) \} \)
Then \( \sum_{i=1}^{n} x_1 x_2 x_3 x_4 x_5 \in M_0(n-2,2). \)
Hence \( d | M_0(n-3,2,1) : M_0(n-3,2,1) \rightarrow M_0(n-2,2). \) Let \( \tilde{d} = d | M_0(n-3,2,1). \)
Then \( \tilde{d} : M_0(n-3,2,1) \rightarrow M_0(n-2,2) \) such that \( \tilde{d}(x_1 x_2 x_3 x_4 x_5) = 2x_1 x_2 - 2x_1 x_2 x_3. \)
Then \( 2 \tilde{d}(x_1 x_2 x_3 x_4 x_5) = 2x_1 x_2 - 2x_1 x_2 x_3 \in M_0(n-2,2). \)
\( \forall \alpha \in M_0(n-2,2), \exists \frac{\alpha}{2}(x_1 x_2 x_3 x_4 x_5) \in M_0(n-3,2,1) s.t. \)
\( \tilde{d}(\frac{\alpha}{2}(x_1 x_2 x_3 x_4 x_5)) = \alpha(x_1 x_2) \)
and by linearity of \( \tilde{d} \) we get \( \tilde{d} \) is on to. Thus the following sequence
\[ 0 \rightarrow Ker \rightarrow M_0(n-3,2,1) \rightarrow M_0(n-2,2) \rightarrow 0 \]
is exact since the inclusion map \( i \) is one-to-one and \( Im i = Ker \tilde{d}. \) Moreover \( Ker \tilde{d} \subset Ker d. \)

**Corollary 4:** The sequence (2) is split iff \( p \) does not divide \( n-2. \)

**Proof:** Assume \( p \) does not divide \( n-2. \) By theorem (5) we have \( aKS_n - homomorphism \)
\( \phi : M_0(n-3,2,1) \rightarrow M_0(n-2,2) \) s.t.
\( \phi(x_1 x_2) = \frac{1}{2(n-2)} \sum_{i,j,l} x_1 x_2 x_3 x_4 x_5 \) \( \text{Then } \phi(x_1 x_2 x_3 x_4 x_5) = \phi(x_1 x_2) - \phi(x_1 x_2) \)
\( = \frac{1}{2(n-2)} \sum_{i,j,l} x_1 x_2 x_3 x_4 x_5 - \frac{1}{2(n-2)} \sum_{i,j} x_1 x_2 x_3 x_4 x_5 \in M_0(n-3,2,1) \)
i.e. \( \phi | M_0(n-2,2) : M_0(n-2,2) \rightarrow M_0(n-3,2,1). \)
Let \( \bar{\phi} = \phi | M_0(n-2,2). \) Hence \( \bar{\phi} \) is \( aKS_n - homomorphism \) s.t.
\( d \bar{\phi}(x_1 x_2 x_3 x_4 x_5) = \frac{1}{2(n-2)} \sum_{i,j} x_1 x_2 x_3 x_4 x_5 \) \( = \frac{1}{2(n-2)}(2(n-2) x_1 x_2 x_3 x_4 x_5) = \frac{1}{2(n-2)}(2(n-2) x_1 x_2 x_3 x_4 x_5) = x_1 x_2 x_3 x_4 x_5. \)
Then \( \bar{\phi} \) is one on \( M_0(n-2,2). \) Thus the sequence (2) is split i.e.
\( M_0(n-3,2,1) = Ker d \) \( \oplus \bar{\phi}(M_0(n-2,2)). \)

Now assume the sequence (2) is split. Then there exist a \( KS_n - homomorphism \) \( \psi = \phi | M_0(n-2,2) \) where \( \psi \) as its defined in Theorem (5) s.t. \( \bar{\phi} \) is one. i.e.
\( x_1 x_2 x_3 x_4 x_5 = \bar{\phi}(x_1 x_2 x_3 x_4 x_5) = \phi(x_1 x_2 x_3 x_4 x_5) = d \psi(x_1 x_2 x_3 x_4 x_5) = d \psi(x_1 x_2 x_3 x_4 x_5) = d \psi(x_1 x_2 x_3 x_4 x_5) = d \psi(x_1 x_2 x_3 x_4 x_5) = \frac{1}{2(n-2)}(2(n-2) x_1 x_2 x_3 x_4 x_5) = x_1 x_2 x_3 x_4 x_5. \)
Then by equating the coefficient we get \( 2(n-2)k = 1 \) and \( 2(n-2)k_3 = 1. \) In each case we get that \( p \) does not divide \( n-2. \)
Corollary 5: If \( p \neq 2 \) and \( p | (n-1) \) then we get the following composition series

1) \( 0 \subset \overline{N}_0 \subset \overline{N} \subset L_0 \)
2) \( 0 \subset \ker \sigma_3(\overline{N}) \subset \overline{N} \subset L_0 \)

where \( L_0 = M_0(n-2,2) \)

Proof: By corollary \( p | (n-1) \). Then \( p \not| n \) and \( p \not| (n-2) \). Then by corollary \( 4 \) we have \( M_0(n-3,2,1) = \ker d \Leftrightarrow L_0 \) where \( L_0 = M_0(n-2,2) \).

\[
\overline{N}_0 = K \sigma_1(n) ; \quad \overline{C}_1(n) = \sum_{\overline{i}(j,k)} x_i x_j x_k^2 \ . \text{Then the sum of coefficients of} \ \overline{C}_1(n) \text{is} \ \frac{(n-1)(n-2)}{2} \ \text{and since} \ \endgroup
\]

\[
p | (n-1) \text{then} \overline{N}_0 \subset M_0(n-3,2,1) . \text{But} \ d(\overline{C}_1(n)) = 2 \sum_{i,j,k} x_i x_j \neq 0 .
\]

Hence \( \overline{N}_0 \cap \ker d = 0 \), which implies that \( \overline{N}_0 \subset L_0 \). Since \( p \not| n \). Then by proposition \( 1 \) we have \( \overline{N} = \overline{N}_0 \oplus K \sigma_3(n) ; \overline{N}_0 = K \sigma_1(\overline{N}) \). and by theorem \( 3 \) we get \( \overline{N}_0 = N_0 = M_0(n-1,1) \), where \( N_0 = K \sigma_1(n) \).

Let \( g_1 ; K \to K \sigma_3(n) \) defined by \( g_1(k) = k \sigma_3(n) \) then \( g_1 \) is a \( K \sigma_1 \)-isomorphism. Thus \( K \simeq K \sigma_3(n) \).

Let \( g_2 ; K \to K \sigma_3(n) \) defined by \( g_2(k) = k \sigma_3(n) \) then \( g_2 \) is a \( K \sigma_1 \)-isomorphism. Thus \( K \simeq K \sigma_3(n) \).

Hence \( K \sigma_3(n) \simeq K \sigma_3(n) \). Since \( N_0 \simeq M_0(n-1,1) \) (see [4]) and \( \overline{N}_0 = M_0(n-1,1) \) by theorem \( 3 \), then \( \overline{N} = \overline{N}_0 \oplus K \sigma_3(n) \), which implies that \( \overline{N} = \overline{N}_0 \oplus K \sigma_3(n) \).

Let \( \lambda_0 \overline{N} = \overline{N}_0 \oplus K \sigma_3(n) \).

Theorem 6: The following sequence over a field \( K \) is exact.

\[
0 \to M_0(n-3,2,1) \to M(n-3,2,1) \to K \to 0 \quad \text{...(3)}
\]

Proof: Since the inclusion map \( i \) is one-to-one and \( f : M(n-3,2,1) \to K \)

\[
f \left( \sum_{i,j,k} k_{ijk} x_i x_j x_k^2 \right) = \sum_{i,j,k} k_{ijk} \ \text{is onto since} \ \forall k \in K, \exists x_i x_j x_k^2 \in M(n-3,2,1) \quad \text{s.t.}
\]

\[
f \left( k x_i x_j x_k^2 \right) = k . \text{Moreover we have} \ker f = \{ y \in M(n-3,2,1) | f(y) = 0 \}
\]

\[
= \{ \sum_{i,j,k} k_{ijk} x_i x_j x_k^2 \} \}
\]

Then the sum of coefficient of \( C_1(n) = x_1^2(\sigma_2(n) - \sum_{j=2} x_j x_j) = \sum_{i,j,k} x_i x_j x_k^2 \)

\[
\text{Then the sum of coefficient} \ \frac{(n-1)(n-2)}{2} \ \text{which implies that} \ C_1(n) \in M_0(n-3,2,1) \ \text{when} \ p \ \text{divides} \ (n-1) \ . \text{Then we get that} \ \overline{N} \in M_0(n-3,2,1) \ . \text{Since} \ u_{12}(n) = C_1(n) - C_2(n) \ . \text{Then} \ \overline{N}_0 = K u_{12}(n) = K \sigma_1(n) - C_2(n) \) \text{and hence} \ \overline{N}_0 \subset \overline{N} . \text{Since} \ p \neq 2 \ \text{and} \ p \ \text{divides} \ (n-1) \ . \text{Then} \ p \ \text{does not divide} \ n \ \text{which implies by corollary} \ (1) \ \text{that} \ \overline{N}_0 \text{is irreducible submodule over} \ K \sigma_1 \text{,and} \ p \ \text{does not divide} \ (n-2) \ \text{which implies that} \ K \sigma_3 \not\subset \ker d . \text{Since} \ \overline{N} = \overline{N}_0 \oplus K \sigma_3 \text{when} \ p \ \text{does not divide} \ n \ \text{by proposition} \ (1) \text{,and both} \ \overline{N}_0 \text{and} \ K \sigma_3 \text{are irreducible modules.}
\]

DOI: 10.9790/5728-1202022734
Then \( \bar{N} \cap \text{ker} \, d = 0 \). Therefore we get the following series if \( p \neq 2 \) and \( p \mid (n - 1) \)

1) \( 0 \subset \bar{N}_0 \subset \bar{N} \subset \bar{N} \oplus \text{ker} \, d \subset M_0(n - 3,2,1) \subset M(n - 3,2,1) \).
2) \( 0 \subset \bar{N}_0 \subset \bar{N}_0 \oplus \text{ker} \, d \subset \bar{N} \oplus \text{ker} \, d \subset M_0(n - 3,2,1) \subset M(n - 3,2,1) \).
3) \( 0 \subset K \sigma_3 \subset K \sigma_3 \oplus \text{ker} \, d \subset \bar{N} \oplus \text{ker} \, d \subset M_0(n - 3,2,1) \subset M(n - 3,2,1) \).
4) \( 0 \subset K \sigma_3 \subset \bar{N} \subset \bar{N} \oplus \text{ker} \, d \subset M_0(n - 3,2,1) \subset M(n - 3,2,1) \).
5) \( 0 \subset \text{ker} \, d \subset \bar{N}_0 \oplus \text{ker} \, d \subset \bar{N} \oplus \text{ker} \, d \subset M_0(n - 3,2,1) \subset M(n - 3,2,1) \).
6) \( 0 \subset \text{ker} \, d \subset K \sigma_3 \oplus \text{ker} \, d \subset \bar{N} \oplus \text{ker} \, d \subset M_0(n - 3,2,1) \subset M(n - 3,2,1) \).
7) \( 0 \subset \bar{N}_0 \subset \bar{N} \subset L_0 \subset M_0(n - 3,2,1) \subset M(n - 3,2,1) \).

References