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Edge Regular Property of Alpha Product, Beta Product and Gamma Product of Two Fuzzy Graphs

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Abstract: *In this paper, we determined that the alpha product, beta product and gamma product of two edge regular fuzzy graphs need not be edge regular and that if these operations of two fuzzy graphs are edge regular, then G_1 (or) G_2 need not be edge regular. A necessary and sufficient condition for alpha product and gamma product of two fuzzy graphs to be edge regular fuzzy graph is determined.*
Key Words: *Alpha product, Beta product, Gamma product, Regular fuzzy graph, Edge regular fuzzy graph.*
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I. Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975 [11]. Mordeson. J. N and Peng. C. S introduced the concept of operations on fuzzy graphs [2]. The degree of a vertex in fuzzy graphs which are obtained from two given fuzzy graphs using the operations of alpha product, beta product and gamma product was discussed by Nagoor Gani. A and Fathima Kani. B [3]. Radha. K and Kumaravel. N introduced the concept of degree of an edge and total degree of an edge in fuzzy graphs [8]. We study about edge regular fuzzy graphs which are obtained from two given fuzzy graphs using the operations of alpha, beta and gamma product. In general, alpha, beta and gamma product of two edge regular fuzzy graphs G_1 and G_2 need not be edge regular. In this paper, we find necessary and sufficient condition for alpha product, beta product and gamma product of two fuzzy graphs to be edge regular fuzzy graph. First we go through some basic concepts which can be found in [1] – [14].

A fuzzy subset of a set V is a mapping σ from V to [0, 1]. A fuzzy graph G is a pair of functions $G:(\sigma,\mu)$ where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ, (i.e.) $\mu(xy)\le\sigma(x)\wedge\sigma(y)$ for all $x,y\in V$. The underlying crisp graph of $G:(\sigma,\mu)$ is denoted by $G^*:(V,E)$ where $E\subseteq V\times V$. Throughout this paper, $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ denote two fuzzy graphs with underlying crisp graphs $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ with $|V_i|=p_i, i=1,2$. Also $d_{G_i^*}(u_i)$ denotes the degree of u_i in G_i^* and $d_{\overline{G}_i^*}(u_i)$ denotes the degree of u_i in $\overline{G}_i^*$, where $\overline{G}_i^*$ is the complement of G_i^*.

Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. The degree of a vertex u is $d_G(u)=\sum_{u\ne v}\mu(uv)$. The minimum degree of G is $\delta(G)=\wedge\{d_G(v),\forall\ v\in V\}$ and the maximum degree of G is $\Delta(G)=\vee\{d_G(v),\forall\ v\in V\}$. The total degree of a vertex $u\in V$ is defined by $td_G(u)=\sum_{u\ne v}\mu(uv)+\sigma(u)$. If each vertex in G has same degree k, then G is said to be a regular fuzzy graph or *k* – regular fuzzy graph. If each vertex in G has same total degree k, then G is said to be a totally regular fuzzy graph or *k* – totally regular fuzzy graph.

The order and size of a fuzzy graph G are defined by $O(G)=\sum_{u\in V}\sigma(u)$ and $S(G)=\sum_{uv\in E}\mu(uv)$. Let $G^*:(V,E)$ be a graph and let $e=uv$ be an edge in G^*. Then the degree of an edge $e=uv\in E$ is defined by $d_{G^*}(uv)=d_{G^*}(u)+d_{G^*}(v)-2$. If each and every pair of distinct vertices is joined by an edge, then $G^*:(V,E)$ is said to be complete graph.

Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*:(V,E)$. The degree of an edge uv is $d_G(uv)=d_G(u)+d_G(v)-2\mu(uv)$. This is equivalent to $d_G(uv)=\sum_{\substack{uw\in E\\ w\neq v}}\mu(uw)+\sum_{\substack{wv\in E\\ w\neq u}}\mu(wv)$. The total degree of an edge $uv\in E$ is defined by $td_G(uv)=d_G(u)+d_G(v)-\mu(uv)$. This is equivalent to $td_G(uv)=\sum_{\substack{uw\in E\\ w\neq v}}\mu(uw)+\sum_{\substack{wv\in E\\ w\neq u}}\mu(wv)+\mu(uv)=d_G(uv)+\mu(uv)$. The minimum edge degree and maximum edge degree of G are $\delta_E(G)=\wedge\{d_G(uv),\forall\, uv\in E\}$ and $\Delta_E(G)=\vee\{d_G(uv),\forall\, uv\in E\}$. If each edge in G has same degree k, then G is said to be an edge regular fuzzy graph or k – edge regular fuzzy graph. If each edge in G has same total degree k, then G is said to be a totally edge regular fuzzy graph or k – totally edge regular fuzzy graph.

A fuzzy Graph G is said to be strong, if $\mu(xy)=\sigma(x)\wedge\sigma(y)$ for all $xy\in E$. A fuzzy Graph G is said to be complete, if $\mu(xy)=\sigma(x)\wedge\sigma(y)$ for all $x,y\in V$.

Definition [3]: Let $G^*=G_1^*\underset{\alpha}{\times}G_2^*=(V,E)$ be the alpha product of two graphs G_1^* and G_2^*, where $V=V_1\times V_2$ and $E=\{(u_1,u_2)(v_1,v_2):u_1=v_1,u_2v_2\in E_2\ (or)\ u_1v_1\in E_1,u_2=v_2\ (or)\ u_1v_1\in E_1,u_2v_2\notin E_2\ (or)\ u_1v_1\notin E_1,u_2v_2\in E_2\}$. Then the alpha product of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G=G_1\underset{\alpha}{\times}G_2=G_1\underset{\alpha}{\times}G_2:(\sigma_1\underset{\alpha}{\times}\sigma_2,\mu_1\underset{\alpha}{\times}\mu_2)$ defined by

$(\sigma_1\underset{\alpha}{\times}\sigma_2)(u_1,u_2)=\sigma_1(u_1)\wedge\sigma_2(u_2),\forall(u_1,u_2)\in V$ and

$$(\mu_1\underset{\alpha}{\times}\mu_2)((u_1,u_2)(v_1,v_2))=\begin{cases}\sigma_1(u_1)\wedge\mu_2(u_2v_2), & if\ u_1=v_1,u_2v_2\in E_2\\ \mu_1(u_1v_1)\wedge\sigma_2(u_2), & if\ u_1v_1\in E_1,u_2=v_2\\ \mu_1(u_1v_1)\wedge\sigma_2(u_2)\wedge\sigma_2(v_2), & if\ u_1v_1\in E_1,u_2v_2\notin E_2\\ \sigma_1(u_1)\wedge\sigma_1(v_1)\wedge\mu_2(u_2v_2), & if\ u_1v_1\notin E_1,u_2v_2\in E_2\end{cases}.$$

Definition [4]: Let $G^*=G_1^*\underset{\beta}{\times}G_2^*=(V,E)$ be the beta product of two graphs G_1^* and G_2^*, where $V=V_1\times V_2$ and $E=\{(u_1,u_2)(v_1,v_2):u_1v_1\in E_1,u_2v_2\in E_2\ (or)\ u_1v_1\in E_1,u_2v_2\notin E_2\ (or)\ u_1v_1\notin E_1,u_2v_2\in E_2\}$. Then the beta product of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G=G_1\underset{\beta}{\times}G_2=G_1\underset{\beta}{\times}G_2:(\sigma_1\underset{\beta}{\times}\sigma_2,\mu_1\underset{\beta}{\times}\mu_2)$ defined by

$(\sigma_1\underset{\beta}{\times}\sigma_2)(u_1,u_2)=\sigma_1(u_1)\wedge\sigma_2(u_2),\forall(u_1,u_2)\in V$ and

$$(\mu_1\underset{\beta}{\times}\mu_2)((u_1,u_2)(v_1,v_2))=\begin{cases}\mu_1(u_1v_1)\wedge\mu_2(u_2v_2), & if\ u_1v_1\in E_1,\forall u_2v_2\in E_2\\ \mu_1(u_1v_1)\wedge\sigma_2(u_2)\wedge\sigma_2(v_2), & if\ u_1v_1\in E_1,u_2v_2\notin E_2\\ \sigma_1(u_1)\wedge\sigma_1(v_1)\wedge\mu_2(u_2v_2), & if\ u_1v_1\notin E_1,u_2v_2\in E_2\end{cases}.$$

Definition [4]: Let $G^*=G_1^*\underset{\gamma}{\times}G_2^*=(V,E)$ be the gamma product of two graphs G_1^* and G_2^*, where $V=V_1\times V_2$ and $E=\{(u_1,u_2)(v_1,v_2):u_1=v_1,u_2v_2\in E_2\ (or)\ u_1v_1\in E_1,u_2=v_2\ (or)\ u_1v_1\in E_1,u_2v_2\notin E_2\ (or)\ u_1v_1\notin E_1,u_2v_2\in E_2\ (or)\ u_1v_1\in E_1,u_2v_2\in E_2\}$. Then the gamma product of two fuzzy graphs G_1 and G_2 is a fuzzy graph $G=G_1\underset{\gamma}{\times}G_2=G_1\underset{\gamma}{\times}G_2:(\sigma_1\underset{\gamma}{\times}\sigma_2,\mu_1\underset{\gamma}{\times}\mu_2)$ defined by

$(\sigma_1\underset{\gamma}{\times}\sigma_2)(u_1,u_2)=\sigma_1(u_1)\wedge\sigma_2(u_2),\forall(u_1,u_2)\in V$ and

$$(\mu_1 \underset{\gamma}{\times} \mu_2)((u_1,u_2)(v_1,v_2)) = \begin{cases} \sigma_1(u_1) \wedge \mu_2(u_2v_2), & if\ u_1 = v_1, u_2v_2 \in E_2 \\ \mu_1(u_1v_1) \wedge \sigma_2(u_2), & if\ u_1v_1 \in E_1, u_2 = v_2 \\ \mu_1(u_1v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2), & if\ u_1v_1 \in E_1, u_2v_2 \notin E_2 \\ \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \mu_2(u_2v_2), & if\ u_1v_1 \notin E_1, u_2v_2 \in E_2 \\ \mu_1(u_1v_1) \wedge \mu_2(u_2v_2), & if\ u_1v_1 \in E_1, u_2v_2 \in E_2 \end{cases}.$$

II. Edge Regular Properties of Alpha Product of Two Fuzzy Graphs

Remark 2.1:

If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two edge regular fuzzy graphs, then $G_1 \underset{\alpha}{\times} G_2$ is need not be edge regular fuzzy graph.

Example 2.2:

Consider the two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

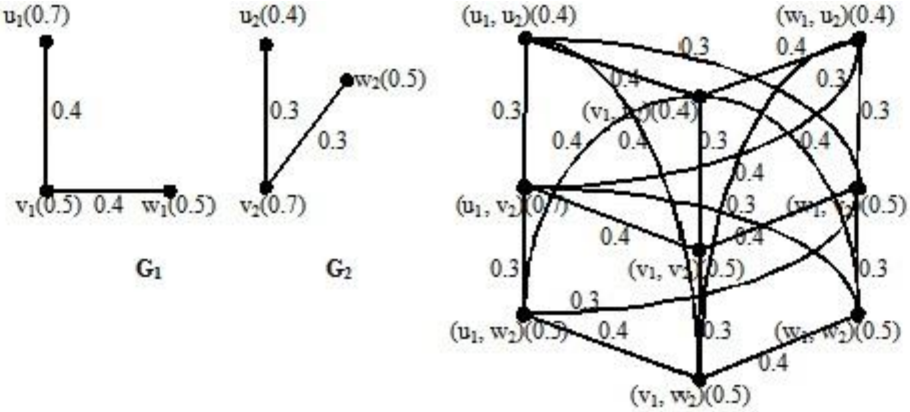

Fig.2.1 $G_1 \underset{\alpha}{\times} G_2$

Here both G_1 and G_2 are edge regular fuzzy graphs of edge degree 0.4 and 0.3. In $G_1 \underset{\alpha}{\times} G_2$, $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2))$ = 2.4 and $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2))$ = 2.5. Hence $G_1 \underset{\alpha}{\times} G_2$ is not an edge regular fuzzy graph.

Remark 2.3:

If $G_1 \underset{\alpha}{\times} G_2$ is an edge regular fuzzy graph, then $G_1 : (\sigma_1, \mu_1)$ (or) $G_2 : (\sigma_2, \mu_2)$ need not be edge regular fuzzy graph.

Example 2.4:

Consider the two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

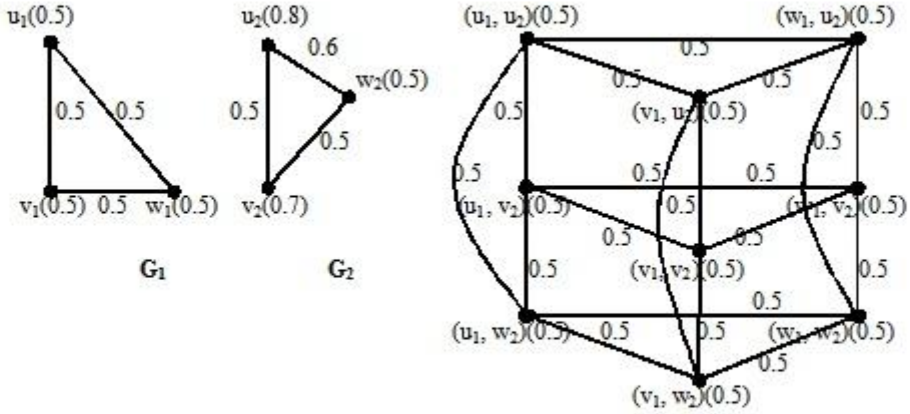

Fig.2.2 $G_1 \underset{\alpha}{\times} G_2$

Here $G_1 \underset{\alpha}{\times} G_2$ is 3 – edge regular fuzzy graph. But G_2 is not an edge regular fuzzy graph.

Theorem 2.5 [10]:

Let G: (σ, μ) be a fuzzy graph on G^*: (V, E) with G^* is k – regular. Then μ is constant if and only if G is both regular and edge regular.

Theorem 2.6[9]:

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs.

Suppose that $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$. Then for any $(u_1, u_2)(v_1, v_2) \in E$,

(1). When $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + d_{G_1}(u_1)(d_{\overline{G_2^*}}(u_2) + d_{\overline{G_2^*}}(v_2) + 2) + d_{\overline{G_1^*}}(u_1)(d_{G_2}(u_2) + d_{G_2}(v_2)),$$

(2). When $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + d_{G_2}(u_2)(d_{\overline{G_1^*}}(u_1) + d_{\overline{G_1^*}}(v_1) + 2) + d_{\overline{G_2^*}}(u_2)(d_{G_1}(u_1) + d_{G_1}(v_1)),$$

(3). When $u_1v_1 \in E_1, u_2v_2 \notin E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + d_{G_2}(u_2)[1 + d_{\overline{G_1^*}}(u_1)] + d_{G_2}(v_2)[1 + d_{\overline{G_1^*}}(v_1)] + d_{\overline{G_2^*}}(u_2)d_{G_1}(u_1)$$
$$+ d_{\overline{G_2^*}}(v_2)d_{G_1}(v_1) \text{ and}$$

(4). When $u_1v_1 \notin E_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_2}(u_2v_2) + d_{G_1}(u_1)[1 + d_{\overline{G_2^*}}(u_2)] + d_{G_1}(v_1)[1 + d_{\overline{G_2^*}}(v_2)] + d_{\overline{G_1^*}}(u_1)d_{G_2}(u_2)$$
$$+ d_{\overline{G_1^*}}(v_1)d_{G_2}(v_2).$$

Theorem 2.7[9]:

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs.

If $\sigma_1 \leq \mu_2$ and σ_1 is a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$, then for any $(u_1, u_2)(v_1, v_2) \in E$,

(a). When $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = c_1(d_{G_2^*}(u_2) + d_{G_2^*}(v_2))(d_{\overline{G_1^*}}(u_1) + 1) - 2c_1$$
$$+ d_{G_1}(u_1)(d_{\overline{G_2^*}}(u_2) + d_{\overline{G_2^*}}(v_2) + 2),$$

(b). When $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + c_1 d_{G_2^*}(u_2)(2 + d_{\overline{G_1^*}}(u_1) + d_{\overline{G_1^*}}(v_1))$$
$$+ d_{\overline{G_2^*}}(u_2)(d_{G_1}(u_1) + d_{G_1}(v_1)),$$

(c). When $u_1v_1 \in E_1, u_2v_2 \notin E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + c_1 d_{G_2^*}(u_2)(1+d_{\overline{G}_1^*}(u_1)) + c_1 d_{G_2^*}(v_2)(1+d_{\overline{G}_1^*}(v_1))$$
$$+ d_{G_1}(u_1) d_{\overline{G}_2^*}(u_2) + d_{G_1}(v_1) d_{\overline{G}_2^*}(v_2) \text{ and}$$

(d). When $u_1v_1 \notin E_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1) + d_{G_1}(v_1) + c_1 d_{G_2^*}(u_2)(1+d_{\overline{G}_1^*}(u_1)) + c_1 d_{G_2^*}(v_2)(1+d_{\overline{G}_1^*}(v_1))$$
$$+ d_{G_1}(u_1) d_{\overline{G}_2^*}(u_2) + d_{G_1}(v_1) d_{\overline{G}_2^*}(v_2) - 2c_1 .$$

Theorem 2.8:

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs and let $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ be complete graphs.

Suppose that $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$. Then for any $(u_1,u_2)(v_1,v_2) \in E$,

(1). When $u_1 = v_1, u_2v_2 \in E_2$, $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + 2d_{G_1}(u_1)$ and

(2). When $u_2 = v_2, u_1v_1 \in E_1$, $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2)) = 2d_{G_2}(u_2) + d_{G_1}(u_1v_1)$.

Proof:

By definition, for any $((u_1,u_2)(v_1,v_2)) \in E$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,v_2)) = \sum_{(u_1,u_2)(w_1,w_2)\in E} (\mu_1 \underset{\alpha}{\times} \mu_2)((u_1,u_2)(w_1,w_2)) + \sum_{(w_1,w_2)(v_1,v_2)\in E} (\mu_1 \underset{\alpha}{\times} \mu_2)((w_1,w_2)(v_1,v_2))$$
$$-2(\mu_1 \underset{\alpha}{\times} \mu_2)((u_1,u_2)(v_1,v_2))$$
$$= \sum_{u_2w_2\in E_2, u_1=w_1} \sigma_1(u_1) \wedge \mu_2(u_2w_2) + \sum_{u_1w_1\in E_1, u_2=w_2} \mu_1(u_1w_1) \wedge \sigma_2(u_2)$$
$$+ \sum_{u_1w_1\in E_1, u_2w_2\notin E_2} \mu_1(u_1w_1) \wedge \sigma_2(u_2) \wedge \sigma_2(w_2) + \sum_{u_1w_1\notin E_1, u_2w_2\in E_2} \sigma_1(u_1) \wedge \sigma_1(w_1) \wedge \mu_2(u_2w_2)$$
$$+ \sum_{w_2v_2\in E_2, w_1=u_1} \sigma_1(v_1) \wedge \mu_2(w_2v_2) + \sum_{w_1v_1\in E_1, w_2=v_2} \mu_1(w_1v_1) \wedge \sigma_2(v_2) + \sum_{w_1v_1\in E_1, w_2v_2\notin E_2} \mu_1(w_1v_1) \wedge \sigma_2(w_2) \wedge \sigma_2(v_2)$$
$$+ \sum_{w_1v_1\notin E_1, w_2v_2\in E_2} \sigma_1(w_1) \wedge \sigma_1(v_1) \wedge \mu_2(w_2v_2) - 2(\mu_1 \underset{\alpha}{\times} \mu_2)((u_1,u_2)(v_1,v_2)) \quad \text{.................................} \quad (2.1)$$

Given G_1^* and G_2^* are complete underlying crisp graphs.

Then edge set of underlying crisp graph of $G_1 \underset{\alpha}{\times} G_2$ is $E = \{(u_1,u_2)(v_1,v_2) : u_1 = v_1, u_2v_2 \in E_2 \quad (or)$ $u_1v_1 \in E_1, u_2 = v_2\}$.

From (2.1), when $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = \sum_{u_2w_2\in E_2, u_1=w_1} \sigma_1(u_1) \wedge \mu_2(u_2w_2) + \sum_{u_1w_1\in E_1, u_2=w_2} \mu_1(u_1w_1) \wedge \sigma_2(u_2)$$
$$+ \sum_{w_2v_2\in E_2, w_1=u_1} \sigma_1(u_1) \wedge \mu_2(w_2v_2) + \sum_{w_1u_1\in E_1, w_2=v_2} \mu_1(w_1u_1) \wedge \sigma_2(v_2) - 2(\mu_1 \underset{\alpha}{\times} \mu_2)((u_1,u_2)(v_1,v_2)).$$
$$= \sum_{u_2w_2\in E_2} \mu_2(u_2w_2) + \sum_{u_1w_1\in E_1} \mu_1(u_1w_1) + \sum_{w_2v_2\in E_2} \mu_2(w_2v_2) + \sum_{w_1u_1\in E_1} \mu_1(w_1u_1) - 2\mu_2(u_2v_2).$$
$$= \sum_{u_2w_2\in E_2} \mu_2(u_2w_2) + \sum_{w_2v_2\in E_2} \mu_2(w_2v_2) - 2\mu_2(u_2v_2) + d_{G_1}(u_1) + d_{G_1}(u_1).$$
$$\therefore d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + 2d_{G_1}(u_1).$$

(2). Proof is similar to the proof of (1).

Theorem: 2.9

Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two regular fuzzy graphs of same degree with $\sigma_1 \geq \mu_2$ and $\sigma_2 \geq \mu_1$. Let $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ be complete graphs. Then G_1 and G_2 are edge regular fuzzy graphs of same degree if and only if $G_1 \underset{\alpha}{\times} G_2$ is an edge regular fuzzy graph.

Proof:

Let $d_{G_1}(u_1)=d_{G_2}(u_2)=m, \forall u_1 \in V_1 \,\&\, u_2 \in V_2$, where m is a constant.

Given G_1^* and G_2^* are complete underlying crisp graphs.

Then edge set of underlying crisp graph of $G_1 \underset{\alpha}{\times} G_2$ is $E=\{(u_1,u_2)(v_1,v_2): u_1=v_1, u_2v_2 \in E_2 \quad (or)$ $u_1v_1 \in E_1, u_2=v_2\}$.

Assume that G_1 and G_2 are k – edge regular fuzzy graphs, where k is a constant.

Then $d_{G_1}(u_1v_1)=d_{G_2}(u_2v_2)=k, \forall u_1v_1 \in E_1 \,\&\, u_2v_2 \in E_2$.

By theorem 2.8, when $u_1=v_1, u_2v_2 \in E_2$,

$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2)+2d_{G_1}(u_1)$ ………………………..…………………..…… (2.2)

$\therefore d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = k+2m$ …………………………………………………………………... (2.3)

By theorem 2.8, when $u_2=v_2, u_1v_1 \in E_1$,

$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2)) = 2d_{G_2}(u_2)+d_{G_1}(u_1v_1)$ …………..…...………………………………….… (2.4)

$\therefore d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2)) = 2m+k$ ………………………………………………………………….….. (2.5)

From (2.3) and (2.5), $G_1 \underset{\alpha}{\times} G_2$ is an edge regular fuzzy graph of degree $k+2m$.

Conversely, assume that $G_1 \underset{\alpha}{\times} G_2$ is an edge regular fuzzy graph.

To prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Let $u_1v_1, w_1x_1 \in E_1$ be any two edges of G_1. Fix $u \in V_2$.

Then $(u_1,u)(v_1,u) \,\&\, (w_1,u)(x_1,u) \in E$, $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u)(v_1,u)) = d_{G_1 \underset{\alpha}{\times} G_2}((w_1,u)(x_1,u))$.

$$2d_{G_2}(u)+d_{G_1}(u_1v_1) = 2d_{G_2}(u)+d_{G_1}(w_1x_1) \quad \text{(Using (2.4))}$$

$$2m+d_{G_1}(u_1v_1) = 2m+d_{G_1}(w_1x_1)$$

$$d_{G_1}(u_1v_1) = d_{G_1}(w_1x_1), \ \forall \ u_1v_1 \,\&\, w_1x_1 \in E_1.$$

$\therefore$ G_1 is an edge regular fuzzy graph.

Similarly, G_2 is an edge regular fuzzy graph.

Now, to prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Suppose that G_1 is k_1 – edge regular fuzzy graph and G_2 is k_2 – edge regular fuzzy graph with $k_1 \neq k_2$.

$\therefore$ $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = 2d_{G_1}(u_1)+d_{G_2}(u_2v_2)$ (Using (2.2))

$= 2m+k_2$ ……..…………………………..……...…………….……………. (2.6)

$\therefore$ $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1)+2d_{G_2}(u_2)$ (Using (2.4))

$= k_1+2m$ …………….………………….………......………….…..…………. (2.7)

From (2.6) and (2.7), $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) \neq d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2))$, since $k_1 \neq k_2$.

This is a contradiction to our assumption that $G_1 \underset{\alpha}{\times} G_2$ is an edge regular fuzzy graph.

Hence G_1 and G_2 are edge regular fuzzy graphs of same degree.

Theorem: 2.10

Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two regular fuzzy graphs of same degree with $\sigma_1\geq\mu_2$ and $\sigma_2\geq\mu_1$. Let $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ be regular graphs of same degree with $|V_1|=|V_2|$. Then G_1 and G_2 are edge regular fuzzy graphs of same degree if and only if $G_1\underset{\alpha}{\times}G_2$ is an edge regular fuzzy graph.

Proof:

Let $d_{G_1}(u_1)=d_{G_2}(u_2)=m$, $d_{G_1^*}(u_1)=d_{G_2^*}(u_2)=n,$ $d_{\overline{G_1^*}}(u_1)=d_{\overline{G_2^*}}(u_2)=p-n,$ $\forall u_1\in V_1\,\&\,u_2\in V_2$, where $m,n\,\&\,|V_1|=|V_2|=p$ are constants.

Assume that G_1 and G_2 are $k-$edge regular fuzzy graphs.

Then $d_{G_1}(u_1v_1)=d_{G_2}(u_2v_2)=k,\forall u_1v_1\in E_1\,\&\,u_2v_2\in E_2$, where k is a constant.

By theorem 2.6, When $u_1=v_1, u_2v_2\in E_2$,

$$d_{G_1\underset{\alpha}{\times}G_2}((u_1,u_2)(u_1,v_2))=d_{G_2}(u_2v_2)+d_{G_1}(u_1)(d_{\overline{G_2^*}}(u_2)+d_{\overline{G_2^*}}(v_2)+2)+d_{\overline{G_1^*}}(u_1)(d_{G_2}(u_2)+d_{G_2}(v_2)).$$

$=k+m(p-n+p-n+2)+(p-n)(m+m)$.

$=k+2m(p-n+1)+(p-n)(2m)$.

$=k+2m(2(p-n)+1)$ ………………………………………………………………………….... (2.8)

By theorem 2.6, When $u_2=v_2, u_1v_1\in E_1$,

$$d_{G_1\underset{\alpha}{\times}G_2}((u_1,u_2)(v_1,u_2))=d_{G_1}(u_1v_1)+d_{G_2}(u_2)(d_{\overline{G_1^*}}(u_1)+d_{\overline{G_1^*}}(v_1)+2)+d_{\overline{G_2^*}}(u_2)(d_{G_1}(u_1)+d_{G_1}(v_1)).$$

$=k+m(p-n+p-n+2)+(p-n)(m+m)$.

$=k+2m(p-n+1)+(p-n)(2m)$.

$=k+2m(2(p-n)+1)$ ………………………………………………………………………….... (2.9)

By theorem 2.6, for any $(u_1,u_2)(v_1,v_2)\in E$,

$$d_{G_1\underset{\alpha}{\times}G_2}((u_1,u_2)(v_1,v_2))=d_{G_1}(u_1v_1)+d_{G_2}(u_2)[1+d_{\overline{G_1^*}}(u_1)]+d_{G_2}(v_2)[1+d_{\overline{G_1^*}}(v_1)]+d_{\overline{G_2^*}}(u_2)d_{G_1}(u_1)$$
$$+d_{\overline{G_2^*}}(v_2)d_{G_1}(v_1),\text{ when } u_1v_1\in E_1, u_2v_2\notin E_2.$$

$=k+m(1+p-n)+m(1+p-n)+(p-n)m+(p-n)m$.

$=k+2m(p-n+1)+(p-n)(2m)$.

$=k+2m(2(p-n)+1)$ ………………………………………………………………………….. (2.10)

By theorem 2.6, for any $(u_1,u_2)(v_1,v_2)\in E$,

$$d_{G_1\underset{\alpha}{\times}G_2}((u_1,u_2)(v_1,v_2))=d_{G_2}(u_2v_2)+d_{G_1}(u_1)[1+d_{\overline{G_2^*}}(u_2)]+d_{G_1}(v_1)[1+d_{\overline{G_2^*}}(v_2)]+d_{\overline{G_1^*}}(u_1)d_{G_2}(u_2)$$
$$+d_{\overline{G_1^*}}(v_1)d_{G_2}(v_2),\text{ when } u_1v_1\notin E_1, u_2v_2\in E_2.$$

$=k+m(1+p-n)+m(1+p-n)+(p-n)m+(p-n)m$.

$=k+2m(p-n+1)+(p-n)(2m)$.

$=k+2m(2(p-n)+1)$ ………………………………………………………………………….. (2.11)

From (2.8), (2.9), (2.10) and (2.11), $G_1\underset{\alpha}{\times}G_2$ is an edge regular fuzzy graph.

Conversely, assume that $G_1\underset{\alpha}{\times}G_2$ is an edge regular fuzzy graph.

To prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Let $u_1v_1, w_1x_1\in E_1$ be any two edges of G_1. Fix $u\in V_2$.

Then $(u_1,u)(v_1,u)\,\&\,(w_1,u)(x_1,u)\in E$, $d_{G_1\underset{\alpha}{\times}G_2}((u_1,u)(v_1,u))=d_{G_1\underset{\alpha}{\times}G_2}((w_1,u)(x_1,u))$.

$$d_{G_1}(u_1v_1)+d_{G_2}(u)(d_{\overline{G_1^*}}(u_1)+d_{\overline{G_1^*}}(v_1)+2)+d_{\overline{G_2^*}}(u)(d_{G_1}(u_1)+d_{G_1}(v_1))$$

$$= d_{G_1}(w_1x_1) + d_{G_2}(u)(d_{\overline{G}_1^*}(w_1) + d_{\overline{G}_1^*}(x_1) + 2) + d_{\overline{G}_2^*}(u)(d_{G_1}(w_1) + d_{G_1}(x_1))$$

$$d_{G_1}(u_1v_1) + m(p-n+p-n+2) + (p-n)(m+m)$$

$$= d_{G_1}(w_1x_1) + m(p-n+p-n+2) + (p-n)(m+m)$$

$$d_{G_1}(u_1v_1) = d_{G_1}(w_1x_1),\ \forall\ u_1v_1\ \&\ w_1x_1 \in E_1.$$

$\therefore\ G_1$ is an edge regular fuzzy graph.

Similarly, G_2 is an edge regular fuzzy graph.

Now, to prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Suppose that G_1 is k_1 – edge regular fuzzy graph and G_2 is k_2 – edge regular fuzzy graph with $k_1 \neq k_2$.

$$\therefore\ d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + d_{G_1}(u_1)(d_{\overline{G}_2^*}(u_2) + d_{\overline{G}_2^*}(v_2) + 2)$$

$$+ d_{\overline{G}_1^*}(u_1)(d_{G_2}(u_2) + d_{G_2}(v_2))$$

$$= k_2 + m(p-n+p-n+2) + (p-n)(m+m)$$

$$= k_2 + 2m(2(p-n)+1) \quad \text{……………………………………………………. (2.12)}$$

$$\therefore\ d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + d_{G_2}(u_2)(d_{\overline{G}_1^*}(u_1) + d_{\overline{G}_1^*}(v_1) + 2)$$

$$+ d_{\overline{G}_2^*}(u_2)(d_{G_1}(u_1) + d_{G_1}(v_1))$$

$$= k_1 + m(p-n+p-n+2) + (p-n)(m+m)$$

$$= k_1 + 2m(2(p-n)+1) \quad \text{……………………………………………………. (2.13)}$$

From (2.12) and (2.13), $d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) \neq d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2))$, since $k_1 \neq k_2$.

This is a contradiction to our assumption that $G_1 \underset{\alpha}{\times} G_2$ is an edge regular fuzzy graph.

Hence G_1 and G_2 are edge regular fuzzy graphs of same degree.

Theorem: 2.11

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $\sigma_1 \leq \mu_2$ and let σ_1 be a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$. Let $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ be regular underlying crisp graphs of same degree with $|V_1| = |V_2|$. If G_1 is strong, then $G_1 \underset{\alpha}{\times} G_2$ is an edge regular fuzzy graph.

Proof:

Given $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $\sigma_1 \leq \mu_2$ and σ_1 is a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$ & G_1^* and G_2^* are regular underlying crisp graphs of same degree.

Since G_1 is strong, μ_1 is a constant function.

Then by theorem 2.5, $G_1 : (\sigma_1, \mu_1)$ is both regular and edge regular.

Thus $d_{G_1}(u_1v_1) = k, d_{G_1}(u_1) = m$, $d_{G_1^*}(u_1) = d_{G_2^*}(u_2) = n,$ $d_{\overline{G}_1^*}(u_1) = d_{\overline{G}_2^*}(u_2) = p-n,$ $\forall u_1 \in V_1, u_1v_1 \in E_1$ & $u_2 \in V_2$, where k, m, n & $|V_1| = |V_2| = p$ are constants.

By theorem 2.7, for any $(u_1,u_2)(v_1,v_2) \in E$,

Case 1: When $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(u_1,v_2)) = c_1(d_{G_2^*}(u_2) + d_{G_2^*}(v_2))(d_{\overline{G}_1^*}(u_1) + 1) - 2c_1$$

$$+\ d_{G_1}(u_1)(d_{\overline{G}_2^*}(u_2) + d_{\overline{G}_2^*}(v_2) + 2).$$

$$= c_1(2n)(p-n+1) - 2c_1 + m(2(p-n)+2).$$

$$= 2(m + c_1n)(p-n+1) - 2c_1 \quad \text{……………………………………………………………………. (2.14)}$$

Case 2: When $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + c_1 d_{G_2^*}(u_2)(2 + d_{\overline{G_1^*}}(u_1) + d_{\overline{G_1^*}}(v_1))$$

$$+ d_{\overline{G_2^*}}(u_2)(d_{G_1}(u_1) + d_{G_1}(v_1)).$$

$= k + c_1 n(2 + 2(p-n)) + (p-n)(2m)$.

$= 2m - 2c_1 + c_1 n(2 + 2(p-n)) + (p-n)(2m)$. (By definition of edge degree $k = 2m - 2c_1$)

$= 2(m + c_1 n)(p - n + 1) - 2c_1$ ………………………………………………………………………. (2.15)

Case 3: When $u_1v_1 \in E_1, u_2v_2 \notin E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + c_1 d_{G_2^*}(u_2)(1 + d_{\overline{G_1^*}}(u_1)) + c_1 d_{G_2^*}(v_2)(1 + d_{\overline{G_1^*}}(v_1))$$

$$+ d_{G_1}(u_1) d_{\overline{G_2^*}}(u_2) + d_{G_1}(v_1) d_{\overline{G_2^*}}(v_2).$$

$= k + c_1 n(1 + p - n) + c_1 n(1 + p - n) + m(p-n) + m(p-n)$.

$= 2m - 2c_1 + 2c_1 n(1 + p - n) + 2m(p-n)$.

$= 2(m + c_1 n)(p - n + 1) - 2c_1$ ………………………………………………………………………. (2.16)

Case 4: When $u_1v_1 \notin E_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\alpha}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1) + d_{G_1}(v_1) + c_1 d_{G_2^*}(u_2)(1 + d_{\overline{G_1^*}}(u_1)) + c_1 d_{G_2^*}(v_2)(1 + d_{\overline{G_1^*}}(v_1))$$

$$+ d_{G_1}(u_1) d_{\overline{G_2^*}}(u_2) + d_{G_1}(v_1) d_{\overline{G_2^*}}(v_2) - 2c_1.$$

$= m + m + c_1 n(1 + p - n) + c_1 n(1 + p - n) + m(p-n) + m(p-n) - 2c_1$.

$= 2m + 2c_1 n(1 + p - n) + 2m(p-n) - 2c_1$.

$= 2(m + c_1 n)(p - n + 1) - 2c_1$ ………………………………………………………………………. (2.17)

From (2.14), (2.15), (2.16) and (2.17), $G_1 \underset{\alpha}{\times} G_2$ is an edge regular fuzzy graph.

III. Edge Regular Properties of Beta Product of Two Fuzzy Graphs

Remark 3.1:

If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two edge regular fuzzy graphs, then $G_1 \underset{\beta}{\times} G_2$ is need not be edge regular fuzzy graph.

Example 3.2:

Consider the two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

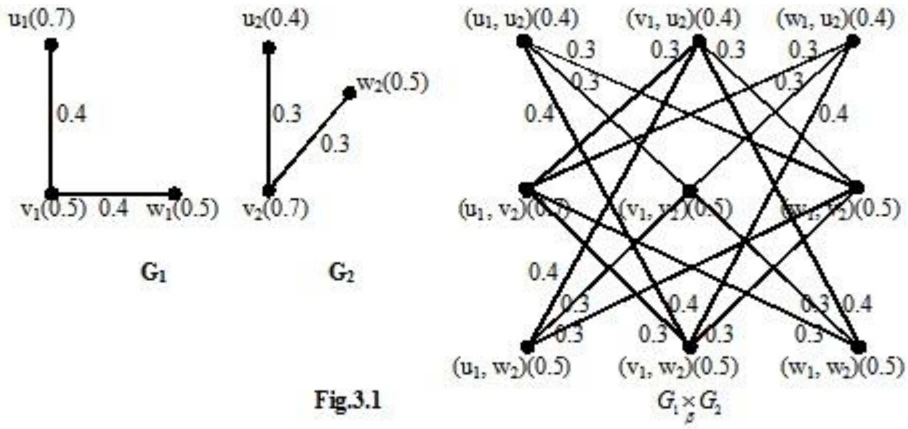

Fig.3.1

Here both G_1 and G_2 are edge regular fuzzy graphs of edge degree 0.4 and 0.3. In $G_1 \underset{\beta}{\times} G_2$, $d_{G_1 \underset{\beta}{\times} G_2}((u_1,u_2)(v_1,v_2))$ = 1.6 and $d_{G_1 \underset{\beta}{\times} G_2}((v_1,u_2)(w_1,v_2))$ = 2.0. Hence $G_1 \underset{\beta}{\times} G_2$ is not an edge regular fuzzy graph.

Remark 3.3:

If $G_1 \underset{\beta}{\times} G_2$ is an edge regular fuzzy graph, then $G_1 : (\sigma_1, \mu_1)$ (or) $G_2 : (\sigma_2, \mu_2)$ need not be edge regular fuzzy graph.

Example 3.4:

Consider the two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$.

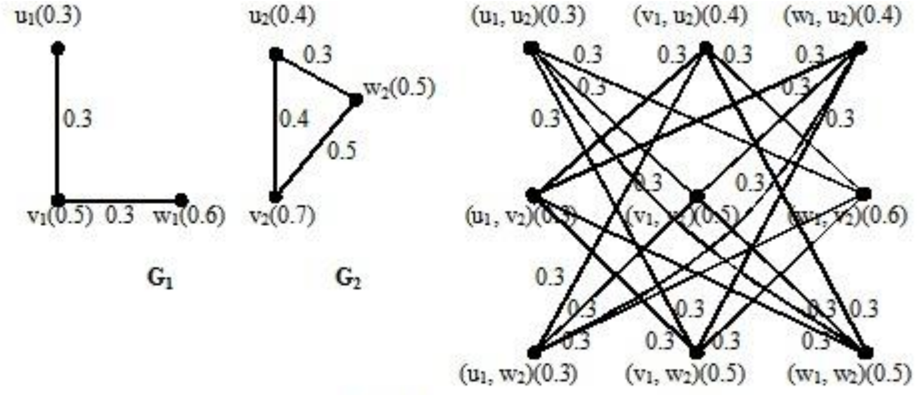

Fig.3.2 $G_1 \underset{\beta}{\times} G_2$

Here $G_1 \underset{\beta}{\times} G_2$ is 1.8 – edge regular fuzzy graph, but G_1 is not an edge regular fuzzy graph.

Theorem 3.5[9]:

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs.

If $\sigma_1 \leq \mu_2$ and σ_1 is a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$, then for any $(u_1, u_2)(v_1, v_2) \in E$,

(a). When $u_1v_1 \in E_1, u_2v_2 \notin E_2$ & $u_1v_1 \in E_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\beta}{\times} G_2}((u_1, u_2)(v_1, v_2)) = (p_2 - 2)(d_{G_1}(u_1) + d_{G_1}(v_1)) + c_1(d_{\overline{G_1^*}}(u_1) d_{G_2^*}(u_2) + d_{\overline{G_1^*}}(v_1) d_{G_2^*}(v_2))$$
$$+ d_{G_1}(u_1v_1) \text{ and}$$

(b). When $u_1v_1 \notin E_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\beta}{\times} G_2}((u_1, u_2)(v_1, v_2)) = (p_2 - 1)(d_{G_1}(u_1) + d_{G_1}(v_1)) + c_1(d_{\overline{G_1^*}}(u_1) d_{G_2^*}(u_2) + d_{\overline{G_1^*}}(v_1) d_{G_2^*}(v_2) - 2).$$

Theorem: 3.6

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $\sigma_1 \leq \mu_2$ and let σ_1 be a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$. Let $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ be regular underlying crisp graphs. If G_1 is strong, then $G_1 \underset{\beta}{\times} G_2$ is an edge regular fuzzy graph.

Proof:

Given $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $\sigma_1 \leq \mu_2$ and let σ_1 be a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$ & $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ be regular underlying crisp graphs.

Since G_1 is strong, μ_1 is a constant function.

Then by theorem 2.5, $G_1 : (\sigma_1, \mu_1)$ is both regular and edge regular.

Thus $d_{G_1}(u_1v_1) = k, d_{G_1}(u_1) = m, d_{G_1^*}(u_1) = n_1, d_{G_2^*}(u_2) = n_2$, $d_{\overline{G_1^*}}(u_1) = p - n_1, d_{\overline{G_2^*}}(u_2) = q - n_2$,

$\forall u_1, v_1 \in V_1$ & $u_2 \in V_2$, where k, m, n_1, n_2 & $|V_1| = p, |V_2| = q$ are constants.

From theorem (3.5), for any $(u_1, u_2)(v_1, v_2) \in E$,

Case 1: $d_{G_1 \underset{\beta}{\times} G_2}((u_1, u_2)(v_1, v_2)) = (p_2 - 2)(d_{G_1}(u_1) + d_{G_1}(v_1)) + c_1(d_{\overline{G_1^*}}(u_1) d_{G_2^*}(u_2) + d_{\overline{G_1^*}}(v_1) d_{G_2^*}(v_2))$

$+d_{G_1}(u_1v_1)$, when $u_1v_1 \in E_1, u_2v_2 \notin E_2$ & $u_1v_1 \in E_1, u_2v_2 \in E_2$.

$=(p_2-2)(2m)+c_1(2(p-n_1)n_2+2m-2c_1$.

$=(p_2-1)(2m)+2c_1((p-n_1)n_2-1)$ …………………………………………………………………. (3.1)

Case 2: $d_{G_1 \underset{\beta}{\times} G_2}((u_1,u_2)(v_1,v_2))=(p_2-1)(d_{G_1}(u_1)+d_{G_1}(v_1))$

$+c_1(d_{\overline{G_1^*}}(u_1)d_{G_2^*}(u_2)+d_{\overline{G_1^*}}(v_1)d_{G_2^*}(v_2)-2)$, when $u_1v_1 \notin E_1, u_2v_2 \in E_2$.

$=(p_2-1)(2m)+c_1(2(p-n_1)n_2-2)$.

$=(p_2-1)(2m)+2c_1((p-n_1)n_2-1)$ …………………………………………………………………... (3.2)

From (3.1) & (3.2) $G_1 \underset{\beta}{\times} G_2$ is an edge regular fuzzy graph.

IV. Edge Regular Properties of Gamma Product of Two Fuzzy Graphs

Remark 4.1:

If $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two edge regular fuzzy graphs, then $G_1 \underset{\gamma}{\times} G_2$ is need not be edge regular fuzzy graph.

Remark 4.2:

If $G_1 \underset{\gamma}{\times} G_2$ is an edge regular fuzzy graph, then $G_1:(\sigma_1,\mu_1)$ (or) $G_2:(\sigma_2,\mu_2)$ need not be edge regular fuzzy graph.

Example 4.3:

Consider the two fuzzy graphs $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$.

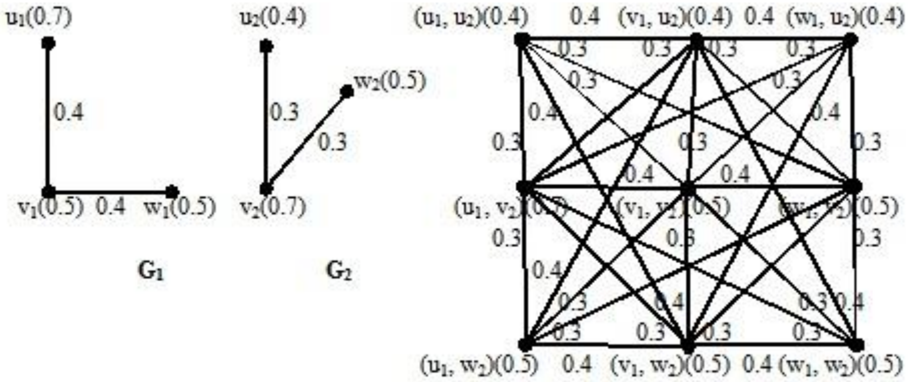

Fig.4.1 $G_1 \underset{\gamma}{\times} G_2$

Here both G_1 and G_2 are edge regular fuzzy graphs of edge degree 0.4 and 0.3. In $G_1 \underset{\gamma}{\times} G_2$, $d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2))$ = 3.7 and $d_{G_1 \underset{\gamma}{\times} G_2}((v_1,u_2)(w_1,v_2))$= 4.1. Hence $G_1 \underset{\gamma}{\times} G_2$ is not an edge regular fuzzy graph.

Example 4.4:

Consider the two fuzzy graphs $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$.

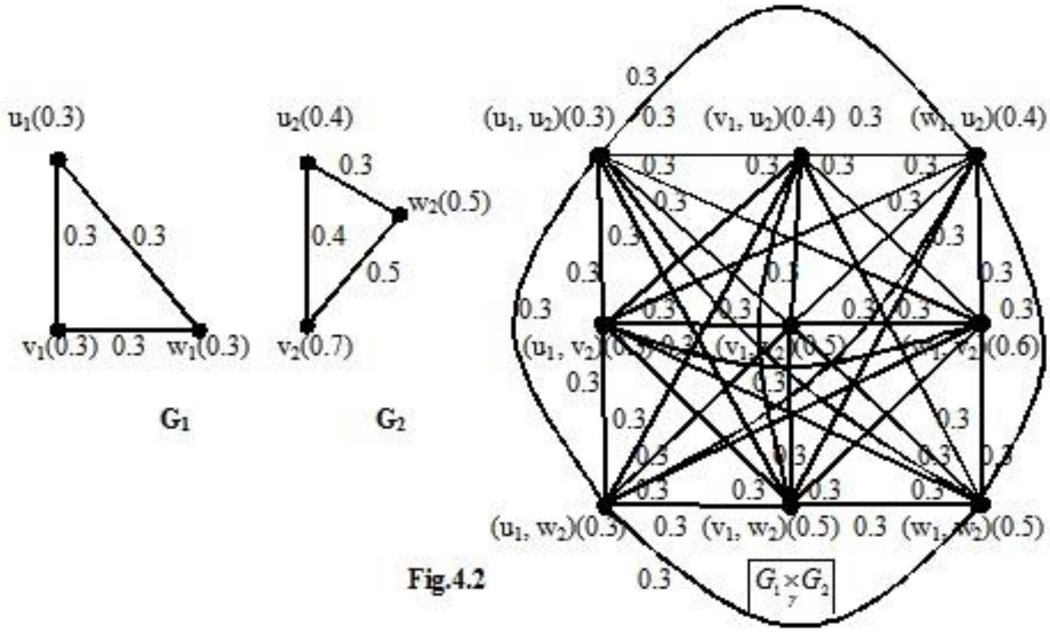

Fig.4.2

Here $G_1 \underset{\gamma}{\times} G_2$ is 1.8 – edge regular fuzzy graph, but G_1 is not an edge regular fuzzy graph.

Theorem 4.5[9]:

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs.

If $\sigma_1 \geq \mu_2, \sigma_2 \geq \mu_1$ and $\mu_1 \geq \mu_2$, then for any $(u_1,u_2)(v_1,v_2) \in E$,

(i). When $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + d_{G_1}(u_1)(d_{\overline{G_2^*}}(u_2) + d_{\overline{G_2^*}}(v_2) + 2) + (p_1 - 1)(d_{G_2}(u_2) + d_{G_2}(v_2)),$$

(ii). When $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + 2p_1 d_{G_2}(u_2) + d_{\overline{G_2^*}}(u_2)(d_{G_1}(u_1) + d_{G_1}(v_1)),$$

(iii). When $u_1v_1 \in E_1, u_2v_2 \notin E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + d_{G_2}(u_2) + d_{G_2}(v_2) + d_{\overline{G_2^*}}(u_2)d_{G_1}(u_1) + d_{\overline{G_2^*}}(v_2)d_{G_1}(v_1)$$
$$+ (p_1 - 1)(d_{G_2}(u_2) + d_{G_2}(v_2)) \text{ and}$$

(iv). When $u_1v_1 \notin E_1, u_2v_2 \in E_2$ & $u_1v_1 \in E_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1) + d_{G_1}(v_1) + d_{G_2}(u_2v_2) + d_{\overline{G_2^*}}(u_2)d_{G_1}(u_1) + d_{\overline{G_2^*}}(v_2)d_{G_1}(v_1)$$
$$+ (p_1 - 1)(d_{G_2}(u_2) + d_{G_2}(v_2)).$$

Theorem 4.6[9]:

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs.

If $\sigma_1 \leq \mu_2$ and σ_1 is a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$, then for any $(u_1,u_2)(v_1,v_2) \in E$,

(i). When $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(u_1,v_2)) = 2p_2 d_{G_1}(u_1) + c_1(d_{G_2^*}(u_2) + d_{G_2^*}(v_2))(d_{\overline{G_1^*}}(u_1) + 1) - 2c_1,$$

(ii). When $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + c_1 d_{G_2^*}(u_2)(2 + d_{\overline{G_1^*}}(u_1) + d_{\overline{G_1^*}}(v_1))$$
$$+ (p_2 - 1)(d_{G_1}(u_1) + d_{G_1}(v_1)),$$

(iii). When $u_1v_1 \in E_1, u_2v_2 \notin E_2$ & $u_1v_1 \in E_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + (p_2 - 1)(d_{G_1}(u_1) + d_{G_1}(v_1)) + c_1 d_{G_2^*}(u_2)(1 + d_{\overline{G_1^*}}(u_1))$$
$$+ c_1 d_{G_2^*}(v_2)(1 + d_{\overline{G_1^*}}(v_1))$$ and

(iv). When $u_1v_1 \notin E_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2)) = p_2(d_{G_1}(u_1) + d_{G_1}(v_1)) + c_1 d_{G_2^*}(u_2)(1 + d_{\overline{G_1^*}}(u_1)) + c_1 d_{G_2^*}(v_2)(1 + d_{\overline{G_1^*}}(v_1))$$
$$-2c_1.$$

Theorem 4.7:

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs and let $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ be complete graphs.

Suppose that $\sigma_1 \geq \mu_2$, $\sigma_2 \geq \mu_1$ and $\mu_1 \geq \mu_2$. Then for any $(u_1,u_2)(v_1,v_2) \in E$,

(1). When $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + 2d_{G_1}(u_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(u_1)d_{G_2}(v_2),$$

(2). When $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,u_2)) = 2d_{G_2}(u_2) + d_{G_1}(u_1v_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(v_1)d_{G_2}(u_2)$$ and

(3). When $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1) + d_{G_1}(v_1) + d_{G_2}(u_2v_2) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(v_1)d_{G_2}(v_2).$$

Proof:

By definition, for any $((u_1,u_2)(v_1,v_2)) \in E$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2)) = \sum_{(u_1,u_2)(w_1,w_2)\in E} (\mu_1 \underset{\gamma}{\times} \mu_2)((u_1,u_2)(w_1,w_2)) + \sum_{(w_1,w_2)(v_1,u_2)\in E} (\mu_1 \underset{\gamma}{\times} \mu_2)((w_1,w_2)(v_1,v_2))$$
$$-2(\mu_1 \underset{\gamma}{\times} \mu_2)((u_1,u_2)(v_1,v_2))$$
$$= \sum_{u_2w_2\in E_2, u_1=w_1} \sigma_1(u_1) \wedge \mu_2(u_2w_2) + \sum_{u_1w_1\in E_1, u_2=w_2} \mu_1(u_1w_1) \wedge \sigma_2(u_2)$$
$$+ \sum_{u_1w_1\in E_1, u_2w_2\notin E_2} \mu_1(u_1w_1) \wedge \sigma_2(u_2) \wedge \sigma_2(w_2) + \sum_{u_1w_1\notin E_1, u_2w_2\in E_2} \sigma_1(u_1) \wedge \sigma_1(w_1) \wedge \mu_2(u_2w_2)$$
$$+ \sum_{u_1w_1\in E_1, u_2w_2\in E_2} \mu_1(u_1w_1) \wedge \mu_2(u_2w_2) + \sum_{w_2v_2\in E_2, w_1=u_1} \sigma_1(v_1) \wedge \mu_2(w_2v_2) + \sum_{w_1v_1\in E_1, w_2=v_2} \mu_1(w_1v_1) \wedge \sigma_2(v_2)$$
$$+ \sum_{w_1v_1\in E_1, w_2v_2\notin E_2} \mu_1(w_1v_1) \wedge \sigma_2(w_2) \wedge \sigma_2(v_2) + \sum_{w_1v_1\notin E_1, w_2v_2\in E_2} \sigma_1(w_1) \wedge \sigma_1(v_1) \wedge \mu_2(w_2v_2)$$
$$+ \sum_{w_1v_1\in E_1, w_2v_2\in E_2} \mu_1(w_1v_1) \wedge \mu_2(w_2v_2) - 2(\mu_1 \underset{\gamma}{\times} \mu_2)((u_1,u_2)(v_1,v_2)) \quad \text{.........} \quad (4.1)$$

Given G_1^* and G_2^* are complete underlying crisp graphs.

Then edge set of underlying crisp graph of $G_1 \underset{\gamma}{\times} G_2$ is $E = \{(u_1,u_2)(v_1,v_2) : u_1 = v_1, u_2v_2 \in E_2 \ (or)$ $u_1v_1 \in E_1, u_2 = v_2 \ (or) \ u_1v_1 \in E_1, u_2v_2 \in E_2\}$.

From (4.1), when $u_1 = v_1, u_2v_2 \in E_2$,

$$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(u_1,v_2)) = \sum_{u_2w_2\in E_2, u_1=w_1} \sigma_1(u_1) \wedge \mu_2(u_2w_2) + \sum_{u_1w_1\in E_1, u_2=w_2} \mu_1(u_1w_1) \wedge \sigma_2(u_2)$$
$$+ \sum_{u_1w_1\in E_1, u_2w_2\in E_2} \mu_1(u_1w_1) \wedge \mu_2(u_2w_2) + \sum_{w_2v_2\in E_2, w_1=u_1} \sigma_1(u_1) \wedge \mu_2(w_2v_2) + \sum_{w_1u_1\in E_1, w_2=v_2} \mu_1(w_1u_1) \wedge \sigma_2(v_2)$$

$$+ \sum_{w_1u_1\in E_1, w_2v_2\in E_2} \mu_1(w_1u_1)\wedge\mu_2(w_2v_2) - 2(\mu_1 \underset{\gamma}{\times} \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_2w_2\in E_2}\mu_2(u_2w_2) + \sum_{u_1w_1\in E_1}\mu_1(u_1w_1) + \sum_{u_1w_1\in E_1, u_2w_2\in E_2}\mu_2(u_2w_2) + \sum_{w_2v_2\in E_2}\mu_2(w_2v_2) + \sum_{w_1u_1\in E_1}\mu_1(w_1u_1)$$

$$+ \sum_{w_1u_1\in E_1, w_2v_2\in E_2}\mu_2(w_2v_2) - 2(\sigma_1(u_1)\wedge\mu_2(u_2v_2))$$

$$= \sum_{u_2w_2\in E_2}\mu_2(u_2w_2) + \sum_{w_2v_2\in E_2}\mu_2(w_2v_2) - 2\mu_2(u_2v_2) + \sum_{u_1w_1\in E_1}\mu_1(u_1w_1) + d_{G_1^*}(u_1)\sum_{u_2w_2\in E_2}\mu_2(u_2w_2)$$

$$+ \sum_{w_1u_1\in E_1}\mu_1(w_1u_1) + d_{G_1^*}(u_1)\sum_{w_2v_2\in E_2}\mu_2(w_2v_2)$$

$$= d_{G_2}(u_2v_2) + d_{G_1}(u_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1}(u_1) + d_{G_1^*}(u_1)d_{G_2}(v_2)$$

$$\therefore\quad d_{G_1\underset{\gamma}{\times}G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + 2d_{G_1}(u_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(u_1)d_{G_2}(v_2).$$

From (4.1), when $u_2 = v_2, u_1v_1 \in E_1$,

$$d_{G_1\underset{\gamma}{\times}G_2}((u_1,u_2)(v_1,u_2)) = \sum_{u_2w_2\in E_2, u_1=w_1}\sigma_1(u_1)\wedge\mu_2(u_2w_2) + \sum_{u_1w_1\in E_1, u_2=w_2}\mu_1(u_1w_1)\wedge\sigma_2(u_2)$$

$$+ \sum_{u_1w_1\in E_1, u_2w_2\in E_2}\mu_1(u_1w_1)\wedge\mu_2(u_2w_2) + \sum_{w_2u_2\in E_2, w_1=u_1}\sigma_1(v_1)\wedge\mu_2(w_2u_2) + \sum_{w_1v_1\in E_1, w_2=u_2}\mu_1(w_1v_1)\wedge\sigma_2(u_2)$$

$$+ \sum_{w_1v_1\in E_1, w_2u_2\in E_2}\mu_1(w_1v_1)\wedge\mu_2(w_2u_2) - 2(\mu_1 \underset{\gamma}{\times} \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{u_2w_2\in E_2}\mu_2(u_2w_2) + \sum_{u_1w_1\in E_1}\mu_1(u_1w_1) + \sum_{u_1w_1\in E_1, u_2w_2\in E_2}\mu_2(u_2w_2) + \sum_{w_2u_2\in E_2}\mu_2(w_2u_2) + \sum_{w_1v_1\in E_1}\mu_1(w_1v_1)$$

$$+ \sum_{w_1v_1\in E_1, w_2u_2\in E_2}\mu_2(w_2u_2) - 2(\mu_1(u_1v_1)\wedge\sigma_2(u_2))$$

$$= \sum_{u_2w_2\in E_2}\mu_2(u_2w_2) + \sum_{u_1w_1\in E_1}\mu_1(u_1w_1) + \sum_{w_1v_1\in E_1}\mu_1(w_1v_1) - 2\mu_1(u_1v_1) + d_{G_1^*}(u_1)\sum_{u_2w_2\in E_2}\mu_2(u_2w_2)$$

$$+ \sum_{w_2u_2\in E_2}\mu_2(w_2u_2) + d_{G_1^*}(v_1)\sum_{w_2u_2\in E_2}\mu_2(w_2u_2)$$

$$= d_{G_2}(u_2) + d_{G_1}(u_1v_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_2}(u_2) + d_{G_1^*}(v_1)d_{G_2}(u_2)$$

$$\therefore\quad d_{G_1\underset{\gamma}{\times}G_2}((u_1,u_2)(v_1,u_2)) = 2d_{G_2}(u_2) + d_{G_1}(u_1v_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(v_1)d_{G_2}(u_2).$$

From (4.1), for any $(u_1,u_2)(v_1,v_2)\in E$, when $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$,

$$d_{G_1\underset{\gamma}{\times}G_2}((u_1,u_2)(v_1,v_2)) = \sum_{u_2w_2\in E_2, u_1=w_1}\sigma_1(u_1)\wedge\mu_2(u_2w_2) + \sum_{u_1w_1\in E_1, u_2=w_2}\mu_1(u_1w_1)\wedge\sigma_2(u_2)$$

$$+ \sum_{u_1w_1\in E_1, u_2w_2\in E_2}\mu_1(u_1w_1)\wedge\mu_2(u_2w_2) + \sum_{w_2v_2\in E_2, w_1=u_1}\sigma_1(v_1)\wedge\mu_2(w_2v_2) + \sum_{w_1v_1\in E_1, w_2=v_2}\mu_1(w_1v_1)\wedge\sigma_2(v_2)$$

$$+ \sum_{w_1v_1\in E_1, w_2v_2\in E_2}\mu_1(w_1v_1)\wedge\mu_2(w_2v_2) - 2(\mu_1 \underset{\gamma}{\times} \mu_2)((u_1,u_2)(v_1,v_2))$$

$$= \sum_{w_2\in V_2}\mu_2(u_2w_2) + \sum_{w_1\in V_1}\mu_1(u_1w_1) + \sum_{u_1w_1\in E_1, u_2w_2\in E_2}\mu_2(u_2w_2) + \sum_{w_2\in V_2}\mu_2(w_2v_2) + \sum_{w_1\in V_1}\mu_1(w_1v_1)$$

$$+ \sum_{w_1v_1\in E_1, w_2v_2\in E_2}\mu_2(w_2v_2) - 2\mu_2(u_2v_2)$$

$$= d_{G_2}(u_2) + d_{G_1}(u_1) + \sum_{u_1w_1\in E_1, w_2\in V_2}\mu_2(u_2w_2) + d_{G_2}(v_2) + d_{G_1}(v_1) + \sum_{w_1v_1\in E_1, w_2\in V_2}\mu_2(w_2v_2) - 2\mu_2(u_2v_2)$$

$$= d_{G_2}(u_2) + d_{G_2}(v_2) + d_{G_1}(u_1) + d_{G_1}(v_1) + d_{G_1^*}(u_1)\sum_{w_2\in V_2}\mu_2(u_2w_2) + d_{G_1^*}(v_1)\sum_{w_2\in V_2}\mu_2(w_2v_2) - 2\mu_2(u_2v_2)$$

$\therefore \quad d_{G_1 \times_\gamma G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1) + d_{G_1}(v_1) + d_{G_2}(u_2v_2) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(v_1)d_{G_2}(v_2)$.

Theorem 4.8:

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two regular fuzzy graphs of same degree with $\sigma_1 \geq \mu_2$, $\sigma_2 \geq \mu_1$ and $\mu_1 \geq \mu_2$. Let $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ be complete graphs. Then G_1 and G_2 are edge regular fuzzy graphs of same degree if and only if $G_1 \times_\gamma G_2$ is an edge regular fuzzy graph.

Proof:

Let $d_{G_1}(u_1) = d_{G_2}(u_2) = m, \forall u_1 \in V_1 \,\&\, u_2 \in V_2$, where m is a constant.

Given G_1^* and G_2^* are complete underlying crisp graphs.

Then edge set of underlying crisp graph of $G_1 \times_\gamma G_2$ is $E = \{(u_1,u_2)(v_1,v_2) : u_1 = v_1, u_2v_2 \in E_2 \ \ (or)$ $u_1v_1 \in E_1, u_2 = v_2 \ (or) \ u_1v_1 \in E_1, u_2v_2 \in E_2\}$.

Assume that G_1 and G_2 are k – edge regular fuzzy graphs, where k is a constant.

Then $d_{G_1}(u_1v_1) = d_{G_2}(u_2v_2) = k, \forall u_1v_1 \in E_1 \,\&\, u_2v_2 \in E_2$.

By theorem 4.7, when $u_1 = v_1, u_2v_2 \in E_2$,

$d_{G_1 \times_\gamma G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + 2d_{G_1}(u_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(u_1)d_{G_2}(v_2)$ ……………... (4.2)

$= k + 2m + n_1m + n_1m$

$= k + 2m(n_1 + 1)$ ……………………...………………………………………………...………. (4.3)

By theorem 4.7, when $u_2 = v_2, u_1v_1 \in E_1$,

$d_{G_1 \times_\gamma G_2}((u_1,u_2)(v_1,u_2)) = d_{G_2}(u_2) + d_{G_1}(u_1v_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_2}(u_2) + d_{G_1^*}(v_1)d_{G_2}(u_2)$ ….. (4.4)

$= m + k + n_1m + m + n_1m$

$= k + 2m(n_1 + 1)$ ……………………...………………………………………………………. (4.5)

By theorem 4.7, for any $(u_1,u_2)(v_1,v_2) \in E$, when $u_1v_1 \in E_1$ and $u_2v_2 \in E_2$,

$d_{G_1 \times_\gamma G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1) + d_{G_1}(v_1) + d_{G_2}(u_2v_2) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(v_1)d_{G_2}(v_2)$ …... (4.6)

$= m + m + k + n_1m \ n_1m$

$= k + 2m(n_1 + 1)$ ……………………...……………………..…………………………………… (4.7)

From (4.3), (4.5) & (4.7), $G_1 \times_\gamma G_2$ is an edge regular fuzzy graph.

Conversely, assume that $G_1 \times_\gamma G_2$ is an edge regular fuzzy graph.

To prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Let $u_1v_1, w_1x_1 \in E_1$ be any two edges of G_1. Fix $u \in V_2$.

Then $(u_1,u)(v_1,u) \,\&\, (w_1,u)(x_1,u) \in E$, $d_{G_1 \times_\gamma G_2}((u_1,u)(v_1,u)) = d_{G_1 \times_\gamma G_2}((w_1,u)(x_1,u))$.

$$d_{G_2}(u) + d_{G_1}(u_1v_1) + d_{G_1^*}(u_1)d_{G_2}(u) + d_{G_2}(u) + d_{G_1^*}(v_1)d_{G_2}(u)$$
$$= d_{G_2}(u) + d_{G_1}(w_1x_1) + d_{G_1^*}(w_1)d_{G_2}(u) + d_{G_2}(u) + d_{G_1^*}(x_1)d_{G_2}(u) \quad \text{(Using (4.4))}$$

$$d_{G_1}(u_1v_1) + 2m(n_1+1) = d_{G_1}(w_1x_1) + 2m(n_1+1)$$

$$d_{G_1}(u_1v_1) = d_{G_1}(w_1x_1), \ \forall \ u_1v_1 \,\&\, w_1x_1 \in E_1.$$

$\therefore$ G_1 is an edge regular fuzzy graph.

Similarly, G_2 is an edge regular fuzzy graph.

Now, to prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Suppose that G_1 is k_1 – edge regular fuzzy graph and G_2 is k_2 – edge regular fuzzy graph with $k_1 \neq k_2$.

$\therefore\ d_{G_1 \times_\gamma G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + 2d_{G_1}(u_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_1^*}(u_1)d_{G_2}(v_2)$ (Using (4.2))

$= k_2 + 2m(n_1+1)$ …………………………………………………………….. (4.8)

$\therefore d_{G_1 \times_\gamma G_2}((u_1,u_2)(v_1,u_2)) = d_{G_2}(u_2) + d_{G_1}(u_1v_1) + d_{G_1^*}(u_1)d_{G_2}(u_2) + d_{G_2}(u_2) + d_{G_1^*}(v_1)d_{G_2}(u_2)$

(Using (4.4))

$= k_1 + 2m(n_1+1)$ …………………………………………………………….. (4.9)

From (4.8) and (4.9), $d_{G_1 \times_\gamma G_2}((u_1,u_2)(u_1,v_2)) \neq d_{G_1 \times_\gamma G_2}((u_1,u_2)(v_1,u_2))$, since $k_1 \neq k_2$.

This is a contradiction to our assumption that $G_1 \times_\gamma G_2$ is an edge regular fuzzy graph.

Hence G_1 and G_2 are edge regular fuzzy graphs of same degree.

Theorem: 4.8

Let $G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ be two regular fuzzy graphs of same degree with $\sigma_1 \geq \mu_2, \sigma_2 \geq \mu_1$ and $\mu_1 \geq \mu_2$. Let $G_1^*:(V_1,E_1)$ and $G_2^*:(V_2,E_2)$ be regular graphs of same degree with $|V_1| = |V_2|$. Then G_1 and G_2 are edge regular fuzzy graphs of same degree if and only if $G_1 \times_\gamma G_2$ is an edge regular fuzzy graph.

Proof:

Let $d_{G_1}(u_1) = d_{G_2}(u_2) = m$, $d_{G_1^*}(u_1) = d_{G_2^*}(u_2) = n,$ $d_{\overline{G}_1^*}(u_1) = d_{\overline{G}_2^*}(u_2) = p - n,$ $\forall u_1 \in V_1 \,\&\, u_2 \in V_2$, where $m, n \,\&\, |V_1| = |V_2| = p$ are constants.

Assume that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Then $d_{G_1}(u_1v_1) = d_{G_2}(u_2v_2) = k, \forall u_1v_1 \in E_1 \,\&\, u_2v_2 \in E_2$, where k is a constant.

By theorem 4.5, when $u_1 = v_1, u_2v_2 \in E_2$,

$d_{G_1 \times_\gamma G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + d_{G_1}(u_1)(d_{\overline{G}_2^*}(u_2) + d_{\overline{G}_2^*}(v_2) + 2) + (p-1)(d_{G_2}(u_2) + d_{G_2}(v_2))$

$= k + m(p-n+p-n+2) + (p-1)(m+m)$

$= k + 2m(2p-n)$ …………………………………………………………………………………. (4.10)

By theorem 4.5, when $u_2 = v_2, u_1v_1 \in E_1$,

$d_{G_1 \times_\gamma G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + 2pd_{G_2}(u_2) + d_{\overline{G}_2^*}(u_2)(d_{G_1}(u_1) + d_{G_1}(v_1))$

$= k + 2pm + (p-n)(m+m)$

$= k + 2m(2p-n)$ …………………………………………………………………………………. (4.11)

By theorem 4.5, when $u_1v_1 \in E_1, u_2v_2 \notin E_2$,

$d_{G_1 \times_\gamma G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + d_{G_2}(u_2) + d_{G_2}(v_2) + d_{\overline{G}_2^*}(u_2)d_{G_1}(u_1) + d_{\overline{G}_2^*}(v_2)d_{G_1}(v_1)$

$+ (p-1)(d_{G_2}(u_2) + d_{G_2}(v_2))$

$= k + m + m + (p-n)m + (p-n)m + (p-1)(m+m)$

$= k + 2m(2p-n)$ …………………………………………………………………………………...... (4.12)

By theorem 4.5, when $u_1v_1 \notin E_1, u_2v_2 \in E_2$ & $u_1v_1 \in E_1, u_2v_2 \in E_2$,

$d_{G_1 \times_\gamma G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1) + d_{G_1}(v_1) + d_{G_2}(u_2v_2) + d_{\overline{G}_2^*}(u_2)d_{G_1}(u_1) + d_{\overline{G}_2^*}(v_2)d_{G_1}(v_1)$

$+ (p-1)(d_{G_2}(u_2) + d_{G_2}(v_2))$

$= m + m + k + (p-n)m + (p-n)m + (p-1)(m+m)$

$= k+2m(2p-n)$ ……………………..……………………………………………………………..… (4.13)

From (4.10), (4.11), (4.12) & (4.13), $G_1 \underset{\gamma}{\times} G_2$ is an edge regular fuzzy graph.

Conversely, assume that $G_1 \underset{\gamma}{\times} G_2$ is an edge regular fuzzy graph.

To prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Let $u_1v_1, w_1x_1 \in E_1$ be any two edges of G_1. Fix $u \in V_2$.

Then $(u_1,u)(v_1,u)$ & $(w_1,u)(x_1,u) \in E$, $d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u)(v_1,u)) = d_{G_1 \underset{\gamma}{\times} G_2}((w_1,u)(x_1,u))$.

$$d_{G_1}(u_1v_1) + 2pd_{G_2}(u) + d_{\overline{G}_2^*}(u)(d_{G_1}(u_1) + d_{G_1}(v_1))$$
$$= d_{G_1}(w_1x_1) + 2pd_{G_2}(u) + d_{\overline{G}_2^*}(u)(d_{G_1}(w_1) + d_{G_1}(x_1))$$
$$d_{G_1}(u_1v_1) + 2pm + (p-n)(m+m) = d_{G_1}(w_1x_1) + 2pm + (p-n)(m+m)$$
$$d_{G_1}(u_1v_1) = d_{G_1}(w_1x_1),\ \forall\ u_1v_1\ \&\ w_1x_1 \in E_1.$$

$\therefore$ G_1 is an edge regular fuzzy graph.

Similarly, G_2 is an edge regular fuzzy graph.

Now, to prove that G_1 and G_2 are edge regular fuzzy graphs of same degree.

Suppose that G_1 is k_1 – edge regular fuzzy graph and G_2 is k_2 – edge regular fuzzy graph with $k_1 \neq k_2$.

$$\therefore\ d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(u_1,v_2)) = d_{G_2}(u_2v_2) + d_{G_1}(u_1)(d_{\overline{G}_2^*}(u_2) + d_{\overline{G}_2^*}(v_2) + 2)$$
$$+ (p-1)(d_{G_2}(u_2) + d_{G_2}(v_2))$$
$$= k_2 + 2m(2p-n) \text{ ……......……………..…………………….………….. (4.14)}$$
$$\therefore d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + 2pd_{G_2}(u_2) + d_{\overline{G}_2^*}(u_2)(d_{G_1}(u_1) + d_{G_1}(v_1))$$
$$= k_1 + 2m(2p-n) \text{ ………….……….…..………......……….……………….. (4.15)}$$

From (4.14) and (4.15), $d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(u_1,v_2)) \neq d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,u_2))$, since $k_1 \neq k_2$.

This is a contradiction to our assumption that $G_1 \underset{\gamma}{\times} G_2$ is an edge regular fuzzy graph.

Hence G_1 and G_2 are edge regular fuzzy graphs of same degree.

Theorem: 4.9

Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $\sigma_1 \le \mu_2$ and let σ_1 be a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$. Let $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ be regular underlying crisp graphs of same degree with $|V_1| = |V_2|$. If G_1 is strong, then $G_1 \underset{\gamma}{\times} G_2$ is an edge regular fuzzy graph.

Proof:

Given $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with $\sigma_1 \le \mu_2$ and σ_1 is a constant function with $\sigma_1(u) = c_1$ for all $u \in V_1$ & G_1^* and G_2^* are regular underlying crisp graphs of same degree.

Since G_1 is strong, μ_1 is a constant function.

Then by theorem 2.5, $G_1 : (\sigma_1, \mu_1)$ is both regular and edge regular.

Thus $d_{G_1}(u_1v_1) = k, d_{G_1}(u_1) = m, d_{G_1^*}(u_1) = d_{G_2^*}(u_2) = n,\quad d_{\overline{G}_1^*}(u_1) = d_{\overline{G}_2^*}(u_2) = p-n,$ $\forall u_1, v_1 \in V_1$ & $u_2 \in V_2$, where k, m, n & $|V_1| = |V_2| = p$ are constants.

By theorem 4.6, for any $(u_1,u_2)(v_1,v_2) \in E$,

Case 1: When $u_1 = v_1, u_2v_2 \in E_2$,

$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(u_1,v_2)) = 2p_2 d_{G_1}(u_1) + c_1(d_{G_2^*}(u_2) + d_{G_2^*}(v_2))(d_{\overline{G}_1^*}(u_1)+1) - 2c_1$.

$= 2p_2 m + c_1(n+n)(p-n+1) - 2c_1$.

$= 2[p_2 m + nc_1(p-n+1) - c_1]$ ……………………………………………………………………… (4.16)

Case 2: When $u_2 = v_2, u_1v_1 \in E_1$,

$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,u_2)) = d_{G_1}(u_1v_1) + c_1 d_{G_2^*}(u_2)(2 + d_{\overline{G}_1^*}(u_1) + d_{\overline{G}_1^*}(v_1))$

$+ (p_2 - 1)(d_{G_1}(u_1) + d_{G_1}(v_1))$.

$= k + c_1 n(2 + p - n + p - n) + (p_2 - 1)(m+m)$.

$= 2m - 2c_1 + c_1 n(2 + p - n + p - n) + (p_2 - 1)(m+m)$. (By definition of edge degree $k = 2m - 2c_1$)

$= 2m - 2c_1 + 2c_1 n(1 + p - n) + 2p_2 m - 2m$.

$= 2[p_2 m + nc_1(p-n+1) - c_1]$ ……………………………………………………………………... (4.17)

Case 3: When $u_1v_1 \in E_1, u_2v_2 \notin E_2$ & $u_1v_1 \in E_1, u_2v_2 \in E_2$,

$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2)) = d_{G_1}(u_1v_1) + (p_2 - 1)(d_{G_1}(u_1) + d_{G_1}(v_1)) + c_1 d_{G_2^*}(u_2)(1 + d_{\overline{G}_1^*}(u_1))$

$+ c_1 d_{G_2^*}(v_2)(1 + d_{\overline{G}_1^*}(v_1))$.

$= k + (p_2 - 1)(m+m) + c_1 n(1 + p - n) + c_1 n(1 + p - n)$.

$= 2m - 2c_1 + 2p_2 m - 2m + 2c_1 n(1 + p - n)$. (By definition of edge degree $k = 2m - 2c_1$)

$= 2[p_2 m + nc_1(p-n+1) - c_1]$ ……………………………………………………………………… (4.18)

Case 4: When $u_1v_1 \notin E_1, u_2v_2 \in E_2$,

$d_{G_1 \underset{\gamma}{\times} G_2}((u_1,u_2)(v_1,v_2)) = p_2(d_{G_1}(u_1) + d_{G_1}(v_1)) + c_1 d_{G_2^*}(u_2)(1 + d_{\overline{G}_1^*}(u_1))$

$+ c_1 d_{G_2^*}(v_2)(1 + d_{\overline{G}_1^*}(v_1)) - 2c_1$.

$= 2p_2 m + c_1 n(1 + p - n) + c_1 n(1 + p - n) - 2c_1$.

$= 2[p_2 m + nc_1(p-n+1) - c_1]$ ……………………………………………………………………… (4.19)

From (4.16), (4.17), (4.18) & (4.19), $G_1 \underset{\gamma}{\times} G_2$ is an edge regular fuzzy graph.

V. Conclusion

In this paper, we have found a necessary and sufficient condition for alpha product and gamma product of two fuzzy graphs to be edge regular fuzzy graph.

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