On the Non Homogeneous Ternary Quadratic Equation

\[ x^2 + xy + y^2 = 12z^2 \]

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**ABSTRACT:** The Ternary Quadratic Diophantine Equation given by \( x^2 + xy + y^2 = 12z^2 \) is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

**Keyword:** Quadratic equation with three unknowns, Integral Solutions.

**I. Introduction**

The Theory of Diophantine equation offers a rich variety of fascinating problems. In particular, quadratic equations, homogeneous and non-homogeneous have aroused the interest of numerous Mathematicians. Since ambiguity [1-3]. This paper concerns with the problem of determining non-trivial solutions of the non-homogeneous quadratic equation with three unknowns given by \( x^2 + xy + y^2 = 7z^2 \). A few relations among the solutions are presented.

**NOTATIONS USED**

- \( T_{m,n} \) - Polygonal number of rank \( n \) with size \( m \).
- \( P_n^m \) - Pyramidal number of rank \( n \) with size \( m \).
- \( Pr_n \) - Pronic number of rank \( n \).
- \( SO_n \) - Stella Octangular number of rank \( n \).
- \( Obl_n \) - Oblong number of rank \( n \).
- \( OH_n \) - Octahedral number of rank \( n \).
- \( Tet_n \) - Tetrahedral number of rank \( n \).
- \( PP_n \) - Pentagonal Pyramidal number of rank \( n \).

**II. Method Of Analysis**

The quadratic Diophantine Equation with three Unknowns to be solved for its non-Zero distinct integral solution is

\[ x^2 + xy + y^2 = 7z^2 \]  \( (1) \)

Introduction of the linear transformations

\[ x = u + v \text{ and } y = u - v \]  \( (2) \)

In (1) leads to

\[ 3u^2 + v^2 = 12z^2 \]  \( (3) \)

Different patterns of solutions of (3) and hence that of (1) using (2) are given below.

**PATTERN-I**

\[ x = u + 3v \text{ and } y = u - 3v \]  \( (4) \)

in (1) leads to \( u^2 + 3v^2 = 4z^2 \)  \( (5) \)

Let \( z = z(a,b) = a^2 + 3b^2 \)  \( (6) \)

Write \( 4 = (1 + i\sqrt{3}) (1 - i\sqrt{3}) \)  \( (7) \)

Using (6) and (7) in (5) and applying the method of factorization

\[ u + i\sqrt{3} v = (1 + i\sqrt{3}) (a + i\sqrt{3}b)^2 \]

Equating the real and imaginary parts, we have

\[ u = u(a,b) = a^2 - 6ab - 3b^2 \]

\[ v = v(a,b) = a^2 - 3b^2 + 2ab \]

Substituting the above values of \( u \) and \( v \) in (4) we get

\[ x = x(a,b) = 4a^2 - 12b^2 \]

\[ y = y(a,b) = -2a^2 - 12ab + 6b^2 \]

\[ z = z(a,b) = a^2 + 3b^2 \]
Properties
1. \( y(a, b) + 2z(a, b) - 108t_{3, b} + t_{126, b} \equiv 0 \) (mod 107)
2. \( x(a, a + 1) + 2y(a, a + 1) = 42P_a \)
3. \( x(a, 3) + y(a, 3) + 2z(a, 3) - 188t_{3, b} + t_{180, a} \equiv 0 \) (mod 219)
4. \( y(a, 7a^2 - 4) + 2z(a, 7a^2 - 4) - t_{26, b} + 36Cp_{14} \equiv 0 \) (mod 11)
5. \( x(a, a + 1) - 2y(a, a + 1) - t_{18, a} - 24P_a \equiv 0 \) (mod 7)
6. \( 2z(b(b + 1), b) + y(b(b + 1), b) + 24P^5_b - 116t_{3, b} + t_{94, b} \equiv 0 \) (mod 103)
7. \( x(a, (a + 1)(a + 2)) + 2y(a, (a + 1)(a + 2)) = 144P_a^3 \)
8. \( 3x(a, a) + z(a, a) \) and \( y(a, a) + z(3a, a) \) represents a nasty number.

PATTERN-II
Rewrite (5) as \( \frac{3(v + z)}{v + z} = \frac{p}{a} \) (B ≠ 0)
The above equation is equivalent to the system of equations,
\(-Au + 3Bv + (3B - A)z = 0\)
\(-Bu + Av + (A + B)z = 0\)
This is satisfied by
\( u = 3B^2 + 6AB - A^2 \)
\( v = A^2 + 2AB - 3B^2 \)
\( z = A^2 + 3B^2 \)
Hence in view of (4), the corresponding solutions of (1) are given by
\( x = x(A, B) = 2A^2 - 6B^2 + 12AB \)
\( y = y(A, B) = 12B^2 - 4A^2 \)
\( z = z(A, B) = A^2 + 3B^2 \)

Properties
1. \( x(A, 3) + 2z(A, 3) - 128t_{3, A} + t_{122, A} \equiv 0 \) (mod 87)
2. \( x(B(B + 1), B) - 2z(B(B + 1), B) - 24P^5_B + 88t_{3, B} - t_{66, B} \equiv 0 \) (mod 75)
3. \( x(A, 2A^2 + 1) + 2z(A, 2A^2 + 1) - 36OH_A - t_{130, A} \equiv 0 \) (mod 3)
4. \( 2z(A(A + 1), B) - x(A(A + 1), B) - t_{26, B} + 24Ct_{A, B} \equiv 24 \) (mod 11)
5. \( y(A, A + 1) + 2x(A, A + 1) = 24P_A \)
6. \( x(A, 1) - s_A + t_{10, A} \equiv 7 \) (mod 15)
7. \( 4\{x(b, b) + y(b, b)\} and 12\{x(a, -a) + y(a, a) + z(a, a)\} \) represents a nasty number.

PATTERN-III
Equation (5) is Equivalent to \( u^2 + 3v^2 = (2z^2) \) which is satisfied by
\( u = 3p^2 - q^2 \)
\( v = 2pq \)
\( z = \frac{1}{2}(3p^2 + q^2) \)
Put \( P = 2A, q = 2B \)
In view (4), the non zero distinct integral solutions of (1) are
\( x(A, B) = 12A^2 - 4B^2 + 24AB \)
\( y(A, B) = 12A^2 - 4B^2 - 24AB \)
\( z(A, B) = 6A^2 + 2B^2 \)

Properties
2. \( y(A, 2A^2 + 1) + 2z(A, 2A^2 + 1) - t_{50, A} + 72OH_A \equiv 0 \) (mod 23)
3. \( x(A, (A + 1), B) - 2z(A(A + 1), B) + t_{18, B} - 48Ct_{A, B} \equiv 48 \) (mod 7)
4. \( x(A, 2) + 2z(A, 2) - 16B^2 + 13t_{122, A} \equiv 0 \) (mod 95)
5. \( x(B(B + 1), B) - 2z(B(B + 1), B) - 48P^5_B + 88t_{5, B} - t_{74, B} \equiv 0 \) (mod 79)
6. \( x(A, 3) - 88t_{3, A} + t_{66, A} \equiv 36 \) (mod 3)
7. \( x(a, a) - z(a, a) \) represents a nasty number.

PATTERN-IV
Rewrite (5) as \( u^2 + 3v^2 = 4z^2 \) \times 1 \)
Write (4) as \( 4 = (1 + i\sqrt{3}) (1 - i\sqrt{3}) \)

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and \[ 1 = \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4} \] (10)

Assume \( z = a + 3b \)

Using (9) and (10) in (8), we get the system of equation,

\[ \begin{align*}
(u + i\sqrt{3}v) &= (1 + i\sqrt{3})\left(\frac{(1+i\sqrt{3})}{2}\right) \cdot (a + i\sqrt{3}b)^2
\end{align*} \]

Equating the real and imaginary parts in the above equation, we get

\[ \begin{align*}
u &= u(a, b) = -a^2 + 3b^2 - 6ab \\
v &= v(a, b) = a^2 - 3b^2 - 2ab
\end{align*} \]

In view of (4), the non-Zero distinct integral solution of (1) are

\[ \begin{align*}
x &= x(a, b) = 2a^2 - 6b^2 - 12ab \\
y &= y(a, b) = -4a^2 + 12b^2 \\
z &= z(a, b) = a^2 + 3b^2
\end{align*} \]

Properties

1. \( x(b + 1, b) - 2z(b + 1, b) + 12(obl)_b + 26t_{ob}, b \equiv 0\) (mod 11)
2. \( y(a, (a + 1)(a + 2)(a + 3)) + 2x(a, (a + 1)(a + 2)(a + 3)) = -576P_t_a \)
3. \( 2(z(a, (a + 1)) + x(a, (a + 1)) + 12P_{r_a} - t_{10}a \equiv 0\) (mod 3)
4. \( x(1, b) - 2z(1, b) + 128t_{23_b} - t_{106}, b \equiv 0\) (mod 103)
5. \( y(a, (a + 1) + 2x(a, (a + 1)) = -48P_5 \)
6. \( 2z(a, 2) + x(a, 2) - 188t_{23_a} + t_{102}, a \equiv 0\) (mod 207)
7. \( 2z(b(b + 1), b) - x(b(b + 1), b) - 24P_5 - 84t_{34_b} + t_{62}, b \equiv 0\) (mod 71)
8. \( x(a, a) + y(a, a) and x(a, a) + z(a, a) represents a nasty numbers. \)

PATTERN-V

Also write 1 as

\[ 1 = 1/49(1 + i\sqrt{3})(1 - 4i\sqrt{3}) \]

In equation (8),

\[ (u + i\sqrt{3}v) = (1 + i\sqrt{3})\left(\frac{(1+i\sqrt{3})}{7}\right) \cdot (a + i\sqrt{3}b)^2 \]

Equating Real and Imaginary parts in the above relation, we get

\[ \begin{align*}
u &= u(a, b) = \frac{1}{7}(33b^2 - 11a^2 - 30ab) \\
v &= v(a, b) = \frac{1}{7}(5a^2 - 12b^2 - 22ab)
\end{align*} \]

Put \( a = 7A, b = 7B \) then

\[ \begin{align*}
u &= u(A, B) = 231B^2 - 77A^2 - 210AB \\
v &= v(A, B) = 35A^2 - 105B^2 - 154AB
\end{align*} \]

In view (4) the non-zero distinct integral solutions of (1) are

\[ \begin{align*}
x &= x(A, B) = 28A^2 - 84B^2 - 672AB \\
y &= y(A, B) = 546B^2 - 182A^2 - 252AB \\
z &= z(A, B) = 49A^2 + 147B^2
\end{align*} \]

Properties

1. \( 49x(B + 1, B) - 28z(B + 1, B) + 32928(obl)_B + t_{16466}, B \equiv 0\) (mod 8281)
3. \( x(A, 1) - S_4 - t_{46_A} \equiv 85\) (mod 645)
4. \( 84z(A, (A + 1)) + 147x(A, (A + 1) - t_{1666}, A + 98784P_P_A \equiv 0\) (mod 8231)
5. \( x(B^2 + 1, B) - x(B^2 - 1, B) - 124Pr_B + t_{26}, B \equiv 0\) (mod 1468)
6. \( \frac{546A(2A^2 - 1) - 147x(A, (2A^2 - 1) - 535080b_6 + 370455a_6 \equiv 8\) (mod 53508) \)

III. Conclusion

In this paper, we have obtained infinitely many non-zero distinct integer solutions to the quadratic equations with three unknowns.

References:


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