Pgrw-Continuous and Pgrw-Irresolute Maps in Topological Spaces

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Abstract: This paper introduces pre generalized regular weakly continuous maps, pgrω-irresolute maps, strongly pgrω-continuous maps, perfectly pgrω-continuous maps and studies some of their properties.

Keywords: pgrω-closed sets, pgrω-open sets, pgrω-continuous maps, pgrω-irresolute maps, strongly pgrω-continuous maps, perfectly pgrω-continuous maps.

I. Introduction

N. Levine[1] introduced Semi-open sets and semi-continuity in topological spaces. The concept of regular continuous and Completely-continuous functions was first introduced by Arya. S. P. and Gupta.R[2]. Later Y. Gnanamal [3] studied the concept of generalized pre regular continuous functions. Also, the concept of ω-continuous functions was introduced by S S Benchali et al [4]. R S Wali et al[5] introduced and studied the properties of pgrω-Continuous and pgrω-Irresolute Maps. Recently R S Wali et al[6] introduced and studied the properties of pgrω-closed sets. The purpose of this paper is to introduce a new class of functions, namely, pgrω-continuous functions and pgrω-irresolute functions, strongly pgrω-continuous maps, perfectly pgrω-continuous maps. Also we study some of the characterization and basic properties of pgrω-continuous functions.

II. Preliminaries

Definition 2.1: A subset A of a topological space (X, T) is called

- a pre-open set[7] if A ⊆ int(cl(A)) and pre-closed set if cl(int(A)) ⊆ A.
- an α-open set[8] if A ⊆ int(cl(int(A))) and α -closed set if cl(int(cl(A))) ⊆ A.
- a semi-preopen set (=β-open)[9] if A⊆cl(int(cl(A))) and a semi-pre closed set (=β-closed) if int(cl(int(A)))⊆A.
- a regular open set[10] if A = int(cl(A)) and a regular closed set if A = cl(int(A)).
- a generalized closed set (briefly g-closed)[11] if cl(A) ⊆ U whenever A ⊆ U and U is open in X.
- a regular generalized closed set(briefly rg-closed)[11] if cl(A)⊆U whenever A⊆U and U is regular open in X.
- a α -generalized closed set(briefly ag -closed)[12] if acl(A)⊆U whenever A⊆U and U is open in X.
- a generalized pre regular closed set(briefly gpr-closed)[3] if pcl(A) ⊆ U whenever A ⊆ U and U is regular open in X.
- a generalized semi-pre closed set(briefly gsp-closed)[13] if spcl(A) ⊆ U whenever A⊆U and U is open in X.
- a regular generalized α-closed set[14] (briefly, rga-closed) if acl (A)⊆ U whenever A⊆ U and U is regular α-open in X.
- an α-regular generalized closed[15] (briefly agr-closed) set if acl(A)⊆ U whenever A⊆ U and U is regular α-open in X.
- a ωα- closed set[16] if acl(A) ⊆ U whenever A⊆ U and U is ω-open in X.
- a generalized pre closed (briefly gp-closed) set[17] if pcl(A)⊆U whenever A⊆U and U is open in X.
- a α-regular w- closed set[5] if acl(A) ⊆ U whenever A⊆U and U is rw -open in X.
- a generalized pre regular weakly closed (briefly gprw-closed) set [18] if pcl(A)⊆U whenever A⊆ U and U is regular semi- open in X.
- a #rg-closed[19] if cl(A)⊆U whenever A⊆U and U is rw-open.
- a regular generalized weak (briefly rgw-closed) set[20] if cl(int(A)) ⊆ U whenever A ⊆ U and U is regular semi open in X.
- ageneralized semi pre regular closed (briefly gspr-closed) set [21] if spcl(A)⊆ U whenever A⊆U and U is regular open in X.

The complements of the above mentioned closed sets in (5) - (18), are called the respective open sets.
Definition 2.2: Let (X, T) be a topological space and A ∈ X. The intersection of all closed (resp pre-closed, α-closed and semi-pre-closed) subsets of the space X containing A is called the closure (resp pre-closure, α-closure and Semi-pre-closure) of A and is denoted by cl(A) (resp pcl(A), acl(A), spcl(A)).

2.3 Pre Generalised Regular Weakly Closed Set:
Definition: A subset A of a topological space (X, T) is called a pre generalised regular weakly closed set [6] if pcl(A) ⊆ U whenever A ⊆ U and U is a rw-open set.

• Theorem: Every pgrw-closed set is gp-closed.
• Theorem: Every pre-closed set is pgrw-closed.
• Corollary: Every α-closed set is pgrw-closed.
• Corollary: Every closed set is pgrw-closed.
• Corollary: Every regular closed set is pgrw-closed.
• Corollary: Every arw-closed set is pgrw-closed.
• Theorem: Every pgrw-closed set is gsp-closed.
• Corollary: Every pgrw-closed set is gspr-closed.
• Corollary: Every pgrw-closed set is gpr-closed.
• Theorem: If A is open and gp-closed, then A is pgrw-closed.
• Theorem: If A is both w-open and wα-closed, then A is pgrw-closed.
• Theorem: If A is both regular-open and rg-closed, then A is pgrw-closed.
• Theorem: If A is both open and g-closed, then A is pgrw-closed.
• Theorem: If A is regular-open and gpr-closed, then it is pgrw-closed.
• Theorem: If A is regular-open and agr-closed, then it is pgrw-closed.
• Theorem: If A is open and αgr-closed, then it is pgrw-closed.
• Theorem: If A is regular open and pgr-closed, then A is p-pre-closed.

2.4: Definition: A subset A of a topological space X is called a pre generalised regular weakly open (briefly pgrw-open) set in X if the complement A^c of A is pgrw-closed in X.

Theorem: (X, T) is a topological space.

i) Every open (α-open, regular-open, ar-o-open, #rg-open, pgr-open) set is pgrw-open.
ii) Every pgrw-open set is gspr–open (gp-open, gp-open and gpr-open).

Definition 2.5: A map f: (X, τ)→(Y, σ) is said to be

• Completely–continuous[22] if f⁻¹ (V) is regular closed in X for every closed subset V of Y
• Strongly–continuous[23] if f⁻¹ (V) is Clopen (both open and closed) in X for every subset V of Y.
• α–continuous[8] if f⁻¹ (V) is α–closed in X for every closed subset V of Y.
• rwg–continuous[24] if f⁻¹ (V) is rwg–closed in X for every closed subset V of Y.
• gp–continuous[25] if f⁻¹ (V) is gp–closed in X for every closed subset V of Y.
• gpr–continuous[3] if f⁻¹ (V) is gpr–closed in X for every closed subset V of Y.
• agr–continuous[15] if f⁻¹ (V) is agr–closed in X for every closed subset V of Y.
• oα–continuous[4] if f⁻¹ (V) is oα–closed in X for every closed subset V of Y.
• gspr–continuous[21] if f⁻¹ (V) is gspr–closed in X for every closed subset V of Y.
• g–continuous[25] if f⁻¹ (V) is g–closed in X for every closed subset V of Y.
• o–continuous[26] if f⁻¹ (V) is o–closed in X for every closed subset V of Y.
• rg–continuous[14] if f⁻¹ (V) is rg–closed in X for every closed subset V of Y.
• gsp–continuous[13] if f⁻¹ (V) is gsp–closed in X for every closed subset V of Y.
• gprw–continuous[18] if f⁻¹ (V) is gprw–closed in X for every closed subset V of Y.
• wrg–continuous[27] if f⁻¹ (V) is wrg–closed in X for every closed subset V of Y.
• #rg–continuous[28] if f⁻¹ (V) is #rg–closed in X for every closed subset V of Y.
• pre–continuous[7] if f⁻¹ (V) is preopen in X for every open set V in Y.
• rg–continuous[29] if the inverse image of every closed set in Y is rg–closed in X.
• semi-pre continuous (β– continuous)[30] if the inverse image of each open set in Y is a semi-preopen set in X.
• semi–generalized continuous (sg–continuous)[31] if for every closed set F of Y the inverse image f⁻¹ (F) is sg–closed in X.
• ro–continuous[32] if f⁻¹ (V) is rw–closed in X for every closed subset V of Y.
• α regular o–continuous (aro–Continuous)[5] if f⁻¹ (V) is aro–Closed in X for every closed set V in Y.
• contra continuous [16] if \( f^{-1}(V) \) is open in \( X \) for every closed subset \( V \) of \( Y \).

**Definition 2.6:** A map \( f: (X, t) \rightarrow (Y, c) \) is said to be
\begin{itemize}
  \item \( \alpha \)-irresolute [8] if \( f^{-1}(V) \) is \( \alpha \)-closed in \( X \) for every \( \alpha \)-closed subset \( V \) of \( Y \).
  \item irresolute [33] if \( f^{-1}(V) \) is semi-closed in \( X \) for every semi-closed subset \( V \) of \( Y \).
  \item contra \( \omega \)-irresolute [26] if \( f^{-1}(V) \) is \( \omega \)-open in \( X \) for every \( \omega \)-open subset \( V \) of \( Y \).
  \item contra irresolute [17] if \( f^{-1}(V) \) is semi-open in \( X \) for every semi-open subset \( V \) of \( Y \).
  \item contra \( r \)-irresolute [34] if \( f^{-1}(V) \) is regular-open in \( X \) for every regular-open subset \( V \) of \( Y \).
\end{itemize}

### III. Pgrw-Continuous Map:

**Definition 3.1:** A map \( f: (X, T_1) \rightarrow (Y, T_2) \) is called a pre generalised regular weakly-continuous map (pgrw-continuous map) if the inverse image \( f^{-1}(V) \) of every closed subset \( V \) of \( Y \) is pgrw-closed in \( X \).

**Example 3.2:** Let \( X = \{a, b, c, d\}, T_1 = \{\{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\} \text{ and } Y = \{a, b, c\} \text{ and } T_2 = \{Y, \emptyset, \{a\}\} \). Define a map \( f: X \rightarrow Y \) by \( f(a) = b, f(b) = c, f(c) = a, f(d) = c \). The closed sets in \( T_2 \) are \( Y, \emptyset, \{b, c\} \). The pgrw-closed sets in \( T_1 \) are \( \emptyset, \{a\}, \{b, c\}, \{a, b\}, \{a, b, c\} \). Inverse images of \( Y, \emptyset, \{b, c\} \) are \( X, \emptyset, \{a, b, d\} \) which are pgrw closed sets.

**Proof:** \( f \) is pgrw-continuous map.

**Theorem 3.3:** A map \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrw-continuous if and only if the inverse image of every open set in \( Y \) is a pgrw-open set in \( X \).

**Proof:** Suppose \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrw-continuous. Let \( U \) be an open set in \( Y \). Then \( U^c \) is closed in \( Y \). Therefore \( f^{-1}(U^c) \) is pgrw-closed in \( X \).

Conversely, suppose \( f: (X, T_1) \rightarrow (Y, T_2) \) is such that the inverse image of every open set is in \( Y \) is pgrw-open in \( X \). Let \( F \) be a closed set in \( Y \). Then \( F^c \) is open in \( Y \). \( f^{-1}(F^c) = X \setminus f^{-1}(F) \). \( f^{-1}(F) \) is pgrw-closed in \( X \). \( f \) is a pgrw-continuous map.

**Theorem 3.4:** If \( f: (X, T_1) \rightarrow (Y, T_2) \) is continuous, then it is pgrw-continuous.

**Proof:** Let \( F \) be a closed subset in \( Y \). \( f \) is continuous. \( f^{-1}(F) \) is a closed set in \( X \). As every closed set is pgrw-closed, \( f^{-1}(F) \) is pgrw-closed. \( f \) is pgrw-continuous map.

The converse is not true.

**Example 3.5:** Consider example 3.2. \( \{b, c\} \) is closed in \( Y \) and its inverse image \( \{a, b, d\} \) is not closed in \( X \).

**Theorem 3.6:** If \( f: (X, T_1) \rightarrow (Y, T_2) \) is completely continuous, then \( f \) is pgrw-continuous.

**Proof:** Assume \( f: (X, T_1) \rightarrow (Y, T_2) \) is completely continuous. Let \( F \) be a closed set in \( Y \). Then \( f^{-1}(F) \) is regular-closed in \( X \).

\( f^{-1}(F) \) is pgrw-closed in \( X \) as every regular-closed set is pgrw-closed.

\( f \) is pgrw-continuous.

The converse is not true.

**Example 3.7:** In the above example 3.2 \( f \) is pgrw-continuous. But not completely continuous.

**Theorem 3.8:** If \( f: (X, T_1) \rightarrow (Y, T_2) \) is pre-continuous, then \( f \) is pgrw-continuous.

**Proof:** A map \( f: (X, T_1) \rightarrow (Y, T_2) \) is pre-continuous. Let \( F \) be a closed set in \( Y \). Then \( f^{-1}(F) \) is pre-closed in \( X \).

\( f \) is pgrw-continuous. The converse is not true.

**Example 3.9:** \( X = \{a, b, c, d\}, T_1 = \{\{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}\} \)
\( Y = \{a, b, c\}, T_2 = \{\{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\} \)
Closed sets in \( T_1 \) are \( X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{c\} \).
Closed sets in \( T_2 \) are \( X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{c\} \).

Define \( f(a) = c, f(b) = a, f(c) = b, f(d) = c \). Inverse images of closed sets in \( Y \) are \( X, \emptyset, \{a\}, \{a, c\}, \{a, b\}, \{a, d\} \).

Then \( f \) is pgrw-continuous. But \( f \) is not pre-continuous since \( f^{-1}\{\{a, c\}\} = \{a, b\} \) is not pre-closed.
Theorem 3.10: If a map \( f : (X, T_1) \to (Y, T_2) \) is \( \alpha \)-continuous, then \( f \) is pgrw-continuous.

Proof: A map \( f : X \to Y \) is \( \alpha \)-continuous.

Let \( F \) be closed in \( Y \). Then \( f^{-1}(F) \) is \( \alpha \)-closed in \( X \).

Then \( f^{-1}(F) \) is pgrw-closed in \( X \) because every \( \alpha \)-closed is pgrw-closed.

\( \therefore \) \( f \) is pgrw-continuous map.

The converse is not true.

Example 3.11: \( X = Y = \{a, b, c, d\} \),
\( T_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \)
\( T_2 = \{Y, \varnothing, \{a\}, \{b\}, \{a, b, c\}\} \)
Closed sets in \( T_1 \) are \( Y, \varnothing, \{a, b, c\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \).

Pgrw-closed sets in \( T_1 \) are \( \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \).

Define \( f(a) = c, f(b) = a, f(c) = b, f(d) = d \). Inverse images of closed sets in \( Y \) are \( X, \varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\} \).

Then \( f \) is pgrw-continuous. But \( f \) is not \( \alpha \)-continuous.

Theorem 3.12: If a map \( f : (X, T_1) \to (Y, T_2) \) is \#rg -continuous, then \( f \) is pgrw-continuous.

The converse is not true.

Example 3.13: \( X = \{a, b, c, d\}, T_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \)
\( Y = \{a, b, c\}, T_2 = \{Y, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \)
Closed sets in \( (Y, T_2) \) are \( Y, \varnothing, \{a\}, \{b\}, \{a, b\} \).

Pgrw-closed sets in \( T_1 \) are \( \{Y, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \).

\#rg-closed sets in \( Y \) are \( \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\).

Define \( f(a) = c, f(b) = a, f(c) = b, f(d) = a \).

Inverse images of closed sets in \( Y \) are \( X, \varnothing, \{a\}, \{a, b\}, f \) is pgrw-continuous but not \#rg -continuous.

Theorem 3.14: If a map \( f : (X, T_1) \to (Y, T_2) \) is \#rgw-continuous, then \( f \) is pgrw-continuous.

The converse is not true.

Example 3.15: \( X = \{a, b, c, d\}, T_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \)
\( Y = \{a, b, c\}, T_2 = \{Y, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \)
Closed sets in \( (Y, T_2) \) are \( Y, \varnothing, \{a\}, \{b\}, \{a, b\} \).

Pgrw-closed sets in \( T_1 \) are \( \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\} \).

\#rgw-closed sets in \( Y \) are \( \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \).

Define \( f(a) = d, f(b) = c, f(c) = a, f(d) = a \).

Inverse images of closed sets in \( Y \) are \( X, \varnothing, \{b\}, f \) is pgrw-continuous but not \#rgw-continuous.

Theorem 16: If a map \( f : (X, T_1) \to (Y, T_2) \) is \( \alpha \)-irresolute, then it is pgrw-continuous.

Proof: Suppose that a map \( f : (X, T_1) \to (Y, T_2) \) is \( \alpha \)-irresolute. Let \( V \) be an open set in \( Y \). Then \( V \) is \( \alpha \)-open in \( Y \). Since \( f \) is \( \alpha \)-irresolute, \( f^{-1}(V) \) is \( \alpha \)-open and hence pgrw-open in \( X \). Thus \( f \) is pgrw-continuous.

Theorem 3.17: If a map \( f : (X, T_1) \to (Y, T_2) \) is pgrw-continuous, then \( f \) is gsp-continuous.

Proof: \( f : X \to Y \) is pgrw-continuous. Let \( F \) be a closed set in \( Y \). Then \( f^{-1}(F) \) is pgrw-closed.

\( \Rightarrow f^{-1}(F) \) is gsp-closed. ‘.’ Every pgrw-closed set is gsp-closed. \( \Rightarrow f \) is gsp-continuous.

Converse is not true.

Example 3.18: \( X = \{a, b, c\}, T_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}\} \)
\( Y = \{a, b, c\}, T_2 = \{Y, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \)
Closed sets in \( T_2 \) are \( Y, \varnothing, \{a\}, \{b\}, \{a, c\} \).

Pgrw-closed sets in \( T_1 \) are \( X, \varnothing, \{a\}, \{b\}, \{a, c\} \).

Define \( f(a) = b, f(b) = c, f(c) = a \). Inverse images of closed sets in \( Y \) are \( X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\} \).

\( f^{-1}(\{a, b\}) = \{a\} \) which is not pgrw-closed. So \( f \) is not pgrw-continuous. gsp-closed sets are \( X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\} \).

\( f \) is gsp-continuous.

Theorem 3.19: If a map \( f : (X, T_1) \to (Y, T_2) \) is pgrw-continuous, then \( f \) is gsp-continuous.

Proof: We can prove it using the fact that every pgrw-closed set is gsp closed.

Converse is not true. For example,
\( X = \{a, b, c, d\}, T_1 = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \)
\( Y = \{a, b, c\}, T_2 = \{Y, \varnothing, \{a\}\} \)
Closed sets in \( T_2 \) are \( Y, \varnothing, \{a\} \).

Pgrw-closed sets in \( T_1 \) are \( \{a\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, d\}, \{b, d\}, \{a, d\}, \{a, b, d\} \).

Define \( f(a) = b, f(b) = c, f(c) = a, f(d) = a \). Inverse images of closed sets are \( X, \varnothing, \{a\} \).

\( f^{-1}(\{a, b\}) = \{a, b\} \) which is not pgrw-closed. So \( f \) is not pgrw-continuous. All subsets of \( X \) are gsp-closed.

\( f \) is gsp-continuous.
Theorem 3.20: If a map f: (X, T₁)→(Y, T₂) is pgrw-continuous, then f is gpr-continuous.
We can prove it using the fact that every pgrw-closed set is gpr-closed.
Converse is not true.

Example 3.21: Consider example 3.18, f is not pgrw-continuous. gpr-closed sets are X, φ, {c}, {d}, {a,b}, {b,c}, {c,d}, {a,a}, {a,d}, {b,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}. f is gpr-continuous.

Theorem 3.22: If a map f: (X, T₁)→(Y, T₂) is pgrw-continuous, then f is gp-continuous.
We can prove it using the fact that every pgrw-closed set is gp-closed.
The converse is not true.

Example 3.23: X={a,b,c}, T₁ = {X, φ, {a}}. Y={a,b,c}, T₂ = {Y, φ, {a}, {b}, {a,b}}.
Closed sets in T₂ are Y, φ, {b,c}, {a,c}. Pgrw-closed sets in T₁ are X, φ, {b}, {c}, {b,c}.
Define f: X→Y as f(a)=c, f(b)=a, f(c)=b. f is gprw-continuous but f is not pgrw-continuous.
Inverse images are X, φ, {a,b}, {b,c}. Then f is gp-continuous but not pgrw-continuous.

Remark: The following examples show that pgrw-continuous map is independent of gprw-continuous, omega-continuous, beta-continuous, wgrw-continuous, sg-continuous, rw-continuous, wa-continuous, gwα-continuous, rwg-continuous.

Example 3.24: Let X={a,b,c,d}, T₁ = {X, φ, {a}, {b}, {a,b,c}}
Y={a,b,c}, T₂ = {Y, φ, {a}, {b}, {a,b}}.
Define f: X→Y as f(a)=c, f(b)=a, f(c)=b, f(d)=c.
Closed sets in T₂ are Y, φ, {b,c}, {a,c}, {c}.
Pgrw-closed sets in T₁ are X, φ, {c}, {d}, {b,c}, {a,d}, {b,d}, {b,c,d}, {a,c,d}, {a,b,d}.
Inverse images are X, φ, {a,c}, {a,d}, {a,b}. Here f is pgrw-continuous but not gprw-continuous, beta-continuous, gwα-continuous, sg-continuous, rw-continuous, wa-continuous, gwα-continuous.

Example 3.25: Let X={a,b,c,d}, T₁ = {X, φ, {a}, {b}, {a,b,c}}
Y={a,b,c}, T₂ = {Y, φ, {a}}.
Define f(a)=b, f(b)=a, f(c)=a, f(d)=c.
Closed sets in T₂ are Y, φ, {b,c}.
Pgrw-closed sets in T₁ are X, φ, {c}, {d}, {b,c}, {a,d}, {b,d}, {b,c,d}, {a,c,d}, {a,b,d}.
Define f(a)=c, f(b)=a, f(c)=b. Inverse images are X, φ, {a,c}. f is pgrw-continuous but not sgw-continuous.

Example 3.27: Let X={a,b,c,d}, T₁ = {X, φ, {a}, {b}, {a,b,c}}
Y={a,b,c}, T₂ = {Y, φ, {a}}.
Closed sets in T₂ are Y, φ, {b,c}.
Pgrw-closed sets in T₁ are X, φ, {c}, {d}, {b,c}, {a,d}, {b,d}, {b,c,d}, {a,c,d}, {a,b,d}.
Define f(a)=b, f(b)=a, f(c)=c, f(d)=b. Inverse images are X, φ, {a,c,d}. f is pgrw-continuous but not rwg-continuous.

Example 3.28: X={a,b,c}, T₁ = {X, φ, {a}, {b}, {a,b}}. Y={a,b,c}, T₂ = {Y, φ, {a}}. Closed sets in T₂ are Y, φ, {b,c}.
Pgrw-closed sets in T₁ are X, φ, {c}, {a,c}, {b,c}. Define f(a)=b, f(b)=c, f(c)=a.
Inverse images are X, φ, {a,b}. f is not pgrw-continuous but f is beta-continuous, gwα-continuous, sg-continuous, rw-continuous, gwα-continuous, rwg-continuous.

Example 3.29: X={a,b,c}, T₁ = {X, φ, {a}}. Y={a,b,c}, T₂ = {Y, φ, {a}, {b,c}}.
Closed sets in T₂ are Y, φ, {b,c}, {a}. Pgrw-closed sets in T₁ are X, φ, {c}, {b}, {b,c}. Define f(a)=b, f(b)=c, f(c)=a.
Inverse images are X, φ, {b,c}. f is not pgrw-continuous but f is gprw-continuous.

Example 3.30: Let X={a,b,c,d}, T₁ = {X, φ, {a}, {b}, {a,b,c}}
Y={a,b,c}, T₂ = {Y, φ, {a}}.
Closed sets in T₂ are Y, φ, {b,c}.
Pgrw-closed sets in T₁ are X, φ, {c}, {d}, {b,c}, {c,d}, {a,d}, {b,d}, {b,c,d}, {a,c,d}, {a,b,d}.
Define f(a)=b, f(b)=c, f(c)=a, f(d)=a. Inverse images are X, φ, {a,b}. f is not pgrw-continuous but f is wa-continuous.
**Remark 3.31:** From the above discussion and known results we have the following implications.

\[ \begin{align*}
\text{regular continuous} & \quad \text{continuous} \quad \text{completely continuous} \quad \text{gp- continuous} \\
\alpha\text{-continuous} & \quad \alpha r w\text{-continuous} \quad \text{pre-continuous} \\
\#\text{rg –continuous} & \quad \text{wgr}\alpha\text{-continuous} \\
\beta\text{-continuous} & \quad \text{rg}\alpha\text{-continuous} \\
\text{w}\alpha\text{-continuous} & \quad \text{sg-continuous} \\
\pw\text{-continuous} &
\end{align*} \]

A → B means A implies B, but converse is not true.

A ↔ B means A and B are independent of each other.

**Theorem 3.32:** If \( f: (X, T_1) \rightarrow (Y, T_2) \) is a map. Then the following statements hold.

1. If \( f \) is \( gprw \)-continuous and contra continuous map, then \( f \) is \( pgrw \)-continuous.
2. If \( f \) is a \( \alpha \)-continuous and contra- \( \alpha \)- irresolute map, then \( f \) is \( pgrw \)-continuous.
3. If \( f \) is a \( \alpha \)-continuous and contra- \( \alpha \)- irresolute map, then \( f \) is \( pgrw \)-continuous.
4. If \( f \) is a \( \alpha \)-continuous and contra- \( \alpha \)- irresolute map, then \( f \) is \( pgrw \)-continuous.
5. If \( f \) is a \( \omega \)-continuous and contra- \( \omega \)- irresolute map, then \( f \) is \( pgrw \)-continuous.
6. If \( f \) is a \( \alpha \)-continuous and contra- \( \alpha \)- irresolute map, then \( f \) is \( pgrw \)-continuous.
7. If \( f \) is a \( \alpha \)-continuous and contra- \( \alpha \)- irresolute map, then \( f \) is \( pgrw \)-continuous.
8. If \( f \) is a \( pgrw \)-continuous and contra- \( \alpha \)- irresolute map, then \( f \) is \( pgrw \)-continuous.

**Proof:**

1. Let \( V \) be a closed set of \( Y \). Then \( f^{-1}(V) \) is open and \( gp \)-closed in \( X \). (\( f \) is \( gp \)-continuous and contra continuous map).
2. Then \( f^{-1}(V) \) is \( pgrw \)-closed in \( X \). (\( f \) is \( pgrw \)-continuous).
3. Thus \( f \) is \( pgrw \)-continuous.
4. Similarly, we can prove (2), (3), (4), (5), (6), (7), (8).

**Theorem 3.33:** If a map \( f: (X, T_1) \rightarrow (Y, T_2) \) is \( pgrw \)-continuous, then \( f(pgrwcl(A)) \subseteq \text{cl}(f(A)) \) for every subset \( A \) of \( X \).

**Proof:**

If \( f(A) \subseteq \text{cl}(f(A)) \) implies that \( A \subseteq f^{-1}(\text{cl}(f(A))) \). Since \( \text{cl}(f(A)) \) is a closed set in \( Y \) and \( f \) is \( pgrw \)-continuous, \( f^{-1}(\text{cl}(f(A))) \) is a \( pgrw \)-closed set in \( X \) containing \( A \). Hence \( pgrwcl(A) \subseteq f^{-1}(\text{cl}(f(A))) \). Therefore \( f(pgrwcl(A)) \subseteq \text{cl}(f(A)) \).
IV. Perfectly Pgrω–Continuous Map:

Definition 4.1: A function \( f: (X, T_1) \rightarrow (Y, T_2) \) is called a perfectly pre generalized regular \( ωeakly-continuous \) function (briefly perfectly pgrω–continuous) function, if \( f^{-1}(V) \) is a clopen (closed and open) set in \( X \) for every pgrω–open set \( V \) in \( Y \).

Theorem 4.2: If a map \( f: (X, T_1) \rightarrow (Y, T_2) \) is perfectly pgrω–continuous, then
(i) \( f \) is pgrω–continuous.
(ii) \( f \) is gsp–continuous.
(iii) \( f \) is gp–continuous.
(iv) \( f \) is gpω–continuous.
(v) \( f \) is g–continuous.

Proof: (i) Let \( F \) be an open set in \( Y \). Then \( F \) is pgrω–open in \( Y \). Since \( F \) is perfectly pgrω–continuous, \( f^{-1}(F) \) is closed and open in \( X \). Hence \( f \) is pgrω–continuous.
(ii) Let \( F \) be an open set in \( Y \). As every open set is pgrω–open in \( Y \) and if is perfectly pgrω–continuous and so \( f^{-1}(F) \) is both closed and open in \( X \), as every open set is pgrω-open that implies gsp–open. Then \( f^{-1}(F) \) is gsp–open in \( X \). Hence \( f \) is gpω–continuous.

Similarly, we can prove (iii), (iv) and (v).

Theorem 4.3: \((X, τ)\) is a discrete topological space and \((Y, σ)\) is any topological space. Then every function \( f: (X, τ) \rightarrow (Y, σ) \) is perfectly pgrω–continuous.

Proof: Let \( U \) be a pgrω–open set in \( Y \). Since \((X, τ)\) is a discrete space \( f^{-1}(U) \) is both open and closed in \((X, τ)\). Hence \( f \) is perfectly pgrω–continuous.

Theorem 4.4: If \( f: (X, T_1) \rightarrow (Y, T_2) \) is a strongly continuous map, then it is perfectly pgrw–continuous.

Proof: Let \( V \) be a pgrw–open set in \( Y \). As \( f \) is strongly continuous and \( V \) is a subset of \( Y \), \( f^{-1}(V) \) is clopen in \( X \). So \( f \) is perfectly pgrw–continuous.

V. Pgrω*–Continuos Map

Definition 5.1: A function \( f: (X, T_1) \rightarrow (Y, T_2) \) is called a pre generalized regular \( ωeakly*-continuous \) function (pgrω*–continuous function) if \( f^{-1}(F) \) is a pgrω–closed set in \( X \) for every pre closed set \( V \) in \( Y \).

Theorem 5.2: If a map \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrω*–continuous, then it is pgrω–continuous.

Proof: \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrω*–continuous. Let \( F \) be any closed set in \( Y \). Then \( F \) is pre closed in \( Y \). Since \( f \) is pgrω*–continuous, the inverse image \( f^{-1}(F) \) is pgrω–closed in \( X \). Therefore \( f \) is pgrω–continuous.

VI. Pgrω–Irresolute map

Definition 6.1: A map \( f: (X, T_1) \rightarrow (Y, T_2) \) is called a pre generalized regular \( ωeakly-irresolute \) (pgrω–irresolute) map if \( f^{-1}(F) \) is a pgrω–closed set in \( X \) for every pgrω–closed set \( V \) in \( Y \).

Theorem 6.2: A map \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrω–irresolute if and only if the inverse image \( f^{-1}(V) \) is pgrω–open in \( X \) for every pgrω–open set \( V \) in \( Y \).

Proof: Assume \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrω–irresolute. Let \( G \) be a pgrω–open set in \( Y \). Then \( G^c \) is pgrω–closed in \( Y \). Since \( f \) is pgrω–irresolute, \( f^{-1}(G^c) = f^{-1}(G^c) \cap X = f^{-1}(G) \) is pgrω–closed in \( X \). But \( f^{-1}(G) = X \)–\( f^{-1}(G) \). Therefore \( f \) is pgrω–irresolute.

Theorem 6.3: Every perfectly pgrw–continuous map is pgrw–irresolute.

Proof: Let \( f: (X, T_1) \rightarrow (Y, T_2) \) be a perfectly pgrw–continuous map. Let \( V \) be a pgrw–open set in \( Y \). Then \( f^{-1}(V) \) is clopen in \( X \) and so \( f^{-1}(V) \) is open. As every open set is pgrw–open, \( f^{-1}(V) \) is pgrw–open. Therefore \( f \) is pgrw–irresolute.

Theorem 6.4: If a map \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrω–irresolute, then it is pgrω*–continuous.

Proof: \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrω–irresolute. Let \( F \) be any pre closed set in \( Y \). Then \( F \) is pgrω–closed in \( Y \). Since \( f \) is pgrω–irresolute, \( f^{-1}(F) \) is pgrω–closed in \( X \). Therefore \( f \) is pgrω*–continuous.

Theorem 6.5: If a map \( f: (X, T_1) \rightarrow (Y, T_2) \) is pgrω–irresolute, then it is pgrω–continuous.
Proof: $f: (X, T_1) \rightarrow (Y, T_2)$ is a pgr–irresolute map. Let $F$ be any closed set in $Y$. Then $F$ is pgr–closed in $Y$. Since $f$ is pgr–irresolute, the inverse image $f^{-1}(F)$ is pgr–closed in $X$. Therefore $f$ is pgr–continuous.

**Theorem 6.6:** If a map $f: (X, T_1) \rightarrow (Y, T_2)$ is pgr–irresolute, then for every subset $A$ of $X$, \( f(pgrocl(A)) \subseteq pcl(f(A)) \).

**Proof:** \( A \subseteq X \). Then $pcl(f(A))$ is pgr–closed in $Y$. Since $f$ is pgr–irresolute, $f^{-1}(pcl(f(A)))$ is pgr–closed in $X$. Further $A \subseteq f^{-1}(F(\alpha) \subseteq f^{-1}(pcl(f(A)))$. Therefore by definition of pgr–closure $pgrocl(A) \subseteq f^{-1}(pcl(f(A)))$, consequently $f(pgrocl(A)) \subseteq pcl(f(A))$.

**Theorem 7.6:** If a map $f: (X, T_1) \rightarrow (Y, T_2)$ is pgr–irresolute, then for every subset $A$ of $X$, \( f(pgrocl(A)) \subseteq pcl(f(A)) \).

**Proof:** \( A \subseteq X \). Then $pcl(f(A))$ is pgr–closed in $Y$. Since $f$ is pgr–irresolute, $f^{-1}(pcl(f(A)))$ is pgr–closed in $X$. Further $A \subseteq f^{-1}(F(\alpha) \subseteq f^{-1}(pcl(f(A)))$. Therefore by definition of pgr–closure $pgrocl(A) \subseteq f^{-1}(pcl(f(A)))$, consequently $f(pgrocl(A)) \subseteq pcl(f(A))$.

**Theorem 6.7:** If $f: (X, T_1) \rightarrow (Y, T_2)$ and $g: (Y, T_1) \rightarrow (Z, T_2)$ are maps, then $g \circ f: (X, T_1) \rightarrow (Z, T_2)$ is a strongly pgr–omega–continuous function.

**Proof:** (i) Let $U$ be an open set in $(Z, T_2)$. Since $g$ is r–continuous, $g^{-1}(U)$ is r–open in $(Y, T_1)$. Therefore $g^{-1}(U)$ is an open set in $(Y, T_1)$. Since $f$ is pgr–irresolute, $f^{-1}(g^{-1}(U))$ is an open set in $(X, T_1)$. Hence $(g \circ f)^{-1}(U)$ is r–open in $(X, T_1)$.

**VII. Strongly Pgr–Continuous Map:**

**Definition 7.1:** A map $f: (X, T_1) \rightarrow (Y, T_2)$ is called a strongly pre generalized regular weakly–continuous (strongly pgr–continuous) map if $f^{-1}(V)$ is a pgr–closed set in $X$ for every pgr–closed set $V$ in $Y$.

**Theorem 7.2:** A map $f: (X, T_1) \rightarrow (Y, T_2)$ is strongly pgr–continuous if and only if $f^{-1}(G)$ is an open set in $X$ for every pgr–open set $G$ in $Y$.

**Proof:** $f: (X, T_1) \rightarrow (Y, T_2)$ is strongly pgr–continuous. Let $G$ be pgr–open in $Y$. The $G^c$ is pgr–closed in $Y$. Since $f$ is strongly pgr–continuous, $f^{-1}(G^c)$ is closed in $X$. But $f^{-1}(G) = X - f^{-1}(G^c)$. Therefore $f^{-1}(G)$ is open in $X$.

Assume that the inverse image of every pgr–open set in $Y$ is open in $X$. Let $F$ be any pgr–closed set in $Y$. Then $F^c$ is pgr–open in $Y$. Therefore $f^{-1}(F^c)$ is open in $X$. But $f^{-1}(F) = X - f^{-1}(F^c)$. Therefore $f^{-1}(F)$ is open in $X$ and so $f^{-1}(F)$ is closed in $X$. Therefore $f$ is strongly pgr–continuous.

**Theorem 7.3:** If a map $f: (X, T_1) \rightarrow (Y, T_2)$ is strongly pgr–continuous and $A$ is an open subset of $X$, then the restriction $f|_A: (A, T_2) \rightarrow (Y, T_2)$ is strongly pgr–continuous.

**Proof:** Let $V$ be any pgr–open set of $Y$. Since $f$ is strongly pgr–continuous, $f^{-1}(V)$ is open in $X$. Since $A$ is open in $X$, $(f|_A)^{-1}(V) = A \cap f^{-1}(V)$ is open in $A$. Hence $f|_A$ is strongly pgr–continuous.

**Theorem 7.4:** If a map $f: (X, T_1) \rightarrow (Y, T_2)$ is strongly pgr–continuous, then it is continuous.

**Proof:** Assume that $f: (X, T_1) \rightarrow (Y, T_2)$ is strongly pgr–continuous. Let $F$ be a closed set in $Y$. As every closed set is pgr–closed, $F$ is pgr–closed in $Y$. Since $f$ is strongly pgr–continuous, $f^{-1}(F)$ is closed in $X$. Therefore $f$ is continuous.

**Theorem 7.5:** If a map $f: (X, T_1) \rightarrow (Y, T_2)$ is strongly pgr–continuous, then it is pgr–irresolute.

**Proof:** If $f$ is strongly pgr–continuous map. Then $F$ is a pgr–closed set in $Y$. Then $f^{-1}(F)$ is closed in $X$. Therefore $f$ is continuous.

**Theorem 7.6:** Every perfectly pgr–continuous map is strongly pgr–continuous.
Proof: Let \( f: (X, T_1) \rightarrow (Y, T_2) \) be a perfectly pgr-continuous map. Let \( U \) be a pgro-open set in \( Y \). As \( f \) is perfectly pgr-continuous \( f^{-1}(U) \) is both open and closed in \( (X, \tau) \). \( f^{-1}(U) \) is open in \( (X, \tau) \). Hence \( f \) is strongly pgro-continuous.

Theorem 7.7: If a map \( f: (X, T_1) \rightarrow (Y, T_2) \) is strongly continuous, then it is strongly pgro-continuous.

Proof: Let \( f: (X, T_1) \rightarrow (Y, T_2) \) be strongly continuous. Let \( G \) be pgro-open in \( Y \). As \( f \) is strongly continuous and \( G \) is a subset of \( Y \), \( f^{-1}(G) \) is clopen in \( X \) and so open in \( X \). Therefore \( f \) is strongly pgro-continuous.

Theorem 7.8: For all discrete spaces \( X \) and \( Y \), if a map \( f: (X, T_1) \rightarrow (Y, T_2) \) is strongly pgro-continuous, then it is strongly continuous.

Proof: Let \( F \) be a subset of \( Y \). As \( Y \) is a discrete space, \( F \) is clopen.

\[
\Rightarrow \{ \begin{array}{l}
F \text{ is open} \Rightarrow f^{-1}(F) \text{ is open.} \\
F \text{ is closed} \Rightarrow f^{-1}(F) \text{ is closed.}
\end{array} \}
\]

\( \Rightarrow f^{-1}(F) \) is clopen. Hence \( f \) is strongly continuous.

Theorem 7.9: If a map \( f: (X, T_1) \rightarrow (Y, T_2) \) is strongly pgro-continuous, then it is pgro-continuous.

Proof: Let \( G \) be an open set in \( Y \). As every open set is pgro-open, \( G \) is pgro-open in \( Y \). Since \( f \) is strongly pgro-continuous, \( f^{-1}(G) \) is open in \( X \). As every open set is pgro-open, \( f^{-1}(G) \) is pgro-open in \( X \). Hence \( f \) is pgro-continuous.

Theorem 7.10: \( f: (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) are two functions.

(i) If \( f \) and \( g \) are strongly pgro-continuous, then \( g \circ f: (X, \tau) \rightarrow (Z, \eta) \) is strongly pgro-continuous.

(ii) If \( f \) is continuous and \( g \) is strongly pgro-continuous, then \( g \circ f: (X, \tau) \rightarrow (Z, \eta) \) is strongly pgro-continuous.

(iii) If \( f \) is pgro-continuous and \( g \) is strongly pgro-continuous, then \( g \circ f: (X, \tau) \rightarrow (Z, \eta) \) is pgro-irresolute.

(iv) If \( f \) is strongly pgro-continuous and \( g \) is pgro-continuous, then \( g \circ f: (X, \tau) \rightarrow (Z, \eta) \) is pgro-irresolute.

Proof: (i) Let \( U \) be a pgro-open set in \( (Z, \eta) \). Since \( g \) is strongly pgro-continuous, \( g^{-1}(U) \) is open in \( (Y, \sigma) \). As every open set is pgro-open, \( g^{-1}(U) \) is pgro-open in \( (Y, \sigma) \). Hence \( f \) is strongly pgro-continuous.

(ii) Let \( U \) be a pgro-open set in \( (Z, \eta) \). Since \( g \) is strongly pgro-continuous, \( g^{-1}(U) \) is open in \( (Y, \sigma) \). Since \( f \) is continuous, \( f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \). Thus \( (gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \) and hence \( gof \) is strongly pgro-continuous.

(iii) Let \( U \) be a pgro-open set in \( (Z, \eta) \). Since \( g \) is pgro-continuous, \( g^{-1}(U) \) is a pgro-open set in \( (Y, \sigma) \). Thus \( (gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is a pgro-open set in \( (X, \tau) \) and hence \( gof \) is pgro-irresolute.

(iv) Let \( U \) be an open set in \( (Z, \eta) \). Since \( g \) is pgro-continuous, \( g^{-1}(U) \) is a pgro-open set in \( (Y, \sigma) \). Since \( f \) is strongly pgro-continuous, \( f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \). Thus \( (gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \) and hence \( gof \) is continuous.

Theorem 7.11: \( f: (X, \tau) \rightarrow (Y, \sigma) \) and \( g: (Y, \sigma) \rightarrow (Z, \eta) \) are two functions.

1. If \( f \) is continuous and \( g \) is perfectly pgro-continuous, then \( gof: (X, \tau) \rightarrow (Z, \eta) \) is strongly pgro-continuous.
2. If \( f \) is perfectly pgro-continuous and \( g \) is strongly pgro-continuous, then \( gof: (X, \tau) \rightarrow (Z, \eta) \) is perfectly pgro-continuous.

Proof: (1) Let \( U \) be a pgro-open set in \( (Z, \eta) \). Since \( g \) is perfectly pgro-continuous, \( g^{-1}(U) \) is clopen in \( (Y, \sigma) \). Since \( f \) is continuous, \( f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \). Thus \( (gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \) and hence \( gof \) is pgro-irresolute.

2. Let \( U \) be a pgro-open set in \( (Z, \eta) \). Since \( g \) is strongly pgro-continuous, \( g^{-1}(U) \) is an open set in \( (Y, \sigma) \) and so pgro-open. Since \( f \) is perfectly pgro-continuous, \( f^{-1}(g^{-1}(U)) \) is a clopen set in \( (X, \tau) \). Thus \( (gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is an open set in \( (X, \tau) \) and hence \( gof \) is perfectly pgro-continuous.

3. Let \( U \) be a pgro-open set in \( Z \). As \( g \) is perfectly pgro-continuous, \( g^{-1}(U) \) is clopen in \( Y \) and so open. As every open set is pgro-open, \( g^{-1}(U) \) is pgro-open in \( Y \). As \( f \) is perfectly pgro-continuous, \( f^{-1}(g^{-1}(U)) \) is clopen in \( X \). Hence \( (gof)^{-1}(U) = f^{-1}(g^{-1}(U)) \) is clopen in \( X \). Hence \( gof \) is perfectly pgro-continuous.
The following diagram shows the relation between above discussed maps.

![Diagram](image-url)

References


K. Kannanand K. Chandrasekhara Rao Pasting Lemmas for Some Continuous Functions ThaiJournal


S. S. Benchalli and R.S Wali on rω- Closed sets is Topological Spaces, Bull, Malays, Math, sci, soc30 (2007), 99-110
