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Numerical Solution Of Parabolic Initial – Boundary Value Problem With Crank-Nicolson's Finite Difference Equations

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Abstract: *The finite difference method is a direct interpretation of the differential equation into a discrete domain so that it can be solved using a numerical method. It is a direct representation of the governing equation* $(\frac{\partial f}{\partial x}) = (f_{i+1} - f_i)/(x_{i+1} - x_i)$. *Using the discontinuous but connected regions, the governing equation is defined within the interval.In this paper, an initial-Boundary value problem of the parabolic type is investigated. The explicit and implicit schemes were established. The numerical solution obtained using Crank-Nicolson's finite difference equations is found to agree with existing analyzing results at discretized nodes of uniform interval.*

I. Introduction

Parabolic equations arise in the study of heat conduction and diffusion processes [i]. The simplest example is the one dimensional heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad [1.1]$$

The two kinds of fundamental parabolic equation are the initial value and the initial-boundary value problem. An initial value problem here is that of finding a function $u(x,t)$ which

(a) Is defined and continuous for $x \in (-\infty,\infty)$,t ≥ 0
(b) Satisfies the equation (1.1) for $x \in (-\infty,\infty)$, t > 0
(c) Satisfies the initial condition $u(x,0) = f(x)$ for
  $x \in (-\infty,\infty)$, where $f(x)$ is a given continuous.

An initial-boundary value problem for the parabolic equation (1.1) is that of finding a function $u(x,t)$ which

(i) is defined and continuous for $x \in [o\,,a], t \geq 0$
(ii) satisfies equation (1.1) for $x \in [o\,,a]$, t > 0
(iii) satisfies the initial and boundary conclusions

$$u(x,\text{o}) = f(x), 0 \leq x \leq a \qquad (1.2)$$

$$u(x,\text{o}) = g_1(t) \; and \; \text{U(a,t)} = g_2(t), t \geq 0 \qquad (1.3)$$

$a > 0$ *and the function* $f(x), g_1(t), g_2$(t) are given and must satisfy.

$$g_{1\,(0)} = f(0), g_{2\,(0)} = f(a) \qquad (1.4)$$

Although analytical solution may exist the numerical treatment is achieved by replacing the unbounded range $0 \leq t \leq \infty$ *by* $0 \leq t \leq T$. The finite difference approximation of (1.1) may be carried out in several ways resulting into explicit and implicit schemes.

Explicit scheme concerns calculating data at the next time level from an explicit formular involving data from previous time level. This scheme leads to (stability) restriction on the maximum allowable time step, Δt.

Implicit scheme concerns data from the next time level occurring on both sides of the difference scheme. That necessitates solving a system of linear equations. There is no stability restriction on the maximum time step.

II. Crank- Nicolson Scheme (Implicit scheme)

Consider the one dimensional that conduction equation

$$\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$$

Where $C^2 = \frac{K}{sp}$ *is the diffusivity of he substance measured in* cm^2/sec.

From Taylor's formula [2]

$$u(x+h,y) = u(x,y) + h\frac{\partial u}{\partial x} + \frac{1}{2!}h^2\,\frac{\partial^2 u}{\partial x^2} + \ldots \qquad (2.2)$$

Truncate after $\frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{u(x+h,y) - u(x,y)}{h} \quad \text{Forward difference formula} \qquad (2.3)$$

Similarly:

$$\frac{\partial u}{\partial x} = \frac{u(x+h,y) - u(x-h,y)}{h} \quad \text{Backward difference formula} \qquad (2.4)$$

Also $$\frac{\partial u}{\partial x} = \frac{u(x,+y+,k) - u(x,y)}{k} \qquad (2.5)$$

$$\frac{\partial u}{\partial x} = \frac{u(x+h,y) - u(x-h,y)}{2h} \quad \text{Central difference formula} \qquad (2.6)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x-h,y) - 2u(x.y) + u(x-h,y)}{h^2} \qquad (2.7)$$

When we substitute the above formula in the parabolic equation and setting

$\propto \ = \frac{\boldsymbol{kc^2}}{\boldsymbol{h}}$, we obtain

$$u_{i,j+1} \ \propto u_{i-1.j} \ + (1-2\propto)u_{i,j} + \propto u_{i-1.j} \qquad (2.8)$$

This is the explicit finite difference scheme for the one dimension heat conduction equation. This explicit scheme is not stable except for $\propto \leq \frac{1}{2}$

For no restriction on $\propto$, the Crank-Nicolson's method is used. Here, $\frac{\partial^2 u}{\partial x^2}$ is replaced by the average of its central- difference approximations on the J^{th} and $(J^{+1})^{th}$ time rows.

Thus:

$$\frac{u_{i,J+1} - u_{rij}}{k} = \frac{C^2}{2h^2}\,[u_{i-i,j} \ - \ 2u_{i,j} \ + u_{i+1,j} \ - u_{i-1,j+i} + 2u_{i,j+1} + u_{i+1,j+1}] \ldots \ (2.9)$$

Simplify and replace $\propto = \frac{\boldsymbol{Kc^2}}{\boldsymbol{h^2}}$, we obtain

$$-\propto u_{i-1,j+1} + (2+2\propto)\, u_{i,j+1} - \propto u_{i+1,j+1} = \propto u_{i-1,j} + (2+2\propto)u_{i,j} + \propto u_{i+1.j} \ldots (2.10)$$

This is the Crank-Nicolson's formula with computation. The scheme is valid for all finite values of $\propto$. In addition it has a higher degree of accuracy $o(h^2 + k^2)$ [3].

III. Numeric illustration

3.1 Example

Consider the initial- boundary problem

$u_t \ = u_{xx}$

Defined on D = $\{(x,t)/0 < x < 3, t > 0\}$

With initial conditions $u(x,0) = f(x) = x^2, 0 \leq x \leq 3$ *and boundary conditions*

$u(o,t) = g_{1\ (t)} \ = 0, u(3,t) = g_2(t) = 9, t > 0$

Solution

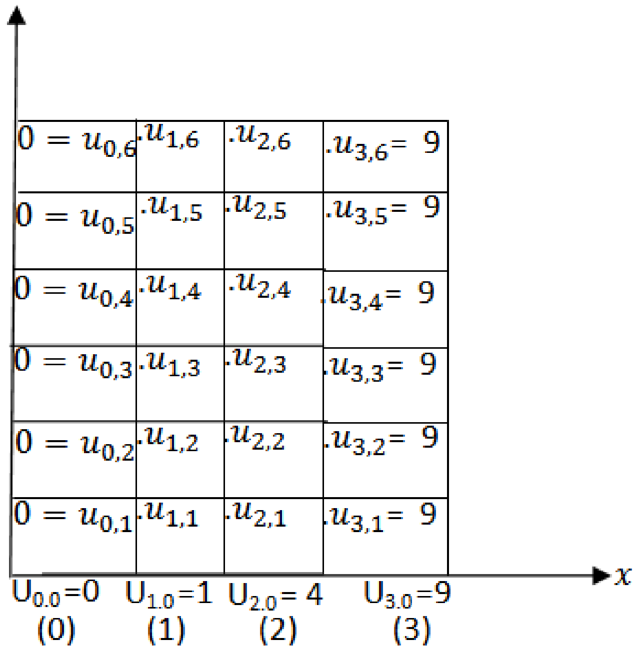
Suppose $T = 30$, *set* $h = 1, k = 5$

$\propto \ = \frac{Kc^2}{h^2} = 5, \ \ c = \ 1$

From the Crank-Nicolson's formula (2.10)

Put $\propto = 5$, *we have*

$$-5u_{i-1,\,j+1} + 12u_{i\ 1+1} - 5u_{i+1,j+1} = 5\,u_{i-1,j} \ - \ 8\,u_{i,j} \ + 5\,u_{i+1,j} \qquad (3.1)$$



For $i = 1, j = 0$

$-5u_{0,1} + 12u_{1,1} - 5u_{2,1} = 5u_{0,0} - 8u_{1,0} + 5u_{2,0}$ (3.2)

Apply given conditions

$-5\,(_{0,1}) + 12u_{1,1} - 5u_{2,1} = 5(0) - 8(1) + 5(4)\; 12u_{1,1} - 5u_{2,1} = 12$ (3.3)

For $i = 2, j = 0$

$-5u_{1,1} + 12u_{2,1} - 5u_{3,1} = 5u_{1,0} - 8u_{2,0} + 5u_{3,0}$ (3.4)

Apply given condition

$-5u_{1,1} + 12u_{2,1} - 5(9) = 5(1) - 8(4) + 5(9)$

$-5u_{1,1} + 12u_{2,1} = 95 - 32 = 63$ (3.4)

Put (3.3) and (3.4) in matrix form

$12u_{1,1} - 5u_{2,1} = 12$

$-5u_{1,1} + 12u_{2,1} = 63$ (3.5)

$$\Rightarrow \begin{pmatrix} 12 & -5 \\ -5 & 12 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix} = \begin{pmatrix} 12 \\ 63 \end{pmatrix}$$

Solving the equation simultaneous for $u_{1,1}$ and $u_{2,1}$

$$\Rightarrow \begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix} = \begin{pmatrix} 3.8571 \\ 6.8571 \end{pmatrix} \quad (3.6)$$

Continuing this way, it is shown that the solution at interior nodes are

$$\begin{pmatrix} u_{2,1} \\ u_{2,2} \end{pmatrix} = \begin{pmatrix} 2.6328 \\ 5.6328 \end{pmatrix}$$

$$\begin{pmatrix} U_{1,3} \\ U_{2,3} \end{pmatrix} = \begin{pmatrix} 3.1574 \\ 6.1574 \end{pmatrix}, \quad \begin{pmatrix} U_{1,4} \\ U_{2,4} \end{pmatrix} = \begin{pmatrix} 2.9325 \\ 5.9325 \end{pmatrix}$$

$$\begin{pmatrix} U_{1.5} \\ U_{2.5} \end{pmatrix} = \begin{pmatrix} 3.0289 \\ 6.0289 \end{pmatrix}, \quad \begin{pmatrix} U_{1.6} \\ U_{2.6} \end{pmatrix}, \quad = \begin{pmatrix} 2.9876 \\ 5.9876 \end{pmatrix}$$

IV. Analytical Evaluation

For the analytical solution

$$u_t = u_{xx}$$

D = $\{(x,t)/0 < x < 3, t > 0\}$

Initial condition: $U(x,o) = x^2$, $0 \leq x \leq 3$

Boundary condition: $u(o,t) = 0$

$u(3,t) = 9$

Solution

Since the boundary condition are not homogeneous, separation of variables method fails. However, it is obvious that the solution will have a steady state solution that varies linearly in x between 0 and 3

$$u(x,t) = k_1 + \frac{x}{L}(k_2 - k_1) + u(x,t) \qquad (4.1)$$

Where $u(x,t) = transient\ part\ of\ solution.$

$k_1 + \frac{x}{L}(k_2 - k_1) = steady\ state\ part.$

In the above problem.

$k_1 = 0, k_2 = 9, L = 3$

$u(x,t) = 3x + u(x,t)$

By substitution (4.2) in the given problem, it transform to

$u_t = u_{xx},\ 0 < x < 3, t > 0$

$boundary\ condition\colon\ u(o,t) = o$

$u(3,t) = 0$

Initial condition: $u(x,o) = x^2 - 3x$

Using method of separation of variables [5]

$$U(x,t) = \sum_{n=1}^{\infty}\left(a_n e^{-(n\pi)2t} + \sin\frac{n\pi x}{3}\right)$$

Where an $= \frac{2}{3}\int_0^3 \phi(\xi)\sin(\frac{n\pi x}{3})d\xi, \phi = x^2 - 3x$

Hence, analytic solution of the problem is

$$U(x,t) = 3x + \sum_{n=1}^{\infty}\left(a_n e^{-(n\pi)2}\,t + \sin(\frac{n\pi x}{3}\right)$$

$$3x + a_1 e^{-}\pi^{2t}\sin\left(\frac{\pi x}{3}\right) + a_2 e^{-4\pi^2}t\sin\left(\frac{2\pi x}{3}\right) + \cdots$$

V. Conclusion

Comparing analytic and numerical results at discrete modes, say (1,1)

$U(1,1) = 3.8571 \quad -numerical$

$U(1,1) = 3.8570 \quad -analytical$

$error =$ 0.0001(infinitesimally small)

Hence, in conclusion, the implicit numerical solution by Crank –Nicolson's finite difference equations agree with the analytic results at internal nodes of uniform interval.

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