On Cubic Implicative Hyper BCK-Ideals of Hyper BCK-Algebras

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Abstract: In this paper, we present the notions of cubic (weak) implicative hyper BCK-ideals of hyper BCK-algebras and then we present some results which characterize the above notions according to the level subsets. In addition, we obtain the relationship among these notions, cubic positive implicative hyper BCK-ideals of types-1, 2...8 and obtain some related results. AMS (2010): 06F35.

Key words: Hyper BCK-algebras, cubic sets, cubic (weak) implicative hyper BCK-ideal.

I. Introduction

BCI/BCK-algebras are generalizations of the concepts of set-theoretic difference and propositional calculi. These two classes of logical algebras were introduced by Imai and Iseki [11] in 1966. It is well known that the class of MV-algebras introduced by Chang in [7] is a proper subclass of the class of BCK-algebras which in turn is a proper subclass of the class of BCI-algebras. Since the introduction of BCI/BCK-algebras, a great deal of literature has been produced, for example see [8, 10, 12, 13, 14]. For the general development of BCI/BCK-algebras, the ideal theory plays an important role. Hence much research emphasis has been on the ideal theory of BCI/BCK-algebras, see [1, 2, 3, 9].

The hyper structure theory was introduced in 1934 by Marty [24] at the 8th Science Congress of Scandinavian Mathematicians. In the following years, several authors have worked on this subject notably in France, United States, Japan, Spain, Russia and Italy. Hyperstructures have many applications in several sectors of both pure and applied sciences. In [9], Jun et. al. applied hyperstructures to BCK-algebras and introduced the notion of hyper BCK-algebras which is a generalization of a BCK-algebra. After the introduction of the concept of fuzzy sets by Zadeh [28], several researchers have carried out the generalizations of fuzzy sets. The idea of cubic set was introduced by Jun et. al. [21], which contains interval valued membership function and non degree membership function. In [15], Jun et. al. introduced the cubic hyper BCK-ideals and obtained some related results. In [27], Satyanarayana and Bindu introduced the concept of cubic hyper BCK-ideals of hyper BCK-algebras. In the present paper, we introduce the concepts of cubic (weak) implicative hyper BCK-ideals of hyper BCK-algebras and we obtain some results which characterize the concept according to level subsets. Also, we obtain the relationship among these concepts, cubic (strong, weak, reflexive) hyper BCK-ideals, cubic positive implicative hyper BCK-ideals of types -1,2,...,8 and some related properties are investigated.

II. Preliminaries

Let H be a non-empty set endowed with hyper operation that is, ° is a function from H × H to P°(H) = P(H)\{∅}. For any two subsets A and B of H, A ° B is denoted by ∪a∈A,b∈B a ° b. We shall use the x ° y instead of x ° {y}, {x} ° y or {x} ° {y}.

Definition 2.1. By a hyper BCK-algebra, we mean a nonempty set H endowed with a hyper operation ° and a constant 0 satisfying the following axioms:
(HK - 1) (x ° z) ° (y ° z) ≪ x ° y,
(HK - 2) (x ° y) ° z = (x ° z) ° y,
(HK - 3) x ° H ≪ {H},
(HK - 4) x ≪ y and y ≪ x implies that x = y for all x,y,z ∈ H.

We can define a relation ≪ on H by letting x ≪ y if and only if 0 ∉ x ° y and for every A,B ⊆ H, A ≪ B is defined for all a ∈ A there exists b ∈ B such that a ≪ b. In such case, we call the relation ≪ the hyper order in H. In the sequel, H will denote hyper BCK-algebra. It should be noted that the condition (HK-3) is equivalent to the condition:
(P1) x ° y ≪ {x}, for all x,y ∈ H.
In any hyper BCK-algebra H, the following hold:
(P2) 0 ° 0 = {0},
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(P3) $0 \ll x$,
(P4) $x \ll x$,
(P5) $A \subseteq B$ implies $A \ll B$,
(P6) $0 \star x = \{0\}$,
(P7) $x \star 0 = \{x\}$,
(P8) $0 \star A = \{0\}$,
(P9) $x \in x \star 0$, for all $x, y, z \in H$ and for any nonempty subsets $A$ and $B$ of $H$.

**Definition 2.2.** Let $I$ be a nonempty subset of hyper BCK-algebra $H$ and $0 \in I$. Then
(I-1) $I$ is called a hyper BCK-subalgebra of $H$ if $x \star y \subseteq I$ for all $x, y \in H$.
(I-2) $I$ is called a weak hyper BCK-ideal of $H$, if $x \star y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.
(I-3) $I$ is called a hyper BCK-ideal of $H$, if $x \star y \subseteq I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.
(I-4) $I$ is called a strong hyper BCK-ideal of $H$, if $x \star y \cap I \neq \phi$ and $y \in I$ imply $x \in I$, for all $x, y \in H$.
(I-5) $I$ is said to be reflexive if $x \star x \subseteq I$, for all $x \in H$.
(I-6) $I$ is said to be $S$-reflexive, if $(x \star y) \cap I \neq \phi \Rightarrow x \star y \subseteq I$ for all $x, y \in H$.
(I-7) $I$ is said to be closed if $x \ll I$ and $y \in I$ implies that $x \in I$ for all $x, y \in H$.

It is easy to see that every $S$-reflexive subset of $H$ is reflexive.

**Definition 2.3.** A hyper BCK-algebra $H$ is called a positive implicative hyper BCK-algebra if for all $x, y, z \in H$, we have $(x \star y) \cap z = (x \star z) \circ (y \star z)$.

If $I$ is a nonempty subset of $H$ and $0 \in I$, then $I$ is called a weak implicative hyper BCK-ideal of $H$ if $(x \star z) \circ (y \star x) \subseteq I$ and $z \in I$ imply that $x \in I$ for all $x, y, z \in H$.

I is said to be an implicative hyper BCK-ideal of $H$, if $(x \star z) \circ (y \star x) \subseteq I$ and $z \in I$ imply that $x \in I$ for all $x, y, z \in H$.

**Definition 2.4.** [25] Let $H$ be a hyper BCK-algebra. A nonempty subset $I$ of $H$ is called a strong implicative hyper BCK-ideal if it satisfies:
(i) $0 \in I$,
(ii) $(x \star z) \circ (y \star x) \subseteq I$ and $z \in I$ imply that $x \in I$ for all $x, y, z \in H$.

**Example 1[25]** Let $H = \{0, 1, 2\}$. Consider the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0}</td>
<td>{1}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{2}</td>
<td>{0,2}</td>
</tr>
</tbody>
</table>

It is clear that $(H, \sigma)$ is a Hyper BCK-algebra. Putting $I = \{0, 2\}$, it can seen that $I$ is a strong implicative Hyper BCK-ideal of $H$.

A fuzzy set in set $X$ is a function $\mu : X \rightarrow [0, 1]$. Let $\mu$ and $\lambda$ be the fuzzy sets of $X$. For $s, t \in [0, 1]$ the set $U(\mu, \lambda) = \{x \in X / \mu(x) \geq s \}$ is called upper $s$-level cut of $\mu$ and set $L(\mu, \lambda) = \{x \in X / \mu(x) \leq t \}$ is called lower $t$-level cut of $\lambda$.

Let $\mu$ be a fuzzy subset of $H$ and $\mu(0) \geq \mu(x)$, for all $x \in H$. Then $\mu$ is called:
(i) a fuzzy weak implicative hyper BCK-ideal of $H$ if $\mu(x) \geq \min\{\inf_{d \in (x \star z) \circ (y \star x)} \mu(d), \mu(z)\}$
(ii) a fuzzy implicative hyper BCK-ideal of $H$ if $\mu(x) \geq \min\{\sup_{d \in (x \star z) \circ (y \star x)} \mu(d), \mu(z)\}$ for all $x, y, z \in H$.

The determination of maximum and minimum between two real numbers is very simple but it is not simple for two intervals. In [5] Biswas described a method to find max/sup and min/inf between two intervals or set of intervals. By the interval number $D$ we mean an interval $[a^-, a^+]$ where $0 \leq a^- \leq a^+ \leq 1$. For interval numbers $D_1 = [a_1^-, b_1^+]$, $D_2 = [a_2^-, b_2^+]$.

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We define the following
\[
\min (D_1, D_2) = D_1 \cap D_2 = \min ([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\min (a_1^-, a_2^-), \min (b_1^+, b_2^+)].
\]
\[
\max (D_1, D_2) = D_1 \cup D_2 = \max ([a_1^-, b_1^+], [a_2^-, b_2^+]) = [\max (a_1^-, a_2^-), \max (b_1^+, b_2^+)].
\]
\[
D_1 + D_2 = [a_1^- + a_2^-, a_1^+ + a_2^+, b_1^- + b_2^-, b_1^+ + b_2^+].
\]
We put
\[
D_1 \leq D_2 \iff a_1^- \leq a_2^- \text{ and } b_1^+ \leq b_2^+.
\]
\[
D_1 = D_2 \iff a_1^- = a_2^- \text{ and } b_1^+ = b_2^+.
\]
\[
D_1 < D_2 \iff D_1 \leq D_2 \text{ and } D_1 \neq D_2.
\]
\[
mD = m[a_1^-, b_1^+] = [m(a_1^-), m(b_1^+)], \text{ where } 0 \leq m \leq 1.
\]

An interval valued fuzzy set \(A\) over \(X\) is an object having the form \(A = \{(x, \bar{\mu}_A(x)) : x \in X\}\) where \(\bar{\mu}_A(x) : X \to \mathbb{D}[0,1]\) is the set of all subintervals of \([0,1]\). The interval \(\bar{\mu}_A(x)\) denotes the intervals of the degree of membership of element \(x\) to the set \(A\) where \(\bar{\mu}_A(x) = [\mu_A(x), \mu_A^+(x)]\) for all \(x \in X\).

**Definition 2.5.** [5] Consider two elements \(D_1, D_2 \in \mathbb{D}[0,1]\). If \(D_1 = [a_1^-, a_1^+]\) and \(D_2 = [a_2^-, a_2^+]\) then \(r\min (D_1, D_2) = [\min (a_1^-, a_2^-), \min (a_1^+, a_2^+)]\) which is denoted by \(D_1 \wedge D_2\). Thus if \(D_i = [a_i^-, a_i^+] \in \mathbb{D}[0,1]\) for \(i = 1, 2, 3, \ldots\) then we define \(r\sup_i(D_i) = [\sup_i a_i^-, \sup_i a_i^+]\) that is \(V^i D_i = [V_i a_i^- V_i a_i^+]\). Now we call \(D_1 \geq D_2\) if and only if \(a_1^- \geq a_2^-\) and \(a_1^+ \geq a_2^+\). Similarly, the relations \(D_1 \leq D_2\) and \(D_1 = D_2\) are defined.

Based on interval valued fuzzy sets, Jun et.al.[21] introduced the notion of (internal, external) cubic sets and investigated several properties.

**Definition 2.6.** [27] Let \(X\) be a nonempty set. A cubic set \(A\) in \(X\) is a Structure \(A = \{(x, \bar{\mu}_A(x), \lambda_A(x)) : x \in X\}\) which is briefly denoted by \(A = (\bar{\mu}_A, \lambda_A)\) where \(\bar{\mu}_A = [\mu^L_A, \mu^U_A]\) is an interval valued fuzzy set in \(X\) and \(\lambda_A\) is fuzzy set in \(X\).

**Definition 2.7.** [27] A cubic set \(A = (\bar{\mu}_A, \lambda_A)\) in \(H\) is called a cubic hyper BCK-ideal of \(H\) if it satisfies the following conditions:

(K1) \(x \ll y\) implies that \(\bar{\mu}_A(x) \geq \bar{\mu}_A(y)\) and \(\lambda_A(x) \leq \lambda_A(y)\).

(K2) \(\bar{\mu}_A(x) \geq \min \{\sup_{b \in \mathbb{D}[0,1]} \bar{\mu}_A(b), \bar{\mu}_A(y)\}\).

(K3) \(\lambda_A(x) \leq \max \{\sup_{b \in \mathbb{D}[0,1]} \lambda_A(b) \lambda_A(y)\}\), for all \(x, y \in H\).

**Definition 2.8.** [27] A cubic set \(A = (\bar{\mu}_A, \lambda_A)\) in \(H\) is called a cubic strong hyper BCK-ideal of \(H\) if it satisfies the following conditions:

(SH1) \(\inf_{a \in \mathbb{D}[0,1]} \bar{\mu}_A(a) \geq \min \{\sup_{b \in \mathbb{D}[0,1]} \bar{\mu}_A(b), \bar{\mu}_A(y)\}\) and

(SH2) \(\sup_{c \in \mathbb{D}[0,1]} \lambda_A(c) \leq \max \{\inf_{d \in \mathbb{D}[0,1]} \lambda_A(d), \lambda_A(y)\}\), for all \(x, y \in H\).

**Definition 2.9.** [27] A cubic set \(A = (\bar{\mu}_A, \lambda_A)\) in \(H\) is called a cubic s-weak hyper BCK-ideal of \(H\) if it satisfies the following conditions:

(S1) \(\bar{\mu}_A(0) \geq \bar{\mu}_A(y)\) and \(\lambda_A(0) \leq \lambda_A(y)\), for all \(x, y \in H\).

(S2) for every \(x, y \in H\) there exists \(a, b \in x \ast y\), \(\bar{\mu}_A(a) \geq \min \{\bar{\mu}_A(a), \bar{\mu}_A(y)\}\) and \(\lambda_A(x) \leq \max \{\lambda_A(b), \lambda_A(y)\}\).

**Definition 2.10.** [27] A cubic set \(A = (\bar{\mu}_A, \lambda_A)\) in \(H\) is called a cubic weak hyper BCK-ideal of \(H\) if it satisfies the following conditions:

(W1) \(\bar{\mu}_A(0) \geq \bar{\mu}_A(y) \geq \min \{\inf_{a \in \mathbb{D}[0,1]} \bar{\mu}_A(a), \bar{\mu}_A(y)\}\) and

(W2) \(\lambda_A(0) \leq \lambda_A(x) \leq \min \{\sup_{b \in \mathbb{D}[0,1]} \lambda_A(b), \lambda_A(y)\}\), for all \(x, y \in H\).

**Definition 2.11.** [27] A cubic set \(A = (\bar{\mu}_A, \lambda_A)\) in \(H\) is said to satisfy the “sup-inf” property if for any subset \(T\) of \(H\) there exist \(x_T, y_T \in T\) such that \(\bar{\mu}_A(x_T) = \sup_{c \in T} \bar{\mu}_A(c)\) and \(\lambda_A(y_T) = \inf_{\gamma \in T} \lambda_A(\gamma)\).

**Theorem 2.12.** [27] Every cubic strong hyper BCK-ideal is a cubic hyper BCK-ideal.
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Definition 3.1. Let \( A = (\bar{\mu}_A,\lambda_A) \) be a cubic subset of \( H \) and \( \bar{\mu}_A(0) \geq \bar{\mu}_A(x), \lambda_A(0) \leq \lambda_A(x) \) for all \( x, y \in H \). Then \( A = (\bar{\mu}_A,\lambda_A) \) is called a cubic positive implicative hyper BCK-ideal of:

(i) Type 1, if for all \( t \in x \ast z \),

\[
\bar{\mu}_A(t) \geq \min \left\{ \inf_{a \in (x \ast z)} \bar{\mu}_A(a), \inf_{b \in y \ast z} \bar{\mu}_A(b) \right\}, \quad \lambda_A(t) \leq \max \{ \sup_{c \in (x \ast y) \ast z} \lambda_A(c), \sup_{d \in y \ast z} \lambda_A(d) \}
\]

(ii) Type 2, if for all \( t \in x \ast z \),

\[
\bar{\mu}_A(t) \geq \min \left\{ \sup_{a \in (x \ast z)} \bar{\mu}_A(a), \inf_{b \in y \ast z} \bar{\mu}_A(b) \right\}, \quad \lambda_A(t) \leq \max \{ \inf_{c \in (x \ast y) \ast z} \lambda_A(c), \inf_{d \in y \ast z} \lambda_A(d) \}
\]

(iii) Type 3, if for all \( t \in x \ast z \),

\[
\bar{\mu}_A(t) \geq \min \{ \sup_{a \in (x \ast y) \ast z} \bar{\mu}_A(a), \sup_{b \in y \ast z} \bar{\mu}_A(b) \}
\]

\[
\lambda_A(t) \leq \max \{ \sup_{c \in (x \ast y) \ast z} \lambda_A(c), \sup_{d \in y \ast z} \lambda_A(d) \}
\]

(iv) Type 4, if for all \( t \in x \ast z \),

\[
\bar{\mu}_A(t) \geq \min \{ \inf_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a), \sup_{b \in y \ast z} \bar{\mu}_A(b) \}
\]

\[
\lambda_A(t) \leq \max \{ \inf_{c \in (x \ast (y \ast z))} \lambda_A(c), \inf_{d \in y \ast z} \lambda_A(d) \}
\]

for all \( x, y, z \in H \).

Definition 3.2. Let \( A = (\bar{\mu}_A,\bar{\lambda}_A) \) be a cubic subset of \( H \). Then \( A = (\bar{\mu}_A,\bar{\lambda}_A) \) is called a cubic positive implicative hyper BCK-ideal of:

(i) Type 5, if there exists \( t \in x \ast z \) such that

\[
\bar{\mu}_A(t) \geq \min \{ \inf_{a \in (x \ast y) \ast z} \bar{\mu}_A(a), \inf_{b \in (y \ast z)} \bar{\mu}_A(b) \}\lambda_A(t) \leq \max \{ \sup_{c \in (x \ast y) \ast z} \lambda_A(c), \sup_{d \in (y \ast z)} \lambda_A(d) \}
\]

(ii) Type 6, if there exists \( t \in x \ast z \) such that

\[
\bar{\mu}_A(t) \geq \min \{ \sup_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a), \sup_{b \in (y \ast z)} \bar{\mu}_A(b) \}\lambda_A(t) \leq \max \{ \inf_{c \in (x \ast (y \ast z))} \lambda_A(c), \inf_{d \in (y \ast z)} \lambda_A(d) \}
\]

(iii) Type 7, if there exists \( t \in x \ast z \) such that

\[
\bar{\mu}_A(t) \geq \min \{ \inf_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a), \sup_{b \in (y \ast z)} \bar{\mu}_A(b) \}\lambda_A(t) \leq \max \{ \sup_{c \in (x \ast (y \ast z))} \lambda_A(c), \sup_{d \in (y \ast z)} \lambda_A(d) \}
\]

(iv) Type 8, if there exists \( t \in x \ast z \) such that

\[
\bar{\mu}_A(t) \geq \min \{ \inf_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a), \inf_{b \in (y \ast z)} \bar{\mu}_A(b) \}\lambda_A(t) \leq \max \{ \inf_{c \in (x \ast (y \ast z))} \lambda_A(c), \inf_{d \in (y \ast z)} \lambda_A(d) \}
\]

for all \( x, y, z \in H \).

Definition 3.3. Let \( A = (\bar{\mu}_A,\lambda_A) \) be a cubic subset of \( H \) and \( \bar{\mu}_A(0) \geq \bar{\mu}_A(x), \lambda_A(0) \leq \lambda_A(x) \) for all \( x, y, z \in H \). Then \( A = (\bar{\mu}_A,\lambda_A) \) is called:

(i) a cubic weak implicative hyper BCK-ideal of \( H \), if \( \bar{\mu}_A(x) \geq \min \{ \inf_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a), \bar{\mu}_A(z) \} \lambda_A(x) \leq \max \{ \sup_{b \in (x \ast (y \ast z))} \lambda_A(b), \lambda_A(z) \} \)

(ii) a cubic implicative hyper BCK-ideal of \( H \), if \( \bar{\mu}_A(x) \geq \min \{ \inf_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a), \bar{\mu}_A(z) \} \lambda_A(x) \leq \max \{ \inf_{b \in (x \ast (y \ast z))} \lambda_A(b), \lambda_A(z) \} \)

for all \( x, y, z \in H \).

Theorem 3.4 Every cubic implicative hyper BCK-ideal of \( H \) is a cubic weak implicative hyper BCK-ideal.

Proof: Suppose \( A = (\bar{\mu}_A,\lambda_A) \) is a cubic implicative hyper BCK-ideal of \( H \). Then for all \( x, y, z \in H \). We have
\[
\inf_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a) \leq \sup_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a)
\]
and also,
\[
\inf_{b \in (x \ast (y \ast z))} \lambda_A(b) \leq \sup_{b \in (x \ast (y \ast z))} \lambda_A(b).
\]
Hence,
\[
\bar{\mu}_A(x) \geq \min \{ \sup_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a), \bar{\mu}_A(z) \}
\]
\[
\geq \min \{ \inf_{a \in (x \ast (y \ast z))} \bar{\mu}_A(a), \bar{\mu}_A(z) \}.
\]
\[
\lambda_A(x) \leq \max \{ \inf_{b \in (x \ast (y \ast z))} \lambda_A(b), \lambda_A(z) \}.
\]
Accordingly, \( A = (\bar{\mu}_A,\lambda_A) \) is a cubic weak implicative hyper BCK-ideal of \( H \).
Example 2. Let \( H = \{0, a, b\} \). Consider the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[0]</td>
<td>[0]</td>
<td>[0]</td>
</tr>
<tr>
<td>a</td>
<td>[a]</td>
<td>[0,a]</td>
<td>[0,a]</td>
</tr>
<tr>
<td>b</td>
<td>[b]</td>
<td>[a]</td>
<td>[0,a]</td>
</tr>
</tbody>
</table>

It follows from [4] that \((H, \circ)\) is a hyper BCK-algebra.

Define a cubic set \( A = (\bar{\mu}_A, \bar{\lambda}_A) \) on \( H \) by \( \bar{\mu}_A(0) = [1, 0.8] \), \( \bar{\mu}_A(a) = [0, 0.2] \), \( \bar{\mu}_A(b) = [0.6, 0.7] \) and \( \bar{\lambda}_A(0) = 0, \bar{\lambda}_A(a) = 1, \bar{\lambda}_A(b) = 0.4 \). Then \( A = (\bar{\mu}_A, \bar{\lambda}_A) \) is a cubic weak implicative hyper BCK-ideal of \( H \) but it is not a cubic implicative hyper BCK-ideal of \( H \), since

\[
\bar{\mu}_A(a) = [0, 0.2] < [1, 0.8] = \bar{\mu}_A(0) = \text{rmin}\{\sup_{t \in [a \circ y] \cap (a \circ x)} \bar{\mu}_A(t), \bar{\mu}_A(0)\},
\]

\[
\bar{\lambda}_A(b) = 0.4 = \text{max}\{\inf_{t \in [a \circ y] \cap (a \circ x)} \bar{\lambda}_A(t), \bar{\lambda}_A(0)\}. \tag{1}
\]

Theorem 3.5.
(i) Every cubic implicative hyper BCK-ideal of \( H \) is a cubic strong hyper BCK-ideal.
(ii) Every cubic weak implicative hyper BCK-ideal of \( H \) is a cubic weak hyper BCK-ideal.

**Proof.** (i) Let \( A = (\bar{\mu}_A, \bar{\lambda}_A) \) be a cubic hyper BCK-ideal of \( H \). Putting \( y = 0 \) and \( z = y \) in Definition 2.8(SH2), we obtain

\[
\bar{\mu}_A(x) \geq \text{rmin}\left\{ \sup_{a \in (x \circ y) \cap (0 \circ x)} \bar{\mu}_A(a), \bar{\mu}_A(y) \right\} = \text{rmin}\{\sup_{a \in (x \circ y)} \bar{\mu}_A(a), \bar{\mu}_A(y)\} \text{ and} \bar{\lambda}_A(x) \leq \text{max}\left\{ \inf_{b \in (x \circ y) \cap (0 \circ x)} \bar{\lambda}_A(b), \bar{\lambda}_A(y) \right\} = \text{max}\{\inf_{b \in (x \circ y)} \bar{\lambda}_A(b), \bar{\lambda}_A(y)\} \tag{1}
\]

First we show that for \( x, y \in H \), if \( x \ll y \) implies that \( \bar{\mu}_A(x) \geq \bar{\mu}_A(y) \) and \( \bar{\lambda}_A(x) \leq \bar{\lambda}_A(y) \).
For this, let \( x, y \in H \) be such that \( x \ll y \). Then \( 0 \in x \circ y \) and from (1), we get

\[
\bar{\mu}_A(x) \geq \text{rmin}\left\{ \sup_{a \in (x \circ y) \cap (0 \circ x)} \bar{\mu}_A(a), \bar{\mu}_A(y) \right\} = \text{rmin}\{\sup_{a \in (x \circ y)} \bar{\mu}_A(a), \bar{\mu}_A(y)\} \text{ and} \bar{\lambda}_A(x) \leq \text{max}\left\{ \inf_{b \in (x \circ y) \cap (0 \circ x)} \bar{\lambda}_A(b), \bar{\lambda}_A(y) \right\} = \text{max}\{\inf_{b \in (x \circ y)} \bar{\lambda}_A(b), \bar{\lambda}_A(y)\} \tag{2}
\]

Let \( x \in H \) and \( a \in x \circ x \). Since \( x \circ x \ll x \), then \( a \ll x \) for all \( a \in x \circ x \) and so by (2), we have
\( \bar{\mu}_A(a) \geq \bar{\mu}_A(x) \) and \( \bar{\lambda}_A(a) \leq \bar{\lambda}_A(x) \) for all \( a \in x \circ x \). Hence
\[
\inf_{a \in (x \circ y) \cap (0 \circ x)} \bar{\mu}_A(a) \geq \bar{\mu}_A(x) \text{ and } \sup_{a \in (x \circ y) \cap (0 \circ x)} \bar{\lambda}_A(a) \leq \bar{\lambda}_A(x) \tag{3}
\]

Combining (1) and (3), we obtain

\[
\inf_{a \in (x \circ y) \cap (0 \circ x)} \bar{\mu}_A(a) \geq \bar{\mu}_A(x) \geq \text{rmin}\{\sup_{b \in (x \circ y) \cap (0 \circ x)} \bar{\mu}_A(b), \bar{\mu}_A(y)\} \text{ and} \sup_{a \in (x \circ y) \cap (0 \circ x)} \bar{\lambda}_A(a) \leq \bar{\lambda}_A(x) \leq \text{max}\{\inf_{b \in (x \circ y) \cap (0 \circ x)} \bar{\lambda}_A(d), \bar{\lambda}_A(y)\} \text{ for all } x, y \in H.
\]

Thus \( A = (\bar{\mu}_A, \bar{\lambda}_A) \) is a cubic strong hyper BCK-ideal of \( H \).
(ii) Let \( A = (\bar{\mu}_A, \bar{\lambda}_A) \) be a cubic weak implicative hyper BCK-ideal of \( H \). Putting \( y = 0 \) and \( z = y \) in Definition 2.8(SH1), we obtain

\[
\bar{\mu}_A(x) \geq \text{rmin}\left\{ \inf_{a \in (x \circ y) \cap (0 \circ x)} \bar{\mu}_A(a), \bar{\mu}_A(y) \right\} = \text{rmin}\{\inf_{a \in (x \circ y)} \bar{\mu}_A(a), \bar{\mu}_A(y)\} \text{ and} \bar{\lambda}_A(x) \leq \text{max}\left\{ \sup_{b \in (x \circ y) \cap (0 \circ x)} \bar{\lambda}_A(b), \bar{\lambda}_A(y) \right\} = \text{max}\{\sup_{b \in (x \circ y)} \bar{\lambda}_A(b), \bar{\lambda}_A(y)\}.
\]
Therefore,
\[
\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x) \geq \min\{\inf_{a \in \mathbb{R}^n} \tilde{\mu}_A(a), \tilde{\mu}_A(y)\}
\]
and
\[
\lambda_A(0) \leq \lambda_A(x) \leq \max\{\sup_{b \in \mathbb{R}^n} \lambda_A(b), \lambda_A(y)\}
\]
for all \(x, y \in H\).
Thus \(A = (\tilde{\mu}_A, \lambda_A)\) is a cubic weak hyper BCK-ideal of \(H\).

**Theorem 3.6.** Let \(A = (\tilde{\mu}_A, \lambda_A)\) be cubic subset of \(H\), then the following statements hold:
(i) If \(A\) is a cubic weak hyper BCK-ideal of \(H\) then for all \(s, t \in [0,1]\),
\(U(\tilde{\mu}_A, \lambda_A) \neq \emptyset \neq L(\lambda_A, t)\) are weak hyper BCK-ideals of \(H\).
(ii) If \(A\) is a cubic implicative hyper BCK-ideal of \(H\), then for all \(s, t \in [0,1]\),
\(U(\tilde{\mu}_A, \lambda_A) \neq \emptyset \neq L(\lambda_A, t)\) are hyper BCK-ideals of \(H\).

**Proof.** (i) Suppose that \(A = (\tilde{\mu}_A, \lambda_A)\) is a cubic weak hyper BCK-ideal of \(H\).
Let \(s, t \in [0,1]\) and \(x, y, z \in H\) be such that \((x \circ z) \circ (y \circ x) \subseteq U(\tilde{\mu}_A, \lambda_A)\) and \(z \in U(\tilde{\mu}_A, \lambda_A)\).
Then \(a \in U(\tilde{\mu}_A, \lambda_A)\) for all \(a \in (x \circ z) \circ (y \circ x)\) which implies that, \(\tilde{\mu}_A(a) \geq \tilde{\mu}_A(x) \geq \tilde{\mu}_A(z) \geq \tilde{\mu}_A(s) \geq \tilde{\mu}_A(s)\). Thus by hypothesis, \(\tilde{\mu}_A(x) \geq \min\{\inf_{a \in (x \circ z) \circ (y \circ x)} \tilde{\mu}_A(a), \tilde{\mu}_A(x)\} \geq \min\{\tilde{\mu}_A(s), \tilde{\mu}_A(z)\} = \tilde{\mu}_A(s)\) and thus \(x \in U(\tilde{\mu}_A, \lambda_A)\).
Now let \(x, y, z \in H\) be such that \((x \circ z) \circ (y \circ x) \subseteq L(\lambda_A, t)\) and \(z \in L(\lambda_A, t)\).
Then \(b \in L(\lambda_A, t)\) and \(b \in (x \circ z) \circ (y \circ x)\) implies \(\lambda_A(b) \leq \lambda_A(z) \leq \lambda_A(t)\).
Hence \(U(\tilde{\mu}_A, \lambda_A) \subseteq L(\lambda_A, t)\).
Hence \(U(\tilde{\mu}_A, \lambda_A) \subseteq L(\lambda_A, t)\)

(ii) Suppose \(A = (\tilde{\mu}_A, \lambda_A)\) is a cubic implicative hyper BCK-ideal of \(H\). Then by theorem 3.6(i), \(A = (\tilde{\mu}_A, \lambda_A)\) is a cubic strong hyper BCK-ideal of \(H\) and so it is a cubic hyper BCK-ideal of \(H\). Then for all \(s, t \in [0,1]\), \(U(\tilde{\mu}_A, \lambda_A) \neq \emptyset \neq L(\lambda_A, t)\) are hyper BCK-ideals of \(H\).

**Theorem 3.7.** Let \(H\) be a positive implicative hyper BCK-algebra. If \(A = (\tilde{\mu}_A, \lambda_A)\) is a cubic weak implicative hyper BCK-ideal of \(H\), then \(A\) is a cubic positive implicative hyper BCK-ideal of \(H\).

**References**

On Cubic Implicative Hyper BCK-Ideals of Hyper BCK-Algebras