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Further results on Antimagic Digraphs

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Abstract: *An antimagic labeling of a digraph D with p vertices and q arcs is a bijection f from the set of all arcs to the set of positive integers f:{1, 2, 3, …,q} such that all the p oriented vertex weights are distinct, where an oriented vertex weight is the sum of the labels of all arcs entering that vertex minus the sum of the labels of all arcs leaving it. A digraph D is called antimagic if it admits an antimagic labeling. In this paper we investigate the existence of antimagic labelings of symmetric digraphs using Skolem sequences.*
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I. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

A *digraph D* consists of a finite nonempty set V of objects called *vertices* and a set E of ordered pairs of distinct vertices. Each element of E is an arc or a directed edge. If a digraph D has the property that for each pair u, v of distinct vertices of D, at most one of (u, v) or (v, u) is an arc of D, then D is an *oriented graph.* An oriented graph can also be obtained by assigning a direction to (that is, orienting) each edge of a graph G. The digraph D is then referred to as an *orientation* of G. A digraph H is called a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$. A digraph D is *symmetric* if whenever (u, v) is an arc of D, then (v, u) is an arc of D as well.

The underlying graph of a digraph D is obtained by removing all directions from the arcs of D and replacing any resulting pair of parallel edges by a single edge. Equivalently, the underlying graph of a digraph D is obtained by replacing each arc $(u,,v)$ or a pair (u, v), (v, u) of arcs by the edge uv. For any graph G the digraph obtained by replacing every edge uv of G by a pair of symmetric arcs uv and vu is denoted by G^* and is called the *symmetric* digraph of G.

A labeling of a graph G is a mapping that assigns integers to the vertices or edges or both, subject to certain conditions. The labeling is called a vertex labeling or an edge labeling or a total labeling according as the domain of the mapping is V or E or $V \cup E$.

HefeTz et al. [3] introduced the concept of antimagic labeling of a digraph. An antimagic labeling of a digraph D with p vertices and q arcs is a bijection from the set of arcs of D to $f:\{1, 2, 3,.., q\}$ such that all p oriented vertex weights are distinct, where an oriented vertex weight is the sum of the labels of all arcs entering that vertex minus the sum of the labels of all arcs leaving it. A digraph D is called *antimagic* if it admits an antimagic labeling. The oriented vertex weight of a vertex $v \in V(D)$ is denoted by $w(v)$. An orientation D of a graph G is called an *antimagic orientation* if the digraph D is antimagic.

HefeTz et al. [3] proved the following theorems, posed a problem and a conjecture.

Theorem 1.1. [3] For every orientation of every undirected graph that belongs to one of the following families, there exists an antimagic labeling.

1. Stars S_n on $n+1$ vertices for every $n \neq 2$.
2. Wheels W_n on $n+1$ vertices for every $n \geq 3$.
3. Cliques K_n on $n+1$ vertices for every $n \neq 3$.

Theorem 1.2. [3] Let $G = (V,E)$ be a $(2d+1)-$regular (not necessarily connected) undirected graph with $d \geq 0$. Then there exists an antimagic orientation of G.

Question 1.3. [3] Is every connected directed graph with at least 4 vertices antimagic?

Conjecture 1.4. [3] Every connected undirected graph admits an antimagic orientation.

They observed that the answer to the Question 1.3 is “No” Indeed if $G = K_{1,2}$ or $K_{3,}$ then G admits an orientation which is not antimagic. Thus not every directed graph is antimagic. Also they proved that the answer to the Conjecture 1.4 is “Yes”.

In this paper we discuss the existence of antimagic labelings for symmetric digraphs and we use the concept of Skolem sequence.

Skolem sequence was introduced by Skolem in 1957, which was used for the construction of Steiner triple systems.

Definition 1.5. [1] A Skolem sequence of order *n* is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ satisfying the following conditions:

1. For every $t \in \{1,2,3,\ldots,n\}$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = t$.
2. If $s_i = s_j = t$. with $i < j$, then $j - i = t.$

Skolem sequences can also be written as a collection of ordered pairs $\{(a_{i,} b_i) : 1 \le i \le n,\ b_i - a_i = i\}$ and $\bigcup_{i=1}^{n} \{a_i, b_i\} = \{1,2,3,\ldots,2n\}.$ For example, when *n = 4* the Skolem sequence S = (1, 1, 3, 4, 2, 3, 2, 4) can be rewritten as {(1, 2), (5, 7), (3, 6), (4, 8)}. When *n = 5*, the Skolem sequence S =(2, 4, 2, 3, 5, 4, 3, 1, 1, 5) can be rewritten as {(8, 9), (1, 3), (4, 7), (2, 6), (5, 10)}.

Theorem 1.6. [5, 1] A Skolem sequence of order *n* exists if and only if $n \equiv 0$ or $1 \pmod 4.$

We now describe a method for the construction of a Skolem sequence of order $n > 5$ where $n \equiv 0$ or $1 \pmod 4.$ We use the ordered pair notation for the Skolem sequence.

When *n = 4s*, a Skolem sequence of order *n* is given by

$$\begin{cases} (4s + r - 1, 8s - r + 1) & 1 \le r \le 2s, \\ (r, 4s - r - 1) & 1 \le r \le s - 2, \\ (s + r + 1, 3s - r) & 1 \le r \le s - 2, \\ (s - 1, 3s), (s, s + 1), (2s, 4s - 1), (2s + 1, 6s). & \end{cases}$$

When *n = 4s + 1*, a Skolem sequence of order *n* is given by

$$\begin{cases} (4s + r + 1, 8s - r + 3) & 1 \le r \le 2s, \\ (r, 4s - r + 1) & 1 \le r \le s, \\ (s + r + 2, 3s - r + 1) & 1 \le r \le s - 2, \\ (s + 1, s + 2), (2s + 1, 6s + 2), (2s + 2, 4s + 1). & \end{cases}$$

II. Main Results

HefeTz et.al [3] introduced the antimagic labelings of digrraphs problems and they proved basic results and proved some families of directed graphs admits an antimagic labelings. Also in [4], Arumugam and Nalliah proved some few families of symmetric directed graphs are antimagic using skolem sequences. Now in this section investigate few families of symmetric directed graphs are admits antimagic labeling using skolem sequences.

Theorem 2.1. Let $G = S_{r,s}$ be the double star where $r \le s$ and $r + s + 1 \equiv 0 \text{ or } 1 \pmod 4$ Then G^* is antimagic.

Proof. Let $V(G)=\{c_1 \cup c_2\} \bigcup \{u_1,u_2,..., u_r\} \cup \{v_1,v_2,..., v_s\}$ be the vertex set of G. Then $E(G)=\{c_1c_2\} \bigcup \{c_1u_1,c_1u_2,..., c_1u_r\} \bigcup \{c_2v_1,c_2v_2,..., c_2v_s\}$. Clearly the size of G is $q=|E(G)|=r+s+1 \equiv 0 \text{ or } 1(\text{mod}4)$. Hence there exists a Skolem sequence S of order q and let $S=\{(a_1,b_1),(a_2,b_2),..., (a_q,b_q)\}$ where $1 \le a_i \le 2q$, $1 \le b_i \le 2q$ and $b_i - a_i = i$. Now we define $f^*: E(G^*) \to \{1,2,3,..., 2q\}$ by

$f^*(c_1c_2)=a_1$,

$f^*(c_2c_1)=b_1$,

$f^*(c_1u_i)=a_{i+1}, 1 \le i \le r$,

$f^*(u_ic_1)=b_{i+1}, 1 \le i \le r$,

$f^*(c_2v_j)=a_{r+1+j}, 1 \le j \le s$,

$f^*(v_jc_2)=b_{r+1+j}, 1 \le j \le s$.

Then $w(c_1)=\dfrac{r(r+1)}{2}$, $w(c_2)=(r+1)s+\dfrac{s(s+1)}{2}-1$, $w(u_i)=-(i+1)$, $1 \le i \le r$ and $w(v_j)=-(r+1+j)$, $1 \le j \le s$. Clearly all the oriented vertex weights are distinct and hence f^* is an antimagic labeling of G^*. □

Example 2.2. Let $G=S_{4,4}$, so that r = 4 and s = 4. Clearly $r+s \equiv 1(\text{mod}4)$. A Skolem sequence of order 9 (in ordered pair notation) is given by $S=\{$(3, 4), (13, 15), (6 ,9), (12, 16), (2, 7), (11, 17), (1, 8), (10, 18), (5 ,14)$\}$. The corresponding antimagic labeling of the symmetric digraph G^* is given in Figure 1.

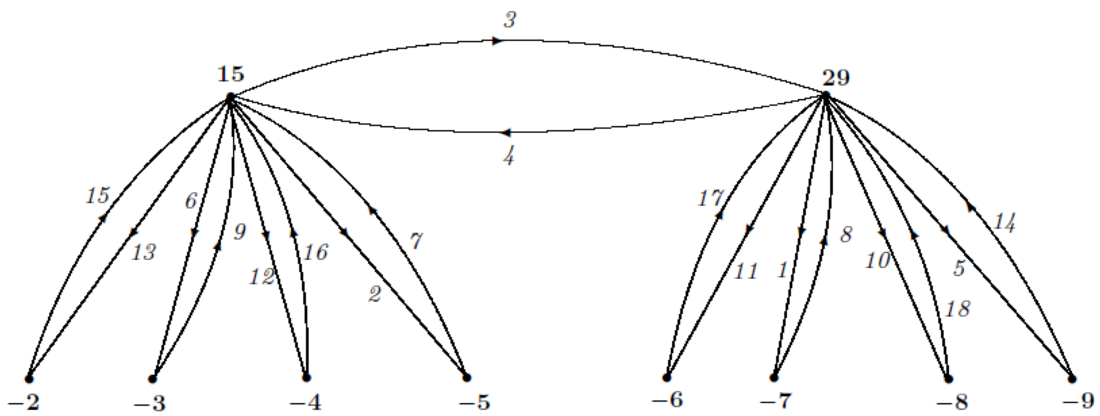

Figure 1: An antimagic labeling of $S^*_{4,4}$.

A graph $G=S_{r,s} \odot K_2$ is obtained from double star $S_{r,s}$ where $r \le s$ attaching K_2 in pendent vertices of $S_{r,s}$. The order and size of G is given by $p=2r+2s+2$ and $q=2r+2s+1$. Suppose $q \equiv 0(\text{mod}4)$. Then $2r+2s+1=4k$, $k \ge 1$ which implies $r+s=2k-\dfrac{1}{2}$, which is impossible. Hence $q \equiv 1\text{mod}4)$.

Theorem 2.3. Let $G=S_{r,s} \odot K_2$ be the double star where $r \le s$ and $2r+2s+1 \equiv 1\text{mod}4)$. Then G^* is antimagic.

Proof. Let $V(G)=\{c_1 \cup c_2\} \bigcup \{u_1,u_2,..., u_r\} \bigcup \{v_1,v_2,..., v_s\} \bigcup \{x_1,x_{.2},..., x_r\} \bigcup \{y_1,y_2,..., y_s\}$ be the vertex set of G. Then $E(G)=\{c_1c_2\} \bigcup \{c_1u_1,c_1u_2,..., c_1u_r\} \bigcup \{c_2v_1,c_2v_2,..., c_2v_s\} \bigcup$

$\{u_1x_1, u_2x_{.2},..., u_rx_r\} \bigcup \{v_1y_1, v_2y_2,..., v_sy_s\}$. Clearly the size of G is $q = |E(G)| = 2r + 2s + 1 \equiv 1 \pmod 4$. Hence there exists a Skolem sequence S of order q and let $S = \{(a_1, b_1), (a_2, b_2),..., (a_q, b_q)\}$ where $1 \le a_i \le 2q$, $1 \le b_i \le 2q$ and $b_i - a_i = i$. Now we define $f^*: E(G^*) \to \{1,2,3,..., 2q\}$ by

$\mathrm{f}^*(\mathrm{c}_1\mathrm{c}_2) = \mathrm{a}_1$,
$\mathrm{f}^*(\mathrm{c}_2\mathrm{c}_1) = \mathrm{b}_1$,
$\mathrm{f}^*(\mathrm{c}_1\mathrm{u}_i) = \mathrm{b}_{r+s+i+1}$, $1 \le i \le r$,
$\mathrm{f}^*(\mathrm{u}_i\mathrm{c}_1) = \mathrm{a}_{r+s+i+1}$, $1 \le i \le r$,
$\mathrm{f}^*(\mathrm{c}_2\mathrm{v}_j) = \mathrm{b}_{2r+s+1+j}$, $1 \le j \le s$,
$\mathrm{f}^*(\mathrm{v}_j\mathrm{c}_2) = \mathrm{a}_{2r+s+1+j}$, $1 \le j \le s$,
$\mathrm{f}^*(\mathrm{u}_i\mathrm{x}_i) = \mathrm{a}_{i+1}$, $1 \le i \le r$,
$\mathrm{f}^*(\mathrm{x}_i\mathrm{u}_i) = \mathrm{b}_{i+1}$, $1 \le i \le r$,
$\mathrm{f}^*(\mathrm{v}_j\mathrm{y}_j) = \mathrm{a}_{r+1+j}$, $1 \le j \le s$,
$\mathrm{f}^*(\mathrm{y}_j\mathrm{v}_j) = \mathrm{b}_{r+1+j}$, $1 \le j \le s$.

Then $w(x_i) = -(i+1)$, $1 \le i \le r$,
$w(y_j) = -(r + j + 1)$, $1 \le j \le s$,
$w(u_i) = r + s + 2 + 2i, 1 \le i \le r$,
$w(v_j) = 3r + s + 2 + 2j, 1 \le j \le s$,

$$w(c_1) = -\left[(r+s+1)r + \frac{r(r+1)}{2} - 1\right] \text{ and}$$

$$w(c_2) = -\left[1 + (2r+s+1)s + \frac{s(s+1)}{2}\right].$$

Clearly all the oriented vertex weights are distinct and hence f^* is an antimagic labeling of G^*.

Example 2.4. Let $G = S_{2,2} \odot K_2$ so that $\mathrm{r} = \mathrm{s} = 2$. Clearly $\mathrm{q} = 2\mathrm{r} + 2\mathrm{s} + 1 \equiv 1 \pmod 4$. A Skolem sequence of order 9 (in order pair notation) is given in $\mathrm{S} = \{$ (3, 4), (13, 15), (6, 9), (12, 16), (2, 7), (11, 17), (1, 8), (10, 18), (5, 14)$\}$. The corresponding antimagic of the symmetric digraph G^* is given in Figure 2.

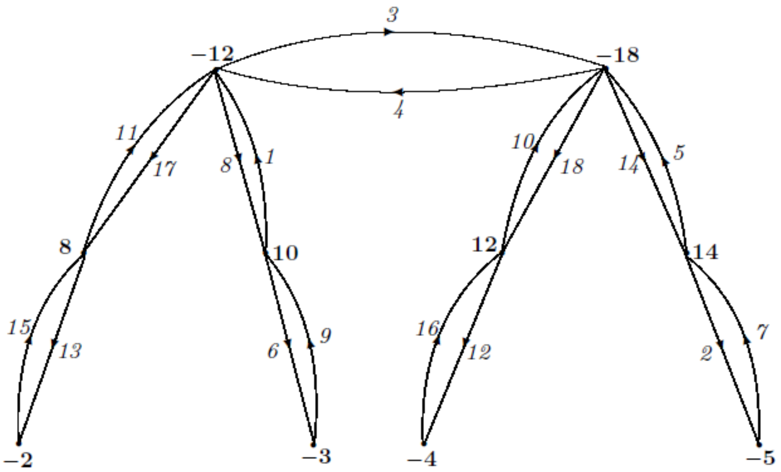

Figure 2: An antimagic labeling of $(S_{2,2} \odot K_2)^*$.

A graph $G = K_{1,n} \odot K_{1,\alpha_i}, 1 \le i \le n$ is obtained from star $K_{1,n}$ with attaching K_{1,α_i} in every pendent vertex of $K_{1,n}$ where $\alpha_1 \le \alpha_2 \le .... \le \alpha_n$.

Theorem 2.5. Let $G = K_{1,n} \odot K_{1,\alpha_i}$ be a graph where $\alpha_1 \le \alpha_2 \le .... \le \alpha_n$ and $\left(\sum_{i=1}^{n} \alpha_i\right) + n \equiv 0$ or $1(\text{mod}4)$. Then G^* is antimagic.

Proof. Let $V(G) = \{c\} \bigcup \{u_i, 1 \le i \le n\} \bigcup \{v_j, 1 \le j \le \alpha_i, 1 \le i \le n\}$ be the vertex set of G Then $E(G) = \{cu_i, 1 \le \mathrm{i} \le \mathrm{n}\} \bigcup \{u_1 v_j, u_2 v_j, ..., u_n v_j\}$, $1 \le \mathrm{i} \le \mathrm{n}$ and $1 \le j \le \alpha_i$. Clearly the size of G is $q = |E(G)| = \left(\sum_{i=1}^{n} \alpha_i\right) + n \equiv 0$ or $1(\text{mod}4)$. Hence there exists a Skolem sequence S of order q (Theorem 1.6). Let $S = \{(a_1, b_1), (a_2, b_2), ..., (a_q, b_q)\}$ where $1 \le a_i \le 2q$, $1 \le b_i \le 2q$ and $b_i - a_i = i$. Now we define $f^* : E(G^*) \to \{1,2,3,..., 2q\}$ by

$f^*(\mathrm{cu_i}) = \mathrm{b}_{\sum_{i=1}^{n} \alpha_i + \mathrm{j}}$, $1 \le \mathrm{i} \le \mathrm{n}$ and $1 \le j \le \alpha_i$,

$f^*(\mathrm{u_i c}) = \mathrm{a}_{\sum_{i=1}^{n} \alpha_i + \mathrm{j}}$, $1 \le \mathrm{i} \le \mathrm{n}$ and $1 \le j \le \alpha_i$,

$f^*(\mathrm{u_i v_j}) = \mathrm{a}_{\left(\sum_{k=1}^{i} \alpha_{k-1}\right) + \mathrm{j}}$, $1 \le \mathrm{i} \le \mathrm{n}$, $1 \le j \le \alpha_i$ and $\alpha_0 = 0$,

$f^*(\mathrm{v_j u_i}) = \mathrm{b}_{\left(\sum_{k=1}^{i} \alpha_{k-1}\right) + \mathrm{j}}$, $1 \le \mathrm{i} \le \mathrm{n}$, $1 \le j \le \alpha_i$ and $\alpha_0 = 0$.

Then $w(c) = -\left[n\left(\sum_{i=1}^{n} \alpha_i\right) + \dfrac{n(n+1)}{2} \right]$,

$w(u_i) = (\alpha_1 + \alpha_2 + ... + \alpha_{i-1})\alpha_i + \dfrac{\alpha_i(\alpha_i + 1)}{2}$, $1 \le \mathrm{i} \le \mathrm{n}$ and $\alpha_0 = 0$,

$w(v_j) = -\left[j + \sum_{k=1}^{i} \alpha_{k-1} \right]$, $1 \le \mathrm{i} \le \mathrm{n}$, $1 \le j \le \alpha_i$ and $\alpha_0 = 0$.

Clearly all the oriented vertex weights are distinct and hence f^* is an antimagic labeling of G^*.

Example 2.6. Let $G = K_{1,3} \odot K_{1,\alpha_i}$ be a graph where, $\alpha_1 = \alpha_2 = \alpha_3 = 2$. Clearly $q = \sum_{k=1}^{3} \alpha_{k-1} + 3 \equiv 1(\text{mod}4)$. A Skolem sequence of order 9 (in order pair notation) is given in in $S = \{$ (3, 4), (13, 15), (6, 9),(12, 16), (2, 7), (11, 17), (1, 8), (10, 18), (5, 14)}. The corresponding antimagic of the symmetric digraph G^* is given in Figure 3.

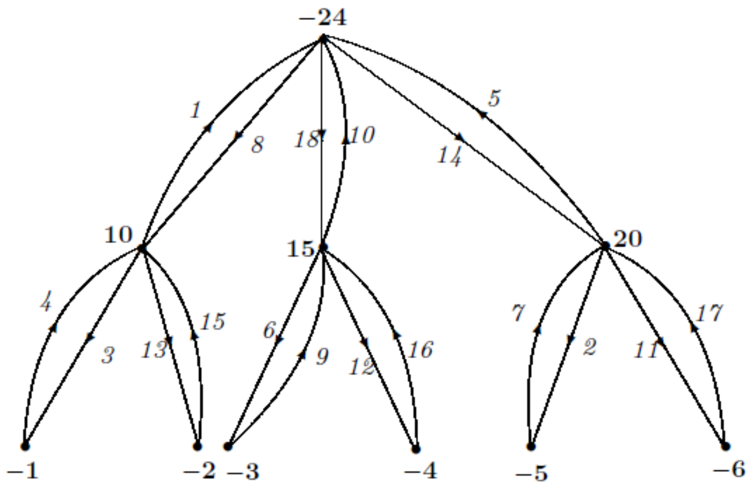

Figure 3: An antimagic labeling of $(K_{1,3} \odot K_{1,\alpha_i})^*$.

III. Conclusion and Scope

In this paper we have discussed the existence of antimagic labelings of digraphs. In particular if G^* is the symmetric digraph associated with an undirected graph G, we have used the concept of Skolem Sequences to prove the existence of antimagic labelings of G^* for several classes of graphs. This proof technique can be used to prove the existence of antimagic labeling of G^* for other families of graphs.

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