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Fourth Order Nonlinear Random Differential Equation

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Abstract:*In this paper, an existence result for a nonlinear fourth order random differential equation is proved under Carathéodory condition. Two existence result for extremal random solutions are also proved for Carathéodory as well as discontinuous cases of nonlinearity involved in the equation. Our investigation are placed in the Banach space of continuous real valued function on closed and bounded interval of the real line together with an application of the random version of the Leray-Schauder Principle.*

Keywords and phrases*:Initial value problem, Random differential equation, Random fixed point theorem, Existence theorem, Extremal solution.*

I. Introduction

Let $\mathbb{R}$ denote the real line and the J = [0, 1] a closed and bounded interval in $\mathbb{R}$. Let C^1 (J, $\mathbb{R}$) denote the class of real valued function defined and continuously differentiable on J. Given a measurable space (Ω, $\mathcal{A}$) and for a given measurable function x: $\Omega \longrightarrow \mathcal{A}C^3$ (J, $\mathbb{R}$). We consider a fourth order nonlinear random differential equation (NRDE)

$$x^{(iv)}(t,\omega) = f(t, x(t,\omega), x'(t,\omega), x''(t,\omega), x'''(t,\omega)) \ , 0 < t < 1 \qquad \ldots \ (1.1)$$

with boundary condition $x(0,\omega) = x'(1,\omega) = x''(0,\omega) = x'''(1,\omega) = 0$
for all $\omega \epsilon\ \Omega$ where f: $J \times \mathbb{R} \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$.

By a random solution of equation (1.1) we mean a measurable function x: $\Omega \longrightarrow \mathcal{A}C^3$ (J, $\mathbb{R}$) that satisfied the equation (1.1) where $\mathcal{A}C^3$ (J, $\mathbb{R}$) is the space of real-valued function whose third derivative exists and is absolutely continuously differentiable on J.

When the random parameter ω is absent, the random (1.1) reduce to the fourth order ordinary differential equation

$$x^{(iv)}(t) = f(t, x(t), x'(t), x''(t), x'''(t)) \ , 0 < t < 1 \qquad \ldots \ (1.2)$$

With boundary condition $x(0) = x'(1) = x''(0) = x'''(1) = 0$

where f: $J \times \mathbb{R} \longrightarrow \mathbb{R}$.

Equation (1.2) has been studied by many authors for different aspects of solution, see for example [3, 4, 5] In this paper we discuss the NRDE (1.1) for existence of solution as well as for existence of the extremal solution, under suitable condition of nonlinearity f which thereby generalize several existence result of the NRDE (1.2) proved in the above mentioned alternatives of Leray- schaudertype[6,7] and algebraic random fixed point theorem of Dhage[6]

II. Existence Result

Let E denote a Banach space with the norm ||.|| and let Q: E⟶E. We further assume that the Banach space E is separable i.e. E has countable dense subset and let β_E be the σ algebra of Borel subset of E. We say a mapping x: $\Omega \longrightarrow$ E is measurable if for any $B \in \beta_E$ one has

$$x^{-1}(B) = \{(\omega, x) \in \Omega \times E : x(\omega, x) \in B\} \in \mathcal{A} \times \beta_E$$

Where $\mathcal{A} \times \beta_E$ is the direct product of the σ algebras A and β_E those defined in Ω and E respectively.

Let Q: $\Omega\times E\longrightarrow E$ be a mapping. Then Q is called a random operator if Q (ω, x) is measurable in ω for all x∈E and it is expressed as Q(ω) x = Q(ω, x). A random operator Q (ω) on E is called continuous (resp. compact, totally bounded and completely continuous) If Q(ω, x) is continuous (resp. compact, totally bounded and completely continuous) in x for all ω ϵ Ω.

Lemma 2.1[2]: LetB_R (0) and $\overline{B_R}(0)$ be the open and closed ball centered at origin of radius R in the separable Banach space E and let Q: $\Omega\times \overline{B_R}(0)\longrightarrow E$ be a compact and continuous random operator. Further suppose that there does not exists an uϵ E with ||u||=R such that Q (ω) u = α u for all α ϵ Ω where α > 1. Then the random equation Q(ω)x = x has a random solution, i.e. there is a measurable function $\xi:\Omega\longrightarrow\overline{B_R}(0)$ such that Q(ω)ξ(ω)=ξ(ω) for all ωϵ Ω .

Lemma 2.2[2]: (Carathéodory) Let Q: Ω × E ⟶ E be a mapping such that Q(.,x) is measurable for all xϵ E and Q(ω,.) is continuous for all ω ϵΩ Then the map (ω, x) ⟶Q(ω, x) is jointly measurable.

We seek random solution of (1.1) in Banach space C (J, ℝ) of continuous real valued function defined on J. We equip the space C (J, ℝ) with the supremum norm||.|| defined by

$\|x\| = \sup_{t\epsilon\mathcal{J}}|x(t)|$

It is known that the Banach space C ($\mathcal{J}$, ℝ) is separable. By L^1 ($\mathcal{J}$, ℝ) we denote the space of Lebesgue measurable real-valued function defined on $\mathcal{J}$. By$\|.\|_{L^1}$ we denote the usual norm in L^1 ($\mathcal{J}$, ℝ) defined by

$\|x\|_{L^1}=\int_0^1|x(t)|dt.$

We need the following definition in the sequel.

Definition 2.3: A function f :$\mathcal{J}\times\mathbb{R}\times\mathbb{R}\times\Omega\longrightarrow\mathbb{R}$ is called random Carathéodory if

i) the map (t, ω)⟶f(t, x, y, ω) is jointly measurable for all (x, y)ϵ$\mathbb{R}^2$, and

ii) the map (x, y)⟶f(t, x, y, ω) is continuous for almost all t ϵ$\mathcal{J}$ and ω ϵ Ω

Definition 2.4: A Carathéodory function f :$\mathcal{J}\times\mathbb{R}\times\mathbb{R}\times\Omega\longrightarrow\mathbb{R}$ is called random L^1- Carathéodory if for each real number r>0 there is a measurable and bounded function $h_r:\Omega\longrightarrow L^1(\mathcal{J},\mathbb{R})$ such that

$|f(t, x, y, \omega)|\le h_r(t, \omega)$ a. e.t ϵ $\mathcal{J}$.

Where|x|, |y|≤ r and for all ω ϵ Ω. Similarly a Carathéodory function f is called $L_{\mathbb{R}}$- Carathéodory if there is a measurable and bounded function h: $\Omega\longrightarrow L^1(\mathcal{J},\mathbb{R})$ such that

$|f(t, x, y, \omega)|\le h(t, \omega)$ a. e. t ϵ $\mathcal{J}$.

for all ω ϵ Ω and (x, y) ϵ $\mathbb{R}^2$

We consider the following set of hypothesis

H_1) The function f is random Carathéodory on $\mathcal{J}\times\mathbb{R}\times\mathbb{R}\times\Omega$

H_2) There exists a measurable and bounded function γ: $\Omega\longrightarrow L^2(\mathcal{J},\mathbb{R})$ and a continuous and non decreasing function ψ: $\mathbb{R}_+\longrightarrow(0,\infty)$ such that

$|f(s, x, x'', \omega)|\le\gamma(t, \omega)\psi(|x|)$, a. e. t ϵ $\mathcal{J}$ For all ω ϵ Ω and xϵℝ

Moreover we assume that $\int_c^\infty\frac{dr}{\psi(r)}=\infty$ for all c≥ 0.

Our main existence result is

Theorem 2.4: Assume that the hypothesis $H_1 - H_2$ hold. Suppose that there exist a real number R>0 such that

$R > r_1\|\gamma(\omega)\|_{L^1}\psi(\mathbb{R})$ … (2.1)

for all ω ϵ Ω where $r_1=\max_{t\epsilon[0,1]} r(t)$, r(t) is in the greens function

Then the (1.1) has a random solution defined on $\mathcal{J}$

Proof: - Set E=C ($\mathcal{J}$, ℝ) and define a mapping Q: Ω × E⟶E by

$$Q(\omega)x(t)=\int_0^1 G(t,s)f(s,x(s,\omega),x'(s,\omega),x''(s,\omega)x'''(s,\omega),\omega)ds \quad \ldots (2.2)$$

for all t ϵ $\mathcal{J}$, ω ϵ Ω . Then the solution of (1.1) is fixed point of operator Q.

Define a closed ball $\overline{B_R}(0)$ in E centered at origin 0 of radius R, Where the real number R satisfies the inequality (2.1). We show that Q satisfies all the condition of lemma 2.1 on $\overline{B_R}(0)$.

First we show that Q is random operator in $\overline{B_R}(0)$Since f (t, x, x′, x′′, x′′′, ω) is random Carathéodory and x (t, ω) is measurable; the map ω⟶ f (t, x, x′, x′′, x′′′, ω) is measurable.

Similarly the production G (t, s) f (s, x(s, ω), x′(s, ω), x′′(s, ω), x′′′(s, ω), ω) of continuous and measurable function is again measurable. Further the integral is a limit of a finite sum of measurable function therefore the map

$\omega\longrightarrow\int_0^1 G(t,s)f(s,x(s,\omega),x'(s,\omega),x''(s,\omega),x'''(s,\omega),\omega)\,ds = Q(\omega)x(t)$

is measurable. As a result Q is random operator on $\Omega\times\overline{B_R}(0)$into E.

Next we show that the random operator Q(ω) is continuous on $\overline{B_R}(0)$. Let x_n be a sequence of point in $\overline{B_R}(0)$converging to the point x in $\overline{B_R}(0)$. Then it is sufficient to prove that

$lim_{n\to\infty} Q(\omega)x_n(t) = Q(\omega)\, X(t)$ for all t∊ $\mathcal{J}$, ω ∊ Ω

By the dominated convergence theorem we obtain

$$\begin{aligned} lim_{n\to\infty} Q(\omega)x_n(t) &= \lim_{n\to\infty} \int_0^1 G(t,s)\, f(s, x_n(s,\omega), x_n'(s,\omega), x_n''(s,\omega), x_n'''(s,\omega), \omega)\, ds \\ &= \int_0^1 G(t,s) \lim_{n\to\infty} f(s, x_n(s,\omega), x_n'(s,\omega), x_n''(s,\omega), x_n'''(s,\omega), \omega)\, ds \\ &= \int_0^1 G(t,s)\, f(s, x(s,\omega), x'(s,\omega), x''(s,\omega), x'''(s,\omega), \omega)\, ds \\ &= Q(\omega)\, x(t) \end{aligned}$$

for all t ∊$\mathcal{J}$, ω ∊Ω . This show that Q (ω) is a continuous random operator on $\overline{B_R}(0)$

Now we show that Q (ω) is compact random operator on $\overline{B_R}(0)$. To finish it, we should prove that Q (ω) $\overline{B_R}(0)$is uniformly bounded and equicontinuous set in E for each ω ∊ Ω. Since the map ω ⟶ γ (t, ω) is bounded and $L^2(\mathcal{J},\mathbb{R}) \subset L^1(\mathcal{J},\mathbb{R})$, by ($H_2$),there is a constant c such that $\|\gamma(\omega)\|_{L^1} \leq c$ for all ω ∊ Ω. Let ω ∊ Ω be fixed then for any x: Ω ⟶$\overline{B_R}(0)$one has

$$\begin{aligned} |Q(\omega)x_n(t)| &\leq \int_0^1 G(t,s)\, |f(s, x(s,\omega), x'(s,\omega), x''(s,\omega), x'''(s,\omega)\omega)|\, ds \\ &\leq \int_0^1 G(t,s)\gamma(s,\omega)\, \psi(|x(s,\omega)|)\, ds \\ &\leq \gamma_1 c\, (\mathbb{R}) = K \end{aligned}$$

for all t ∊ $\mathcal{J}$ and each ω ∊ Ω. This shows that Q (ω) $\overline{B_R}(0)$is uniformly bounded subset of E for each ω ∊ Ω.

Nextwe show thatQ(ω)$\overline{B_R}(0)$is equi-continuous set in E. Forany x∊ $\overline{B_R}(0)$, t_1, t_2 ∊ $\mathcal{J}$ we have

$$\begin{aligned} |Q(\omega)x(t_1) - Q(\omega)x(t_2)| &\leq \int_0^1 |G(t_1,s) - G(t_2,s)|\gamma(s,\omega)\psi(|x(s,\omega)|)\, ds \\ &\leq \int_0^1 |G(t_1,s) - G(t_2,s)|\, \gamma(s,\omega)\psi(R)\, ds \end{aligned}$$

By Höider inequality

$$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \leq \left(\int_0^1 |G(t_1,s) - G(t_2,s)|^2 ds\right)^{1/2} \left(\int_0^1 |\gamma(s,\omega)|^2 ds\right)^{1/2} \psi(R)\, .$$

Hence for all t_1, t2 ∊ $\mathcal{J}$

$|Q(\omega)\, x(t_1) - Q(\omega)\, x(t_2)| \longrightarrow 0$ as $t_1 \longrightarrow t_2$

Uniformly for all x ∊ $\overline{B_R}(0)$. Therefore Q (ω) $\overline{B_R}(0)$is an equi-continuous set in E, then we know it is compact by Arzela-Ascoli theorem for each ω ∊ Ω . Consequently, Q (ω) is a completely continuous random operator on $\overline{B_R}(0)$

Finally, we suppose there exists such an element u in E with ||u|| = R satisfying Q(ω)u(t) = α u(t, ω) for some ω ∊ Ω, where α>1. Now for this ω ∊ Ω we have

$$\begin{aligned} |u(t,\omega)| &\leq \frac{1}{\alpha} |Q(\omega)\, u(t)| \\ &\leq \int_0^1 G(t,s)|\, f(s, u(s,\omega), u'(s,\omega), u''(s,\omega), u'''(s,\omega),\omega)|ds \\ &\leq r_1 \int_0^1 \gamma(s,\omega)\psi|u(s,\omega)|ds \\ &\leq r_1 \|\gamma(\omega)\|_{L^1}\psi\|u(\omega)\| \qquad \text{for all } t \in \mathcal{J} \end{aligned}$$

Taking supremum over t in the above inequality yields.

$R = \|u(\omega)\| \leq r_1 \|\gamma(\omega)\|_{L^1}\psi(R)$

for some ω ∊ Ω. This contradicts to condition (2.1). This all the condition of Lemma2.1 are satisfied. Hence the random equation

Q (ω) x (t) = x (t, ω)

has a random solution in $\overline{B_R}(0)$i.e. there is a measurable function ξ: Ω ⟶$\overline{B_R}(0)$such that Q(ω)ξ(t) = ξ(t, ω) for all t ∊ $\mathcal{J}$, ω ∊Ω . As a result, the random (1.1) has a random solution defined on $\mathcal{J}$. This completes the proof.

III. Extremal Random Solutions

It is sometime desirable to know the realistic behavior of random solution of a given dynamical system. Therefore we prove the existence of extremal positive random solution of (1.1) defines on Ω ×$\mathcal{J}$.

We introduce an order relation ≤ in C($\mathcal{J}$, ℝ) with the help of cone K defined by

K = {x ∊ C ($\mathcal{J}$, ℝ): x (t) ≥ 0 on $\mathcal{J}$}

Let x, y ∊ X then x ≤ y if and only if y - x ∊K. Thus we have x ≤ y↔ x (t) ≤ y (t) for all t ∊ $\mathcal{J}$. It is known that the cone K is normal in C ($\mathcal{J}$, ℝ). For any function a, b: Ω ⟶ C ($\mathcal{J}$, ℝ) we define a random interval [a, b] in C($\mathcal{J}$, ℝ) by

$[a, b] = \{x \in C(\mathcal{J}, \mathbb{R}) : a(\omega) \leq x \leq b(\omega)\ \forall \omega \in \Omega\} = \cap_{\omega\in\Omega} [a(\omega), b(\omega)]$

Definition:3.1: A operator Q: Ω×E ⟶ E is called non decreasing if Q(ω)x≤Q(ω)y for all ω ∊ Ω and for all x, y ∊ E for which x≤y

We use the following random fixed point theorem of Dhage[6]

Theorem 3.2[6]: Let $(\Omega, \mathcal{A})$ be a measurable space and let [a, b] be a random order integral in the separable Banach space E. Let Q: $\Omega\times$ [a, b] $\longrightarrow$ [a, b] be a completely continuous and non decreasing random operator. Then θ has a minimal fixed point x_* and maximal random fixed point y^* in [a, b]. Moreover the sequence $\{Q(\Omega)\, x_n\}$ with x_0 = a and $\{Q(\Omega) y_n\}$ with y_0 = b converge to x_* and y^*.

Definition 3.3: A measurable function α: $\Omega \longrightarrow C(\mathcal{J}, \mathbb{R})$ is called a lower random solution of (1.1) if

$\alpha^{(iv)} \leq f(t, \alpha(t, \omega), \alpha''(t, \omega), \alpha''(t, \omega), \alpha'''(t, \omega), \omega)$ a. e. $t \in \mathcal{J}$

$\alpha(0, \omega) \leq 0, \alpha'(0, \omega) \leq 0, \alpha''(0, \omega) \leq 0; \alpha'''(0, \omega) \leq 0$ for all $t \in \mathcal{J}$ and $\omega \in \Omega$.

Similarly a measurable function β: $\Omega \longrightarrow C(\mathcal{J}, \mathbb{R})$ is called a upper random solution of (1.1) if

$\beta^{(iv)} \geq f(t, \beta(t, \omega), \beta'(t, \omega), \beta''(t, \omega), \beta'''(t, \omega), \omega)$ a. e. $t \in \mathcal{J}$

$\beta(0, \omega) \geq 0, \beta'(0, \omega) \geq 0, \beta''(0, \omega) \geq 0; \beta'''(0, \omega) \geq 0$ for all $t \in \mathcal{J}$ and $\omega \in \Omega$.

Definition 3.4: A random solution θ of (1.1) is called maximal if for all random solution of (1.1), one hasx (t, ω) for all $t \in \mathcal{J}$ and $\omega \in \Omega$.

(H_3) The (1.1) has lower random solution a and an upper random solution b with a $\leq$b on $\mathcal{J}$

(H_4) The function h: $\mathcal{J} \times \Omega \longrightarrow \mathbb{R}_+$ defined by

h (t, ω) = |f(t, a(t, ω),ω)|+ |f(t, b(t, ω),ω)| is Lebegueintegrable in t for all $\omega \in \Omega$.

Hypothesis (H_3) holds in particular when there exist measurable function u, v : $\Omega \longrightarrow C(\mathcal{J}, \mathbb{R})$ such that for each $\omega \in \Omega$ $u(t, \omega) \leq f(t, x, y, \omega)$ for all $t \in \mathcal{J}$ and $x \in \mathbb{R}$. In this case the lower and upper random solution of (1.1) are given by

$a(t, \omega) = \int_0^1 G(t, s)\, u(s, \omega)ds$

And $b(t, \omega) = \int_0^1 G(t, s)\, v(s, \omega)ds$ respectively.

The detail about the lower and upper random solution for differential equation could be found in [8] if f is L^1Carathéodory on $\mathbb{R} \times \Omega$ then (H_4) remains valid.

Theorem3.5: Assume that $H_1 - H_4$ holds; then (1.1) has a minimal random solution $x_*(\omega)$ and maximal solution $y^*(\omega)$ defined on$\mathcal{J}$ moreover ,

$x_*(t, \omega) = lim_{n\to\infty} x_n(t, \omega)$,

$y^*(t, \omega) = lim_{n\to\infty} y_n(t, \omega)$ for all $t \in \mathcal{J}$ and $\omega \in \Omega$ where the random sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are given by

$x_{n+1}(t, \omega) = \int_0^1 G(t, s)\, |f(s\, x_n(s, \omega), x_n'(s, \omega), x_n''(s, \omega), x_n'''(s, \omega), \omega)|\, ds$

for all n$\geq$0 with x_0 = a and

$y_{n+1}(t, \omega) = \int_0^1 G(t, s)|\, f(s\, y_n(s, \omega), y_n'(s, \omega), y_n''(s, \omega), y_n'''(s, \omega), \omega)|ds$

for all n$\geq$0 with y_0 = b and for all $t \in \mathcal{J}$, $\omega \in \Omega$.

Proof: We set E = $C(\mathcal{J}, \mathbb{R})$ and define an operator Q: $\Omega \times [\alpha, \beta] \longrightarrow E$ by [2.2]. We show that Q satisfies all the condition of [3.2] on [a, b]. It can be shown as in the proof of theorem 2.1 that Q is a random operator on $\Omega \times$ [a, b]. We show that it is non decreasing random operator on [a, b]. Let x, y: $\Omega \longrightarrow$ [a, b] be arbitrary such that x $\leq$ y on Ω Then

$Q(\omega)\, x(t) \leq \int_0^1 G(t, s)\, f(s, x(s, \omega), x'(s, \omega), x''(s, \omega), x'''(s, \omega), \omega)\, ds$

$\leq \int_0^1 G(t,)\, f(s, y(s, \omega), y'(s, \omega), y''(s, \omega), y'''(s, \omega), \omega)\, ds$

$\leq Q(\omega)\, y(t)$

for all $t \in \mathcal{J}$ and $\omega \in \Omega$. As a result Q (ω) x$\leq$ Q (ω) y for all $\omega \in \Omega$ and that Q is non decreasing random operator on [a, b]

Second by the hypothesis (H_3)

$a(t, \omega) \leq Q(\omega)a(t)$

$= \int_0^1 G(t, s)\, f(s, a(s, \omega), a'(s, \omega), a''(s, \omega), a'''(s, \omega), \omega)\, ds$

$\leq \int_0^1 G(t, s)\, f(s, x(s, \omega), x'(s, \omega), x''(s, \omega), x'''(s, \omega), \omega)\, ds$

$= Q(\omega)\, x(t)$

$\leq Q(\omega)\, b(t)$

$= \int_0^1 G(t, s)\, f(s, b(s, \omega), b'(s, \omega), b''(s, \omega), b'''(s, \omega), \omega)\, ds$

$\leq b(t, \omega)$

for all $t \in \mathcal{J}$ and $\omega \in \Omega$. As a result Q defines a random operator Q: $\Omega\times$ [a, b] $\longrightarrow$ [a, b]

Next since (H_4) holds, the hypothesis (H_2) is satisfied with $\gamma(t, \omega) = h(t, \omega)$ for all $(t, \omega) \in \mathcal{J} \times \Omega$ and $\psi(r) = 1$ for all real number $r \geq 0$. Now it can be shown as in the proof of Theorem 3.1 that the random operator Q (ω) is completely continuous on [a, b] into itself. Thus the random operator Q (ω) satisfies all the conditions of theorem 3.1 and so the random operator equation $Q(\omega)x = x(\omega)$ has a least and greatest random solution in [a, b]. Consequently the (1.1) has a minimal and a maximal random solution defined on $\mathcal{J}$. This completes the proof.

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