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Numerical Solution of Seepage Problem of groundwater flow

M.S.Prajapati[1], P.H. Bhathawala[2]
[1]*Associate professor, Department of Mathematics, U.B.Science College, Mehsana-pin-384002, Gujarat, India*
[2] *Professor of IT, AURO University, Surat – 394510, Gujarat, India*

Abstract : *In the present paper we have discuss numerical solution of seepage problem of ground water flow using hydraulic theory. We examine the seepage of groundwater down sloping bedrock in heterogeneous soil in vertical direction and numerical solution is obtained by RK-4 method with MatLab coding .*
Keywords: *Ground water, fluid flow, RK-4 method.*

I. Introduction

We have determined the free surface which is concave upwards for the seepage of groundwater in heterogeneous soil on sloping bedrock in terms of special functions. In case of homogeneous soil with small slope the free surface is concave downwards and the pressure head at head reservoir is greater than that of the tail reservoir. In both cases the free surface represents a falling surface.

Dupit's assumptions for small inclinations of free surface of a gravity flow system from the basis of the theory called the hydraulic theory. In 1863, he assumed that for small inclinations of free surface, the flow is horizontal and uniform everywhere in a vertical section and that the velocities at the free surface are proportional to the slope of the free surface but are independent of the depth. We have,

$$u = -k'\frac{\partial h}{\partial x} \quad \text{and} \quad v = -k'\frac{\partial h}{\partial y} \tag{1}$$

where u and v are the steady state velocity components in a column of liquid of height h above the impervious base. Thus the hydraulic theory, in essence, is the theory in which flow is averaged over the depth so that the dimensions of the investigated flow is decreased by one, if the flow takes place in xz-plane, then the flow elements will depend on x-coordinates only, and for flow in three dimensional space the elements depend on the two co-ordinates x and y only.

In case of groundwater flow over large area, the free surface is slightly curved. Examples of such flow are seepage from canal to river, flow in irrigation etc. For such flows the area covered by seepage may measure in square kilometers while the depth of flow is only a few meters or decameters and to a large extent (save regions in immediate neighbourhood of canal or river) varies only a fraction of meter. These are fit cases for the application of hydraulic theory.

We discuss three particular features of the hydraulic theory which are useful to our investigations.

In the general three dimensional flow, head is a function of x,y,z. If the free surface of groundwater is slightly curved then z varies slightly which in turn implies that it oscillates about some average value $\bar{z}$. Expanding h into a power series of $z - \bar{z}$ we have,

$$h(x, y, z) = h(x, y, \bar{z}) + (z - \bar{z})\left(\frac{\partial h}{\partial z}\right)_{z=\bar{z}} + \dots \dots \dots \tag{2}$$

Here we assume that the velocity is almost horizontal, that is vertical velocities are very small which means that $\frac{\partial h}{\partial z}$ is small and there for we neglect its product with the small quantity $z - \bar{z}$. Thus, the real head $h(x, y, z)$ is replaced by the quantity $h(x, y, \bar{z})$ which depends on the seepage depth of flow $\bar{z}$, but not z. This is designated by $h(x, y)$. Thus

$$h = h(x, y) \tag{3}$$

Hence, the head is constant in each vertical section.
Neglecting atmospheric pressure in the value of h, and using the above result we have,

$$z = h(x, y) \tag{4}$$

for free surface. This proves the contention.

Consider a fluid prism, bounded below by a horizontal surface xy lying in an impervious base plane, above by the free surface, and on the sides by vertical planes standing on a rectangle dx.dy in the base. If the flow rates in the direction of x and y are denoted by q_x (per unit length in the x-direction) and q_y (per unit length in the y-direction) respectively then the equation of continuity may be written as,

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0 \tag{5}$$

For a homogeneous medium (k as constant) the values of q_x and q_y are given by,

$$q_x = -k\left(h\frac{\partial h}{\partial x}\right) \quad \text{and} \quad q_y = -k\left(h\frac{\partial h}{\partial y}\right) \tag{6}$$

For porous medium with vertical heterogeneity q_x is computed by dividing h in elements dz, so that the rate through an elementary rectangle is

$$-k(z)\frac{\partial h}{\partial z}dz$$

where k(z) is the seepage coefficient of the porous medium which varies with z. Integrating with respect to z between 0 and h we get,

$$q_x = -\frac{\partial h}{\partial x}\int_0^h k(z)dz \tag{7}$$

where h does not depend upon z. Similarly

$$q_y = -\frac{\partial h}{\partial y}\int_0^h k(z)dz \tag{8}$$

II. Statement Of The Problem

Water from a head reservoir flows into adjacent soil, which stands on inclined bedrock and exhibits heterogeneous in the vertical direction. After seeping over considerable distance it falls into a tail reservoir. We examine here the nature of the free surface of flow in a vertical plane when the seepage face is neglected. We choose a horizontal line at the bottom of the tail reservoir as the x-axis, a vertical line besides it as z-axis. The inclined boundary is line $z = -mx$, where $m = \tan\alpha$, $\left(0 < \alpha < \frac{\pi}{2}\right)$ is the slope of the inclined bedrock.

III. Mathematical Formulation Of The Problem

According to Darcy's law, the seepage velocity is given by,

$$u = -k(z)\frac{\partial h}{\partial x} \tag{9}$$

where h is the piezometric head and k(z) is the seepage coefficient of the porous medium which varies with z linearly.

For definiteness we assume that

$$k(z) = k_0(1 - bz) \tag{10}$$

where k_o and b are non-zero constants and b>0, verma [3].

Since the flow of groundwater takes place over considerable distance, the analysis is based on hydraulic theory. In the hydraulic theory, the h is equal to the height of the free surface (neglecting the atmospheric pressure) and the flow elements depend on x alone. Hence using (7) the flow rate q_x is given by,

$$q_x = -\frac{\partial h}{\partial x}\int_0^h k(z)dz \tag{11}$$

where z=0 is the foot and z=h is the top of the vertical section at a distance x for which q_x is measured, $\frac{dh}{dx}$ is independent of z.

The equation of continuity using (5) is,

$$\frac{\partial q_x}{\partial x} = 0 \tag{12}$$

q_x = constant = q(say) (13)

from equations (10), (11) and (12) we obtain

$$q = -k_0 \frac{dh}{dx} \int_{-mx}^{h} (1 - bz)\,dz \qquad (14)$$

Performing the integration and on rearrangement we obtain,

$$\frac{dx}{dh} = P(h) + Qx + Rx^2 \qquad (15)$$

Where $P(h) = -\frac{k_0}{q}\left(h - \frac{b}{2}h^2\right)$ (16)

$$Q = -\frac{k_0 m}{q} \qquad (17)$$

$$R = -\frac{k_0 b m^2}{2q} \qquad (18)$$

The equation (15) is the generalized Riccati's equation.
Considering P(h) = 1, Q =1 and R=-1, the equation (15) reduces to,

$$\frac{dx}{dh} = 1 + x - x^2 \qquad (19)$$

which is second order linear differential equation, whose solution is obtained using Runge-Kutta 45 – method shown in table 1(a) and A multi-step Adams method shown in table1(b).

IV. Numerical Solution And Interpretation

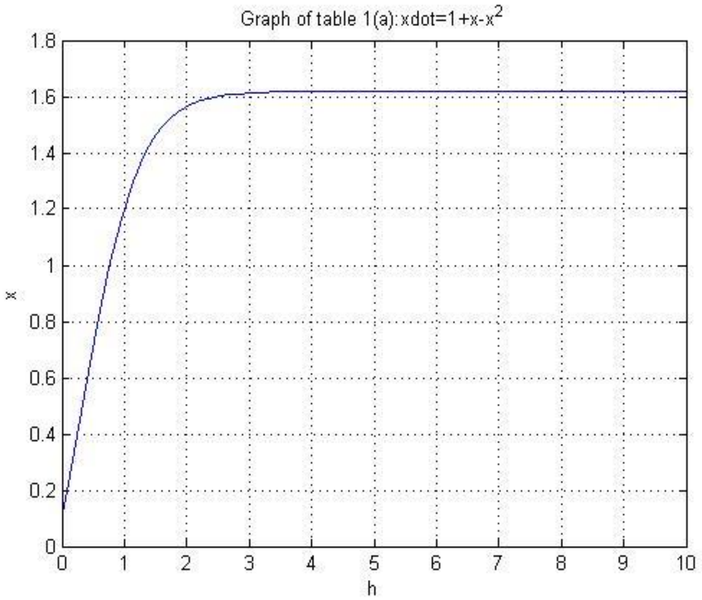
$\frac{dx}{dh} = 1 + x - x^2$ with initial condition x=0.1 when h=0.

Above ordinary differential equation is solved with MATLAB using Runge-Kutta 45 – method, Raj Kumar Bansal [12] and result is shown in following table. i.e. values of x at various values of h.

Table 1(a) Computational results for seepage problem :

h	x		h	x
0	0.100000000000000		3.610893008127416	1.616556776508301
0.004608965929375	0.105033007100305		3.807537856006123	1.616982540043549
0.009217931858751	0.110084371519486		4.057537856006123	1.617510914965923
0.013826897788126	0.115153924156238		4.307537856006123	1.617800697879317
0.018435863717502	0.120241492752247		4.557537856006123	1.617839730121722
0.041480693364378	0.145943333874746		4.807537856006123	1.617840319951375
0.064525523011255	0.172067931102620		5.057537856006123	1.617937628770001
0.087570352658132	0.198590297477710		5.307537856006123	1.617990947876307
0.110615182305009	0.225483615037943		5.557537856006123	1.617998106910962
0.225839330539393	0.364509879340476		5.807537856006123	1.617998237310321
0.341063478773778	0.507830906809162		6.057537856006123	1.618016200146564
0.456287627008162	0.650886489195188		6.307537856006123	1.618026040969727
0.571511775242547	0.789163749839772		6.557537856006123	1.618027361472868
0.759012823128445	0.993644743344057		6.807537856006123	1.618027386273227
0.946513871014343	1.164289617342429		7.057537856006123	1.618030703584289
1.134014918900240	1.297610239459839		7.307537856006123	1.618032520894264
1.321515966786139	1.396270713898784		7.557537856006123	1.618032764724766
1.451993565029754	1.448271460335780		7.807537856006123	1.618032769329592
1.582471163273369	1.488921193481227		8.057537856006123	1.618033382007702
1.712948761516984	1.520143640987856		8.307537856006123	1.618033717646878
1.843426359760599	1.543995762927923		8.557537856006123	1.618033762679021
1.973903958004214	1.562357624667464		8.807537856006123	1.618033763530337
2.104381556247829	1.576265474749194		9.307537856006123	1.618033938678712
2.234859154491444	1.586636088630149		9.057537856006123	1.618033876688261
2.365336752735059	1.594420396840859		9.557537856006123	1.618033946995834
2.529242180674118	1.601749579526405		9.807537856006123	1.618033947153095
2.693147608613177	1.606834327822320		9.855653392004593	1.618033951396439
2.857053036552236	1.610166699931526		9.903768928003061	1.618033955206820
3.020958464491296	1.612448133765366		9.951884464001530	1.618033958628101
3.217603312370002	1.614531426198208		10.000000000000000	1.618033961700401
3.414248160248709	1.615857377395976			

[Table 1(a)]



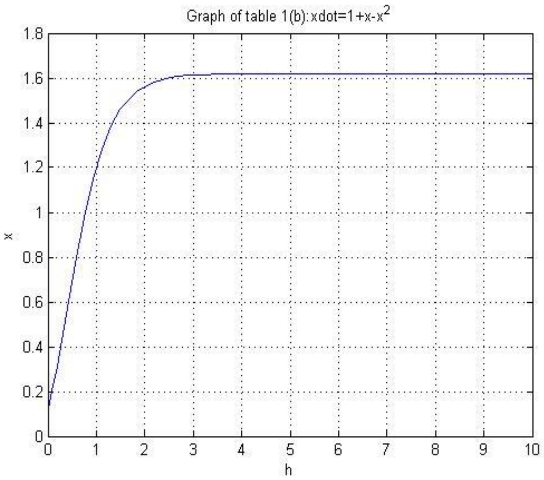
$\frac{dx}{dh} = 1 + x - x^2$ with initial condition x=0.1 when h=0.

Above ordinary differential equation is solved with MATLAB using A multi-step Adams method – Raj Kumar Bansal [12], method and result is shown in following table. i.e. values of x at various values of h.

Table 1(b) Computational results for seepage problem :

h	x		h	x
0	0.100000000000000		2.190226435848247	1.583124266494947
0.000725293041323	0.100790798546186		2.524427978950880	1.601397403656344
0.002175879123969	0.102373769456496		2.858629522053512	1.610618613730645
0.005077051289261	0.105545187509604		3.192831065156145	1.614738485012917
0.010879395619845	0.111909782245729		3.527032608258777	1.616538687522123
0.022484084281014	0.124724809871102		3.861234151361409	1.617411337426366
0.045693461603350	0.150688230598948		4.529637237566674	1.617376585278314
0.092112216248024	0.203862181632498		5.198040323771939	1.617796925595480
0.184949725537371	0.314409974402845		5.866443409977204	1.618955650631942
0.370624744116065	0.544745250242849		6.534846496182468	1.618720292746460
0.556299762694758	0.771331763553410		7.203249582387733	1.617340147214750
0.741974781273452	0.976448124608758		7.871652668592998	1.617146387279476
0.927649799852146	1.148602647673250		8.540055754798262	1.618602751555834
1.113324818430840	1.284196489843127		9.208458841003527	1.619038608259886
1.298999837009534	1.385799226943076		9.876861927208791	1.617488539442059
1.484674855588227	1.459110980479683		10.000000000000000	1.617614885017571
1.856024892745615	1.546540135836375			

[Table 1(b)]



The analytic solution of this problem shows that the effect of heterogeneity of the soil on sloping bedrock is to depress the free surface at each point without reversing its concavity. In case of homogeneous soil with small slope the free surface is concave downward and pressure head at head reservoir is greater than that of the tail reservoir. We have obtained the numerical solution of a seepage problem using RK-4 method and its graphical representation is also obtained using MATLAB coding. The numerical solution is consistent with analytical solution as well as with the physical phenomenon of the seepage problem. Thus the numerical data matches with the physical phenomenon of the seepage problem. The graph (3a) and numerical tables also shows that gradually seepage has been increase as the depth increase and after some specific depth as the saturation occurs the seepage has been constant.

V. Conclusion

Engineers in several fields have to learn the mechanism of drainage and to apply to problems of water supply, land reclamation and stabilization of foundations and sub grade, and also to the fields of petroleum production and agriculture.

Drainage in general is any provision for the removal of excess water. The common objective of land projects to prevent or eliminate either water logging or inundation or otherwise productive land. Drainage of projected land refers principally to the disposal of surplus natural water adversely affecting irrigation. Practically every area where irrigation has been carried on for time has been affected by high water table. Therefore provision for adequate drainage is an essential part of planning, construction and operation of an irrigation project.

For agriculture purpose, the continued presence of water in excess of that needed for vegetation is harmful. Prolonged saturation of soil excludes air essentially for healthy plant growth and the soil becomes cold, sour and unproductive. Consequently unsaturated or irrigated soils is a necessary evil, so to this type of drainage where originally saturation conditions are existing up to the top.

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